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THE NUMBER OF TRADERS REQUIRED TO MAKE A MARKET COMPETITIVE: THE BEGINNINGS OF A THEORY

by

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Introduction

If the number of buyers and sellers trading within a market is large, then the market almost surely becomes perfectly competitive and therefore *ex post* Pareto efficient.[1] Left open by this result is the question of how many traders are required in a market to make it "large." For example, should a market with ten traders on each side be considered large enough to be perfectly competitive for all practical purposes? Evidence from controlled experiments with double auctions that Smith (1982, Proposition 5) has summarized indicates that the answer is yes empirically. Our goals in this theoretical paper are, first, to develop a general technique for studying this question within a simple market and, second, to demonstrate through application of this technique that—consistent with the experimental evidence—five or six traders on each side of the market is enough to generate essentially competitive, *ex post* efficient outcomes.

The simple market we work with is one in which N sellers have one indivisible unit each of the traded commodity, M buyers seek to purchase a single unit each, and buyers pay sellers for their purchases with money. Both buyers and sellers' preferences are fully described by the reservation values they place on a unit of the commodity. Each individual's reservation value is private and unverifiable by the other

[1] Roberts and Postlewaite (1976) show that as an economy is replicated repeatedly the incentive for agents to exhibit price-taking, competitive behavior increases except in special cases.

market participants. This means that the price observed in the market must be based in some manner on the values the individuals report, which may or may not be their true values. Consequently in a small market each individual has some influence on price and may decide to exaggerate his or her value strategically in order to manipulate the price up or down with the hope of securing a greater share of the available gains from trade.

It is this manipulation that causes small markets to be noncompetitive and *ex post* inefficient in their outcomes. This can be seen most clearly by considering the case of a market with a single seller and a single buyer. Suppose the reservation value of the seller is 48¢ and the reservation value of the buyer is 52¢. *Ex post* efficiency requires that the trade be consumated since the object is more valuable to the buyer than the seller. Nevertheless, depending on the seller's and buyer's beliefs about each other's reservation values, the trade may fail to take place. For example, if the buyer is quite confident that the seller's reservation value lies in the interval 25¢ to 55¢, he may hold out for a price less than 50¢. Similarly, if the seller is quite confident that the buyer's reservation value lies in the interval 45¢ to 75¢, then he may hold out for a price greater than 50¢. But if this happens no trade occurs and the outcome is *ex post* inefficient.

How many individuals have to be involved in the market in order to eliminate almost completely this behavior and the resulting inefficiency? We approach this question in three steps. First, we model the trading problem as a game of incomplete information where the appropriate equilibrium concept is the Bayesian Nash equilibrium.[2] Second, we generalize Myerson and Satterthwaite's results (1981) for bilateral trade to the case of multilateral trade. For the case of one buyer and one seller they used the revelation principle to characterize all individually rational, incentive compatible trading mechanisms and developed a technique for calculating <u>ex ante</u> efficient, individually rational, bilateral trading mechanisms. We derive parallel results for arbitrary numbers of buyers and sellers.

An individually rational trading mechanism is a mechanism such that, no matter what a trader's reservation value is, the expected utility of participating in the mechanism and attempting to make a trade is non-negative. An incentive compatible mechanism is a mechanism set up so that, given each individual's subjective prior distribution about every other individual's reservation value, everyone has an incentive to correctly state his or her true reservation value. The revelation principle states that, in terms of outcomes, every allocation mechanism is equivalent to some incentive compatible mechanism. Therefore in searching for <u>ex ante</u> efficient mechanisms only incentive compatible mechanisms need be considered. The revelation principle has its origins in Gibbard's paper (1973) on straightforward mechanisms and was developed by Myerson (1979 and 1981), Harris and Townsend (1981), and Harris and Raviv (1981). It played an essential role in Myerson and

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[2]Harsanyi (1967-68) introduced these concepts.

Satterthwaite (1981) and does so again in this paper.

Our third step is to apply this theory to a specific example. For simple markets ranging up to twelve individuals on each side we calculate the properties of the anonymous, ex ante efficient, incentive compatible trading mechanism.[3] The key assumptions of our example are that each individual's reservation value is drawn from a uniform distribution over the interval [0, 1], every individual knows only his own reservation value, and common knowledge exists among all individuals that every individual's reservation value is drawn uniformly from the unit interval. This, for the case of one buyer and one seller, is precisely the same example that Chatterjee and Samuelson (1979), Myerson and Satterthwaite (1981), and Wilson (1982) have used in their papers.

The results that we obtain as the number of buyers and sellers increase in tandem are striking. For the case of one individual on each side of the market the individually rational, ex ante efficient mechanism realizes in expectation 84.36% of the expected gains from trade that an ex post efficient mechanism would realize if one existed. In other words, if the ex ante efficient mechanism were used repeatedly with the reservation values of buyers and sellers being drawn independently and uniformly from the unit interval, then the total gains from trade realized by the participants would average out over the long

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[3] An anonymous mechanism is a mechanism that treats all buyers and sellers independently of their labels. Thus if buyers one and two report identical reservation values, then the mechanism must accord them equal probabilities of receiving an object and equal expected monetary payment.

run to 84.36% of the value to which it would average out if, for each draw, each individual's reservation value were common knowledge and the traded object were always assigned to the individual with the higher reservation value. If the number of individuals on each side of the market increases to six, then the *ex ante* efficient mechanism realizes in expectation 99.31% of the gains from trade that an *ex post* efficient mechanism would realize. For twelve individuals on each side of the market this number rises to 99.83%. These numerical results indicate that if the number of individuals on each side of the market increases, then the *ex post* gains from trade that the *ex ante* efficient mechanism fails to capture decreases quadratically. Consequently by the time the market reaches five or six individuals per side the degree of inefficiency is inconsequential.

The interest of our results is increased when they are considered in conjunction with Wilson's result (1982) that if individuals' reservation values are uniformly distributed over the unit interval, then the double sealed bid auction is equivalent to an anonymous, *ex ante* efficient, incentive compatible and individually rational trading mechanism. The rules of the double sealed bid auction are that each buyer submits a sealed bid, each seller submits a sealed offer, the bids and offers are arrayed against each other, a market clearing price is computed, and all trades that are feasible at that market clearing price are executed. This auction, which is not incentive compatible, is equivalent to an *ex ante* efficient, incentive compatible mechanism in the sense that both result in the same trades being executed and the same gains from trade being realized. Therefore mechanisms do exist that are used in practice and that are ex ante efficient, at least when buyers' and sellers' reservation values are uniformly distributed. This suggests the following conjecture—which lies quite beyond the scope of this paper—concerning how our and Wilson's results can be jointly generalized: for a large class of distributions of buyer and seller reservation values the double auction asymptotically approaches ex post efficiency quadratically.

Four substantive sections follow. First we present the model, second we characterize incentive compatible, individually rational mechanisms, third we describe how this characterization can be exploited to calculate ex ante efficient trading mechanisms that maximize the expected gains from trade, and fourth we calculate examples of such mechanisms when the number of traders on each side of the market ranges between one and twelve. It is the fourth section that constitutes our main contribution. The other three sections are a generalization to the multilateral case of Myerson and Satterthwaite's results (1981) for the bilateral case.[4] This generalization requires several complicating

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[4]Wilson (1982) also generalized Myerson and Satterthwaite's results (1981) to the multilateral case. The main differences are as follows. First, in our Lemmas 3 and 4 we derive the form that the payment schedules for individually rational, incentive compatible mechanisms must follow. These results have no parallel in his paper. Second, we have written our proofs to be reasonably accessible and convincing to readers who do not have a deep background in Bayesian games and incentive compatible mechanisms. Wilson sketched most of his proofs, which has the virtue of brevity, but also has disadvantage of imposing high costs on many readers. Finally, in our model we permit (i) the reservation value of each buyer i to be drawn from his or her own distribution F_i and (ii) the reservation value of each seller j to be drawn from his or her own

changes in the problem's formulation with consequent changes in Myerson and Satterthwaite's proofs. The proofs' modifications are significant enough that any but the most technically skilled reader would find their construction to be a major task. Therefore we have included complete proofs for each proposition we present.

The Model

The market we study consists of N identical objects, N sellers who each own one of the objects, M buyers who each seek to buy one of the objects, and money. Buyer i's reservation valuation of the object, which is the maximum amount that he can pay to purchase it and not reduce his utiliy, is x_i. He or she knows this value, but it is an unobservable quantity to all sellers and to all other buyers. Sellers and the other buyers regard x_i as distributed with positive density $f_i(\cdot)$ over the interval $[a_i, b_i]$. Similarly seller j knows z_j, his or her own reservation value. Buyers and other sellers regard it as distributed with positive density $h_j(\cdot)$ over $[c_j, d_j]$. Let the distribution functions of these densities be $F_i(\cdot)$ and $H_j(\cdot)$ respectively. We assume that a buyer i and seller j exist such that $b_i > c_j$.[5] All buyers and sellers consider the reservation values of

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distribution H_j. Wilson requires a common F for all buyers and a common H for all sellers.

[5]This assumption rules out the trivial case where trade between buyers and sellers is never in anyone's interest. Specifically, it

other buyers and sellers to be statistically independent both of each other and their own values. These densities and associated cumulative distribution functions constitute the essential data of the trading problem that we consider. We call the pair (F,H), where $F = (F_1, \ldots, F_M)$ and $H = (H_1, \ldots, H_N)$, the trading problem because for the trading situations we consider these M+N distribution functions are the primitive data.

Before defining what we mean by a trading mechanism, we must introduce some notation. Let $x = (x_1, \ldots, x_M)$, $z = (z_1, \ldots, z_N)$, $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_M)$, and $z_{-j} = (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_N)$. The density $g(x,z) = \Pi_{i=1}^{M} f_i(x_i) \cdot \Pi_{j=1}^{N} h_j(z_j)$ describes the joint distribution of all the reservation values, the density $g(x_{-i}, z) = g(x, z)/f_i(x_i)$ describes the distribution of reservation values buyer i perceives himself as facing, and the density $g(x, z_{-j}) = g(x, z)/h_j(z_j)$ describes the distribution of reservation values seller j perceives himself as facing.

A trading mechanism consists of N+M probability schedules and N+M payment schedules that determine the final distribution of money and goods given the N+M declared valuations of the buyers and sellers. Let the probabilities of an object being assigned to buyer i and seller j in the final distribution of goods be $p_i(x,z)$ and $q_j(x,z)$ respectively. Let the payments to buyer i and seller j be $r_i(x,z)$ and $s_j(x,z)$ respectively. A negative value for r_i indicates that buyer i pays

guarantees that with positive probability some buyer's reservation value will be greater than some seller's reservation value.

negative r_i units of money for receiving one unit of the traded object with probability p_i. The r_i and s_j payments, which are not conditional on whether buyer i actually receives an object or seller j actually gives up his object, may be regarded as certainty equivalents of payments that are made only when an individual is involved in a trade. Let a trading mechanism be denoted by the 2M+2N vector (p, q, r, s) of probability and payment schedules. We assume that the joint distribution of reservation values g, the probability schedules p and q, and the payment schedules r and s are common knowledge among all the buyers and sellers.

The payment and probability schedules are constrained so that in the final distribution of goods and money all N objects are assigned to some trader and payments exactly offset receipts. Thus:

$$\sum_{i=1}^{M} p_i(x,z) + \sum_{j=1}^{N} q_j(x,z) = N \tag{1}$$

and

$$\sum_{i=1}^{M} r_i(x,z) + \sum_{j=1}^{N} s_j(x,z) = 0 \tag{2}$$

for all (x, z).[6] The reason for this latter constraint is that trading

[6]Note that (1) requires a balance of goods only in expectation. Balance of goods can always be achieved in fact by making the assignments of the N objects to the N + M individuals correlated across individuals. Thus, for a given set of declared valuations, buyer 1 can be assigned an object with probability p_1 through an independent draw of a random number in the [0, 1] interval. Buyer 2 can next be assigned an object with probability p_2 through a second independent draw, etc. This process of assigning objects through independent draws first to the M buyers and then to the N sellers can be continued until either (a) all N objects have been assigned or (b) K objects remain and exactly K buyers

connotes individuals freely cooperating with one and another without intervention or aid from a third party. The trading process is initiated when all players declare reservation values to an arbiter. Given these submitted bids, money and the N objects are reallocated as the trading mechanism (p, q, r, s) mandates.

Each trader is an expected utility maximizer and has a von Neumann-Morgenstern utility function that is additively separable and linear in money and in the reservation value of the traded object. Thus buyer i's expected utility, given that his true reservation value is x_i and the vectors of declared reservation values are $\hat{x}$ and $\hat{z}$, is

$$\bar{U}_i(x_i, \hat{x}, \hat{z}) = r_i(\hat{x}, \hat{z}) + x_i p_i(\hat{x}, \hat{z}). \tag{3}$$

Seller j's expected utility, given that his true reservation value is z_j and the declared values are $\hat{x}$ and $\hat{z}$, is

$$\bar{V}_j(z_j, \hat{x}, \hat{z}) = s_j(\hat{x}, \hat{z}) - z_j + z_j q_j(\hat{x}, \hat{z}). \tag{4}$$

The buyers' utility functions $\bar{U}_i$ are normalized so that if $(\hat{x}, \hat{z})$ are such that buyer i is certain not to receive an object ($p_i = 0$) and is not required to make a cash payment ($r_i = 0$), then his expected utility is zero. The sellers' utility functions are normalized similarly. Put another way, if a buyer or seller elects not to participate in the trading mechanism, then his or her expected utility is zero.

We place two constraints on the mechanisms that we consider in our

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and sellers remain to have an object assigned to them. If eventuality (a) occurs, then the remaining buyers and sellers should be excluded from receiving an object. If eventuality (b) occurs, then the K remaining buyers and sellers should each receive an object. This rule guarantees that exactly N objects are distributed. The dependence that this rule induces between the probability of buyer 1 being assigned an object and seller N not being assigned an object has no effect on our results.

search for ex ante efficiency. First is individual rationality. It requires for each trader that, given any admissible reservation value, the expected utility of participating in the mechanism is nonnegative. If this constraint were violated, those individuals with unfavorable reservation values would decline to participate in the trading, thus contradicting our assumption that they do participate. Second is incentive compatibility. An incentive compatible mechanism never gives any trader an incentive to declare a reservation value different than his true reservation value, i.e., declaration of true values is always a Bayesian Nash equilibrium if the mechanism is incentive compatible. Imposing this constraint greatly simplifies the analytics of the problem. We lose no generality because the revelation principle states that for every mechanism an equivalent incentive compatible mechanism exists.

Formalization of the individual rationality and incentive compatibility constraints requires additional notation and definitions.[7] Let

$$\bar{p}_i(x_i) = \int\ldots\int p_i(x,z)g(x_{-i},z)dx_{-i}dz, \tag{5}$$

$$\bar{q}_j(z_j) = \int\ldots\int q_j(x,z)g(x,z_{-j})dxdz_{-j}, \tag{6}$$

$$\bar{r}_i(x_i) = \int\ldots\int r_i(x,z)g(x_{-i},z)dx_{-i}dz, \tag{7}$$

and

[7] The definitions that follow are written based on the assumption that all traders will in fact declare their true reservation values. This assumption is legitimate because we are considering only incentive compatible mechanisms.

$$\bar{s}_j(z_j) = \int \ldots \int s_j(x,z)g(x,z_{-j})dxdz_{-j}. \tag{8}$$

Conditional on buyer i's reservation value being x_i, the quantities $\bar{p}_i(x_i)$ and $\bar{r}_i(x_i)$ are respectively his expected probability of receiving an object and his expected money receipts. The quantities $\bar{q}_j$ and $\bar{s}_j$ have identical meanings for seller j. The expected utilities of buyer i and seller j conditional on their reservation values are

$$U_i(x_i) = \bar{r}_i(x_i) + x_i\bar{p}_i(x_i) \tag{9}$$

and

$$V_j(z_j) = z_j(\bar{q}_j(z_j) - 1) + \bar{s}_j(z_j). \tag{10}$$

In terms of these definitions, individual rationality requires that, for all buyers i and all sellers j, $U_i(x_i) \geq 0$ for every $x_i \in [a_i, b_i]$ and $V_j(z_j) \geq 0$ for every $z_j \in [c_j, d_j]$. Incentive compatibility is defined to be that, for every buyer i and all x_i and $\hat{x}_i$ in $[a_i, b_i]$,

$$U_i(x_i) \geq \bar{r}_i(\hat{x}_i) + x_i\bar{p}_i(\hat{x}_i) \tag{11}$$

and, for every seller j and all z and $\hat{z}$ in $[c_j, d_j]$,

$$V_j(z_j) \geq z_j(\bar{q}_j(\hat{z}_j) - 1) + \bar{s}_j(\hat{z}_j). \tag{12}$$

If (11) is violated for some x and $\hat{x}_i$, then buyer i has an incentive to declare $\hat{x}_i$ rather than his or her true reservation value, x_i. The parallel interpretation holds for (12). Inequalities (11) and (12) are therefore a necessary and sufficient condition that the honest declaration of reservation values is a Bayesian Nash equilibrium for the trading mechanism (p, q, r, s).

Characterization of Individually Rational Incentive Compatible Mechanisms

Theorem 1 characterizes all individually rational, incentive compatible mechanisms in a manner that is particularly convenient for computing *ex ante* efficient mechanisms. The theorem exactly generalizes Myerson and Satterthwaite's (1981) Theorem 1 from the bilateral case to the general case of arbitrary numbers of buyers and sellers.

Theorem 1. Let $p(\cdot,\cdot)$ and $q(\cdot,\cdot)$ be the buyers and sellers probability schedules respectively. Functions $r(\cdot,\cdot)$ and $s(\cdot,\cdot)$ exist such that (p,q,r,s) is an incentive compatible and individually rational mechanism if and only if $\bar{p}_i(\cdot)$ is a nondecreasing function for all buyers i, $\bar{q}_j(\cdot)$ is a nondecreasing function for all sellers j, and

$$\begin{aligned}&\sum_{i=1}^{M} \int\cdots\int \left(x_i + \frac{F_i(x_i) - 1}{f_i(x_i)}\right) p_i(x,z) g(x,z)\,dxdz \\ &\quad - \sum_{j=1}^{N} \int\cdots\int \left(z_j + \frac{H_j(z_j)}{h_j(z_j)}\right)[1 - q_j(x,z)] g(x,z)\,dxdz \geq 0.\end{aligned} \tag{13}$$

Furthermore, given any individually rational, incentive compatible mechanism, for all i and j, $U_i(\cdot)$ is nondecreasing, $V_j(\cdot)$ nonincreasing, and

$$\sum_{i=1}^{M} U_i(a_i) + \sum_{j=1}^{N} V_j(d_j) = \sum_{i=1}^{M} \min_{x\varepsilon[a_i,b_i]} U_i(x) + \sum_{j=1}^{N} \min_{z\varepsilon[c_j,d_j]} V_j(z)$$

$$= \sum_{i=1}^{M} \int \ldots \int \left(x_i + \frac{F_i(x_i) - 1}{f_i(x_i)}\right) p_i(x,z) g(x,z) dxdz \tag{14}$$

$$- \sum_{j=1}^{N} \int \ldots \int \left(z_j + \frac{H_j(z_j)}{h_j(z_j)}\right) [1 - q_j(x,z)] g(x,z) dxdz.$$

Proof of the theorem consists of the four lemmas that follow. Lemma 4 is interesting in its own right because it gives an explicit formula for calculating the payment schedules r_i and s_j from any pair of probability schedules p and q satisfying Theorem 1's conditions.

<u>Lemma 1</u>. If (p, q, r, s) is an incentive compatible mechanism, then, for all i and j, $\bar{p}_i(\cdot)$, $\bar{q}_j(\cdot)$, and $U_i(\cdot)$ are nondecreasing and $V_j(\cdot)$ is nonincreasing,

$$U_i(x_i) = U_i(a_i) + \int_{a_i}^{x_i} \bar{p}_i(t)dt, \tag{15}$$

and

$$V_j(z_j) = V_j(d_j) + \int_{z_j}^{d_j} [1 - \bar{q}_j(t)]dt. \tag{16}$$

Proof of Lemma 1. Let (p, q, r, s) be incentive compatible. The definitions of U_i, $\bar{r}_i$, $\bar{p}_i$, and incentive compatibility imply that, for all i, and for all x_i, $\hat{x} \in [a_i, b_i]$,

$$U_i(x_i) = \bar{r}_i(x_i) + x_i\bar{p}_i(x_i) \geq \bar{r}_i(\hat{x}) + x_i\bar{p}_i(\hat{x})$$

and

$$U_i(\hat{x}) = \bar{r}_i(\hat{x}) + \hat{x}\bar{p}_i(\hat{x}) \geq \bar{r}_i(x_i) + \hat{x}\bar{p}_i(x_i).$$

Subtracting the inequalities appropriately yields

$$U_i(x_i) - U_i(\hat{x}) \geq \bar{r}_i(\hat{x}) + x_i\bar{p}_i(\hat{x}) - \bar{r}_i(\hat{x}) - \hat{x}\bar{p}_i(\hat{x})$$
$$= (x_i - \hat{x})\bar{p}_i(\hat{x})$$

and

$$U_i(x_i) - U_i(\hat{x}) \leq \bar{r}_i(x_i) + x_i\bar{p}_i(x_i) - \bar{r}_i(x_i) - \hat{x}\bar{p}_i(x_i)$$
$$= (x_i - \hat{x})\bar{p}_i(x_i).$$

Combining these two inequalities leaves

$$(x_i - \hat{x})\ \bar{p}_i(x_i) \geq U_i(x_i) - U_i(\hat{x}) \geq (x_i - \hat{x})\ \bar{p}_i(\hat{x}). \qquad (17)$$

Inspection of (17) shows that if $x_i > \hat{x}$, then $\bar{p}_i(x_i) \geq p_i(\hat{x})$. Thus $\bar{p}_i(\cdot)$ is a nondecreasing function. Since $\bar{p}_i$ is monotonic, dividing through by $(x_i - \hat{x})$ and taking the limit as $\hat{x} \to x_i$ gives the result $U_i'(x_i) = \bar{p}_i(x_i)$ almost everywhere. Also because of its monotonicity, $\bar{p}_i(\cdot)$ is Riemann integrable. This gives the desired expression:

$$U_i(x_i) = U_i(a_i) + \int_{a_i}^{x_i} \bar{p}_i(t)dt.$$

Inspection of the integral shows that $U_i(\cdot)$ is nondecreasing as asserted. The analogous properties of $\bar{q}_j$ and V_j may be derived through an exactly parallel argument.●

<u>Lemma 2</u>. If (p, q, r, s) is an individually rational and incentive compatible mechanism, then

$$\sum_{i=1}^{M} U_i(a_i) + \sum_{j=1}^{N} V_j(d_j) = \sum_{i=1}^{M} \min_{x\varepsilon[a_i,b_i]} U_i(x) + \sum_{j=1}^{N} \min_{z\varepsilon[c_j,d_j]} V_j(z)$$

$$= \sum_{i=1}^{M} \int\ldots\int(x_i + \frac{F_i(x_i) - 1}{f_i(x_i)})p_i(x,z)g(x,z)dxdz \qquad (18)$$

$$- \sum_{j=1}^{N} \int\ldots\int(z_j + \frac{H_j(z_j)}{h_j(z_j)})[1 - q_j(x,z)]g(x,z)dxdz$$

$$\geq 0.$$

Proof of Lemma 2. Let T be the unconditional, ex ante expected gains from trade:

$$T = \sum_{i=1}^{M} \int_{a_i}^{b_i} U_i(x)f_i(x)dx + \sum_{j=1}^{N} \int_{c_j}^{d_j} V_j(z)h_j(z)dz. \qquad (19)$$

Substitution of equation (2) and equations (5) through (10) into (19) gives:

$$T = \sum_{i=1}^{M} \int\ldots\int[x_i p_i(x,z) + r_i(x,z)]g(x,z)dxdz$$

$$+ \sum_{j=1}^{N}\{\int\ldots\int[z_j q_j(x,z) + s_j(x,z)]g(x,z)dxdz - \int_{c_j}^{d_j} z_j h_j(z)dz\}$$

$$= \sum_{i=1}^{M} \int\ldots\int[x_i p_i(x,z) + r_i(x,z)]g(x,z)dxdz$$

$$+ \sum_{j=1}^{N}\int\ldots\int[z_j(q_j(x,z) - 1) + s_j(x,z)]g(x,z)dxdz$$

$$= \int\ldots\int\{\sum_{i=1}^{M} x_i p_i(x,z) - \sum_{j=1}^{N} z_j[1-q_j(x,z)]\}g(x,z)dxdz. \qquad (20)$$

We obtain an alternative expression for T through substitution of (15) and (16) into (19):

$$T = \sum_{i=1}^{M} \{U_i(a_i) + \int_{a_i}^{b_i}\int_{a_i}^{x} \bar{p}_i(t)f_i(x)dtdx\}$$

$$+ \sum_{j=1}^{N} \{V_j(d_j) + \int_{c_j}^{d_j}\int_{z}^{d_j} [1 - \bar{q}_j(t)]\ h_j(z)dtdz\}$$

$$= \sum_{i=1}^{M} \{U_i(a_i) + \int_{a_i}^{b_i}\int_{t}^{b_i} \bar{p}_i(t) f_i(x)dxdt\}$$

$$+ \sum_{j=1}^{N} \{V_j(d_j) + \int_{c_j}^{d_j}\int_{c_j}^{t} (1 - \bar{q}_j(t))h_j(z)dzdt\}$$

$$= \sum_{i=1}^{M} \{U_i(a_i) + \int_{a_i}^{b_i} \bar{p}_i(t)[1 - F_i(t)]dt\} + \sum_{j=1}^{N} \{V_j(d_j) + \int_{c_j}^{d_j}[1 - \bar{q}_j(t)]H_j(t)dt\}$$

$$= \sum_{i=1}^{M} \{U_i(a_i) + \int\ldots\int(1 - F_i(x_i))p_i(x,z)g(x_{-i},z)dxdz\}$$

$$+ \sum_{j=1}^{N} \{V_j(d_j) + \int\ldots\int[1 - q_j(x,z)]H_j(z_j)g(x,z_{-j})dxdz\}$$

$$= \sum_{i=1}^{M} \{U_i(a_i) + \int\ldots\int p_i(x,z)(\frac{1 - F_i(x_i)}{f_i(x_i)})g(x,z)dxdz\}$$

$$+ \sum_{j=1}^{N} \{V_j(d_j) + \int\ldots\int(1 - q_j(x,z))\frac{H_j(z_j)}{h_j(z_j)}\, g(x,z)dxdz\}. \tag{21}$$

Lines three and four of (21) result from changing the order of integration and lines six and seven result from substituting in (5) and (6).

Equating (20) and (21), which are two alternative expressions for T, and collecting similar terms produces:

$$\sum_{i=1}^{M} U_i(a_i) + \sum_{j=1}^{N} V_j(d_j) = \sum_{i=1}^{M} \int\ldots\int x_i p_i(x,z)g(x,z)dxdz$$

$$+ \sum_{j=1}^{N} \int\ldots\int z_j(q_j(x,z) - 1)g(x,z)dxdz$$

$$+ \sum_{i=1}^{M} \int \ldots \int \left(\frac{F_i(x_i) - 1}{f_i(x_i)}\right) p_i(x,z) g(x,z) dxdz$$

$$- \sum_{j=1}^{N} \int \ldots \int [1 - q_j(x,z)] \frac{H_j(z_j)}{h_j(z_j)} g(x,z) dxdz$$

$$= \sum_{i=1}^{M} \int \ldots \int [x_i + \left(\frac{F_i(x_i) - 1}{f_i(x_i)}\right)] p_i(x,z) g(x,z) dxdz$$

$$- \sum_{j=1}^{N} \int \ldots \int \{z_j + \frac{H_j(z_j)}{h_j(z_j)}\} [1 - q_j(x,z)] g(x,z) dxdz. \tag{22}$$

Lemma 1 showed that, for all i and j, $U_i(\cdot)$ is nondecreasing and $V_j(\cdot)$ is nonincreasing. Consequently U_i and V_j attain their minimum values at a_i and d_j respectively. Therefore

$$\sum_{i=1}^{M} U_i(a_i) + \sum_{j=1}^{N} V_j(c_j)$$

$$= \sum_{i=1}^{M} \min_{x_i \varepsilon [a_i, b_i]} U_i(x_i) + \sum_{j=1}^{N} \min_{z_j \varepsilon [c_j, d_j]} V_j(z_j) \geq 0 \tag{23}$$

where the individual rationality constraints generate the final inequality. Equations (22) and (23) together confirm equation (18).●

<u>Lemma 3</u>. If (p, q, r, s) is an incentive compatible mechanism, then, for all i and j,

$$\bar{r}_i(x) = \int_{a_i}^{x} t d[-\bar{p}_i(t)] + C_i \tag{24}$$

and

$$\bar{s}_j(z) = \int_{z}^{d_j} t \bar{q}_j(t) + D_j \tag{25}$$

where C_i and D_j are constants of integration. Suppose, for all i and j, that $\bar{p}_i(\cdot)$ and $\bar{q}_j(\cdot)$ are nondecreasing and that

$r_i(\cdot, \cdot)$ and $s_j(\cdot, \cdot)$ have the properties that $\bar{r}_i(\cdot)$ and $\bar{s}_j(\cdot)$ are respectively of the form of (24) and (25). Then the mechanism (p, q, r, s) is incentive compatible.

Proof of Lemma 3. As a variation we present the proof for the seller's side of the market. The proof for the buyer's side is analogous. Suppose (p, q, r, s) is incentive compatible. Definitionally, for all j, $V_j(z) = z(\bar{q}_j(z) - 1) + \bar{s}_j(z)$. Rearranged this becomes $\bar{s}_j(z) = V_j(z) + z(1 - \bar{q}_j(z))$. Therefore

$$\bar{s}_j(z) - \bar{s}_j(\hat{z}) = V_j(z) - V_j(\hat{z}) + z(1 - \bar{q}_j(z)) - \hat{z}(1 - \bar{q}_j(\hat{z}))$$

$$= V_j(z) - V_j(\hat{z}) + \int_{\hat{z}}^{z} d[t(1 - q_j(t)].$$

Since (p, q, r, s) is incentive compatible, equation (16) of Lemma 1 implies that

$$V_j(z) - V_j(\hat{z}) = \int_z^{\hat{z}}(1 - \bar{q}_j(t))dt.$$

Consequently,

$$\bar{s}_j(z) - \bar{s}_j(\hat{z}) = \int_z^{\hat{z}}(1 - \bar{q}_j(t))dt + \int_{\hat{z}}^{z} d[t(1 - \bar{q}_j(t))]$$

$$= \int_z^{\hat{z}}(1 - \bar{q}_j(t))dt + \int_{\hat{z}}^{z} (1 - \bar{q}_j(t))dt + \int_{\hat{z}}^{z} [-td\bar{q}_j(t)]$$

$$= \int_z^{\hat{z}} td\bar{q}_j(t).$$

This means that

$$\bar{s}_j(z) = \int_z^{d_j} td\bar{q}_j(t) + D_j,$$

which completes the proof of the Lemma's necessity part.

The sufficiency part's proof is as follows. Assume that, for all j, $\bar{q}_j(\cdot)$ is nondecreasing and that $\bar{s}_j(z)$ has the form of equation (25). Incentive compatibility for sellers demands that, for all j and for all $z, \hat{z} \in [c_j, d_j]$, $V_j(z) \geq z(\bar{q}_j(\hat{z}) - 1) + \bar{s}_j(\hat{z})$. Use of equation (10), the definition of $V_j(\cdot)$, permits us to rewrite the requirement of incentive compatibility as

$$z(\bar{q}_j(z) - 1) + \bar{s}_j(z) \geq z(\bar{q}_j(\hat{z}) - 1) + \bar{s}_j(\hat{z})$$

or, after rearrangement,

$$z(\bar{q}_j(z) - \bar{q}_j(\hat{z})) + \bar{s}_j(z) - \bar{s}_j(\hat{z}) \geq 0. \tag{26}$$

Equation (25) implies that

$$\bar{s}_j(z) - \bar{s}_j(\hat{z}) = \int_z^{\hat{z}} t\,d\bar{q}_j(t).$$

Notice that

$$z(\bar{q}_j(z) - \bar{q}_j(\hat{z})) = z\int_{\hat{z}}^{z} d\bar{q}_j(t).$$

Addition of these last two equations gives the result:

$$\begin{aligned} &z(\bar{q}_j(z) - \bar{q}_j(\hat{z})) + \bar{s}_j(z) - \bar{s}_j(\hat{z}) \\ &\qquad = z\int_{\hat{z}}^{z} d\bar{q}_j(t) + \int_z^{\hat{z}} t\,d\bar{q}_j(t) = \int_{\hat{z}}^{z} (z-t)\,d\bar{q}_j(t) \geq 0 \end{aligned}$$

where the first line is just the left-hand side of the incentive compatibility requirement (26). The inequality at the end of the second line, which is necessary for (26) to be satisfied, follows from $\bar{q}_j(\cdot)$ being nondecreasing and, consequently, the integrand $(z-t)d\bar{q}_j(t)$ being nonnegative for all admissible z and $\hat{z}$.●

<u>Lemma 4</u>. Suppose that the probability schedules $p(\cdot,\cdot)$ and $q(\cdot,\cdot)$ satisfy Lemma 2's equation (18) and have the property that $\bar{p}_i(\cdot)$ and $\bar{q}_j(\cdot)$ are nondecreasing for all i and j. Let r_i and s_j have the forms:

$$r_i(x,z) = \sum_{k=1}^{M} \int_{a_k}^{x_k} t\,d[-\bar{p}_k(t)] + \sum_{\substack{k=1 \\ k\neq i}}^{M} \int_{a_k}^{b_k} t[1 - F_k(t)]d[\bar{p}_k(t)]$$

$$+ \frac{N}{M} \sum_{k=1}^{N} \{\int_{c_k}^{d_k} tH_k(t)d\bar{q}_k(t) - \int_{z_k}^{d_k} t\,d\bar{q}_k(t)\} + C_k, \quad i=1,\ldots,M; \tag{27}$$

$$s_j(x,z) = \sum_{k=1}^{M} \{\int_{a_k}^{x_k} t\,d\bar{p}_k(t) + \int_{a_k}^{b_k} t[1 - F_k(t)]d[-\bar{p}_k(t)]\}$$

$$+ \sum_{k=1}^{N} \int_{z_k}^{d_k} t\,d\bar{q}_k(t) - \sum_{\substack{k=1 \\ k\neq j}}^{N} \int_{c_k}^{d_k} tH_k(t)d\bar{q}_k(t) + D_j, \quad j=1,\ldots,N-1; \tag{28}$$

$$s_N(x,z) = -[\sum_{k=1}^{M} r_k(x,z) + \sum_{k=1}^{N-1} s_k(x,z)]. \tag{29}$$

where the $N+M-1$ constants C_i and D_j are set so that $U_i(a_i) = 0$ $(i=1,\ldots,M)$ and $V_j(d_j) = 0$ $(j=1,\ldots,N-1)$. The resulting mechanism (p, q, r, s) is individually rational and incentive compatible.

Proof of Lemma 4. The first part of the proof consists of integrating (27), (28), and (29) to confirm that, for all i and j, $\bar{r}_i(\cdot)$ and $\bar{s}_j(\cdot)$ are of the forms (25) and (26) that, according to Lemma 3, are necessary for a mechanism to be incentive compatible. The second part of the proof consists of observing that the $M+N-1$ constants, C_i and D_j, can be set to guarantee satisfaction of the $M+N$ individual rationality constraints.

Recall the two definitions:

$$\bar{r}_i(x_i) = \int \ldots \int r_i(x,z)g(x_{-i},z)dx_{-i}dz$$

and

$$\bar{s}_j(z_j) = \int \ldots \int s_j(x,z)g(x,z_{-j})dxdz_{-j}.$$

We begin by integrating $r_i(\cdot, \cdot)$ to obtain $\bar{r}_i(\cdot)$. Integration of the expression within the first summation of (27) gives, for k=i,

$$\int_{a_i}^{x_i} td[-\bar{p}_i(t)].$$

When $k \neq i$ it is

$$\int_{a_k}^{b_k}\int_{a_k}^{x_k} td[-\bar{p}_k(t)]f_k(x_k)dx_k =$$

$$\int_{a_k}^{b_k}\int_{t}^{b_k} tf_k(x_k)dx_k d[-\bar{p}_k(t)] = \int_{a_k}^{b_k} t(1 - F_k(t))d[-\bar{p}_k(t)]$$

where the second line results from changing the order of integration. Therefore integration over the first two summations yields:

$$\int_{a_i}^{x_i} td[-p_i(t)] + \sum_{k\neq i}\int_{a_k}^{b_k} t(1 - F_k(t))d[-p_k(t)] + \sum_{\substack{k=1\\k\neq i}}^{M}\int_{a_k}^{b_k} t(1 - F_k(t))d[\bar{p}_k(t)]$$

$$= \int_{a_i}^{x_i} td[-\bar{p}_i(t)]. \tag{30}$$

Next we integrate the third summation term. Consider its second term:

$$\int \ldots \int\int_{z_k}^{d_k} td\bar{q}_k(t)g(x_{-i},z)dx_{-i}dz$$

$$= \int_{c_k}^{d_k}\int_{z_k}^{d_k} th_k(z_k)d\bar{q}_k(t)dz_k$$

$$= \int_{c_k}^{d_k}\int_{c_k}^{t} th_k(z_k)dz_k d\bar{q}_k(t) = \int_{c_k}^{d_k} tH_k(t)d\bar{q}_k(t). \tag{31}$$

where on the second line we integrate out all the densities except $h_k(\cdot)$ and on the third line we switch the order of integration.

Inspection of the summation shows that its two terms exactly offset each other, i.e., its value is zero. Therefore (30) plus C_j is the result:

$$\bar{r}_i(x_i) = \int \ldots \int r_i(x,z) f(x_{-i},z) dx_{-i} dz$$
$$= \int_{a_i}^{x_i} t d[-\bar{p}_i(t)] + C_i, \qquad i=1,\ldots,M. \tag{32}$$

Note that $\bar{r}_i(\cdot)$ has the form that Lemma 3 requires for incentive compatibility. Therefore the specification of equation (27) for $r_i(x, z)$ is consistent with (p, q, r, s) being an incentive compatible mechanism.

For the first N-1 sellers integrating $s_j(x,z)$ to obtain $\bar{s}_j$ produces analogous results. Thus

$$\bar{s}_j(z_j) = \int_{z_j}^{d_j} t d\bar{q}_j(t) + D_j, \quad j=1,\ldots,N-1,$$

which is consistent with Lemma 3's requirement for incentive compatibility.

To complete our proof that (p, q, r, s) is incentive compatible we must show that $s_N(\cdot,\cdot)$ as specified by equation (29) implies a form for $\bar{s}_N(z_N)$ that is consistent with Lemma 3's requirement for incentive compatibility. Recall that:

$$s_N(x,z) = -[\sum_{i=1}^{M} r_i(x,z) + \sum_{j=1}^{N-1} s_j(x,z)];$$

and

$$s_N(z_N) = \int \ldots \int s_N(x,z) f(x,z_{-N}) dx dz_{-N}.$$

We begin with integration of the individual $r_i(\cdot,\cdot)$ terms:

$$\int \ldots \int r_i(x,z) g(x,z_{-N}) dx dz_{-N}$$
$$= \sum_{k=1}^{M} \int_{a_k}^{b_k} t(1 - F_k(t)) d[-\bar{p}_k(t)] + \sum_{\substack{k=1 \\ k \neq i}}^{M} \int_{a_k}^{b_k} t(1 - F_k(t)) d\bar{p}_k(t)$$

$$+\frac{N}{M}\sum_{k=1}^{N}\int_{c_k}^{d_k} tH_k(t)d\bar{q}_k(t) - \frac{N}{M}\sum_{k=1}^{N-1}\int_{c_k}^{d_k} tH_k(t)d\bar{q}_k(t) \tag{33}$$

$$-\frac{N}{M}\int_{z_N}^{d_N} td\bar{q}_N(t) + C_i$$

$$= \int_{a_i}^{b_i} t(1-F_i(t))d[-\bar{p}_i(t)] + \frac{N}{M}\int_{c_N}^{d_N} tH_N(t)d\bar{q}_N(t) - \frac{N}{M}\int_{z_N}^{d_N} td\bar{q}_N(t) + C_i$$

where $i = 1, \ldots, M$.

Integration of the individual $s_j(x,z)$ terms gives:

$$\int\ldots\int s_j(x,z)g(x,z_{-N})dxdz_{-N}$$

$$= \sum_{k=1}^{M}\{\int_{a_k}^{b_k} t[1 - F_k(t)]d\bar{p}_k(t) + \int_{a_k}^{b_k} t[1 - F_k(t)]d[-\bar{p}_k(t)]\}$$

$$+ \int_{z_N}^{d_N} td\bar{q}_N(t) + \sum_{k=1}^{N-1}\int_{c_k}^{d_k} tH_k(t)d\bar{q}_k(t) - \sum_{\substack{k=1 \\ k\neq j}}^{N}\int_{c_k}^{d_k} tH_k(t)d\bar{q}_k(t) + D_j$$

$$= \int_{c_j}^{d_j} tH_j(t)d\bar{q}_j(t) - \int_{c_N}^{d_N} tH_N(t)d\bar{q}_N(t) + \int_{z_N}^{d_N} td\bar{q}_N(t) + D_j \tag{34}$$

where $j = 1, \ldots, N-1$. Summing (33) and (34) produces:

$$\bar{s}_N(z_N) = -\sum_{k=1}^{M}\{\int_{a_k}^{b_k} t[1 - F_k(t)]d[-\bar{p}_k(t)] + C_k\}$$

$$+ N\{-\int_{c_N}^{d_N} tH_N(t)d\bar{q}_N(t) + \int_{z_N}^{d_N} td\bar{q}_N(t)\}$$

$$- \sum_{k=1}^{N-1}\{\int_{c_k}^{d_k} tH_k(t)d\bar{q}_k(t) + D_k\}$$

$$+ (N - 1)\{\int_{c_N}^{d_N} tH_N(t)d\bar{q}_N(t) - \int_{z_N}^{d_N} td\bar{q}_N(t)\}.$$

If we define

$$D_N = -\sum_{k=1}^{M}\int_{a_k}^{b_k} t[1 - F_k(t)]d[-\bar{p}_k(t)] - \sum_{k=1}^{M} C_k$$

$$- \sum_{k=1}^{N}\int_{c_k}^{d_j} tH_k(t)d\bar{q}_k(t) - \sum_{k=1}^{N-1} D_k, \tag{35}$$

then

$$\bar{s}_N(z_N) = \int_{z_N}^{d_N} td\bar{q}_N(t) + D_N, \tag{36}$$

which fulfills Lemma 3's requirement for incentive compatibility. Therefore (p, q, r, s) is incentive compatible because (r, s) satisfy Lemma 3's requirements and, by assumption, (p, q) satisfy Theorem 1's requirements.

For every seller i and every buyer j, individual rationality requires $U_i(x) \geq 0$ for all $x \in [a_i, b_i]$ and $V_j(z) \geq 0$ for all $z \in [c_j, d_j]$. Since the mechanism is incentive compatible, Lemma 1 implies that both U_i and V_j are monotonic; therefore we need only to set the M+N-1 constants C_i and D_j so that, for all i and j, $U_i(a_i) = 0$ and $V_i(d_i) = 0$. From equations (9) and (10) we know that:

$$U_i(a_i) = \bar{r}_i(a_i) + a_i\bar{p}_i(a_i), \quad i=1,\ldots,M;$$

and

$$V_j(d_j) = d_j(\bar{q}_j(d_j) - 1) + \bar{s}_j(d_j), \quad j=1,\ldots,N.$$

From the first part of this Lemma's proof we also know that:

$$\bar{r}_i(a_i) = \int_{a_i}^{a_i} td[-\bar{p}_i(t)] + C_i = C_i, \quad i=1,\ldots,M;$$

and

$$\bar{s}_j(d_j) = \int_{d_j}^{d_j} td\bar{q}_j(t) + D_j = D_j \quad j=1,\ldots,N-1.$$

For the M buyers and the first N-1 sellers the constants C_i and D_j are arbitrary constants of integration. Therefore let $C_i = -a_i\bar{p}_i(a_i)$, for

$i = 1, \ldots, M$, and let $D_j = d_j(1 - \bar{q}_j(d_j))$, for $j = 1, \ldots, N-1$. This convention guarantees that, for the M buyers, $U(a_i) = 0$ and, for the first N-1 sellers, $V(d_i) = 0$.

This set of M+N-1 constants determines the value of D_N through equation (35). The hypothesis of this Lemma states that the probability schedules (p, q) satisfy equation (18) of Lemma 2:

$$\sum_{i=1}^{M} U(a_i) + \sum_{j=1}^{N} V(d_j) \geq 0. \tag{37}$$

The value of $V(d_N)$ must be nonnegative since the value of every other term in (37) is zero. Therefore the mechanism is individually rational for all buyers and sellers.●

Lemmas 1, 2, and 4 together imply Theorem 1.

Constructing an *Ex Ante* Efficient, Individually Rational, Incentive Compatible Trading Mechanism

A trader's *ex ante* expected utility from participating in trade is his expected utility evaluated before he learns his reservation value for the object. Thus $\tilde{U}_i = \int U_i(t)f_i(t)dt$ and $\tilde{V}_j = \int V_j(t)h_j(t)dt$ are respectively buyer i and seller j's *ex ante* expected utilities. An individually rational trading mechanism is *ex ante* Pareto optimal if no trader's *ex ante* expected utility can be increased without decreasing some other trader's *ex ante* expected utility. As we discussed in the paper's first two sections, individual rationality is necessary for a trading mechanism to be feasible in the sense of securing the traders' voluntary participation. We restrict ourselves to incentive compatible mechanisms because, according to the revelation principle, we lose no generality by doing so.

Within our particular model a mechanism is _ex ante_ Pareto optimal only if it maximizes the sum of the traders' _ex ante_ expected utilities or, equivalently, maximizes the sum of their expected gains from trade.[8] This is a direct consequence of our assumption of transferable utility, i.e. each trader's utility function is additively seperable in money and the traded object's reservation value. An _ex post_ optimal trading mechanism is one that assigns the N traded objects to the N traders who have the highest reservation values. Clearly (for transferable utility) an _ex post_ optimal mechanism is also _ex ante_ optimal.[9]

Theorem 1 in the preceding section characterizes all probability schedules (p, q) that can be the basis of an individually rational, incentive compatible trading mechanism for the trading problem (F, H). In this section we develop a technique for selecting from among that set of probability schedules the pair of schedules that maximizes the expected gains from trade and is therefore _ex ante_ efficient. To accomplish this, we generalize the technique Myerson and Satterthwaite

[8] Recall that every trader's expected utility is normalized to be zero for the case where trade is impossible. Therefore maximizing the expected gains from trade is equivalent to maximizing the sum of all traders _ex ante_ expected utilities.

[9] See Holmstrom and Myerson (1981) for further discussion of _ex ante_ optimality, _ex post_ optimality, and a third optimality concept, _interim_ optimality.

(1981) used for the bilateral case. Our results are summarized in two theorems. Theorem 2 states sufficient conditions for a particular trading mechanism--the α^* mechanism--to be ex ante efficient. Theorem 3 states sufficient conditions for the α^* mechanism to exist and be ex ante efficient.

Two functions play a crucial role in our construction:

$$\psi_i^B(x_i,\alpha) = x_i + \alpha \cdot \left(\frac{F_i(x_i) - 1}{f_i(x_i)}\right), \quad i=1,\ldots,M, \tag{38}$$

and

$$\psi_j^S(z_j,\alpha) = z_j + \alpha \cdot \frac{H_j(z_j)}{h_j(z_j)}, \quad j=1,\ldots,N, \tag{39}$$

where α is a nonnegative scalar. In the terminology of Myerson (1982), for a given α and a given x_i, the quantity ψ_i^B is buyer i's "virtual" reservation value for the traded object. Similarly, ψ_j^S is seller j's virtual reservation value. Let $\psi(x, z, \alpha) = \{\psi_1^B(x_1\ \alpha), \ldots, \psi_N^S(z_N, \alpha)\}$; it is the M+N vector of the traders' virtual reservation values.

Define $R_i(x, z, \alpha)$ to be the rank of the element $\psi_i^B(x_i, \alpha)$ within ψ. Similarly, let $R_j(x, z, \alpha)$ is the rank of the element $\psi_j^S(z_j, \alpha)$ within ψ. For example, if $M = N = 1$ and $\psi = (.2, .4)$, then $R_{i=1} = 2$ and $R_{j=1} = 1$. Given this notation, we can define a class of buyer and seller probability schedules that α parameterizes:[10]

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[10] If several elements of ψ have the same value so that it is ambiguous which buyers and sellers should be classified as having virtual reservation prices as ranking within the top N, then the probability schedules should randomize among the several candidates so as to guarantee that exactly N traders are assigned an object. Thus if seller 2 and buyer 3 are tied for rank N, then each should be given a nonindependent probability of .5 of receiving an object in the final allocation.

$$p_i^\alpha(x,z) = \begin{cases} 1 & \text{if } R_i(x,\ z,\ \alpha) \leqslant N \\ 0 & \text{if } R_i(x,\ z,\ \alpha) > N \end{cases} \qquad i=1,\ldots,M; \tag{40}$$

$$q_j^\alpha(x,z) = \begin{cases} 1 & \text{if } R_j(x,\ z,\ \alpha) \leqslant N \\ 0 & \text{if } R_j(x,\ z,\ \alpha) > N \end{cases} \qquad i=1,\ldots,N. \tag{41}$$

Let $p^\alpha = (p_1^\alpha, \ldots, p_M^\alpha)$ and $q^\alpha = (q_1^\alpha, \ldots, q_N^\alpha)$. These probability schedules assign the N available objects to those N traders for whom the objects have the highest virtual reservation values. Given (p^α, q^α), let $r^\alpha = (r_1^\alpha, \ldots, r_M^\alpha)$ and $s^\alpha = (s_1^\alpha, \ldots, s_N^\alpha)$ be the payment schedules that equations (27), (28), and (29) of Lemma 4 specifies. We call the trading mechanism $(p^\alpha, q^\alpha, r^\alpha, s^\alpha)$ an α-mechanism. If (p^α, q^α) satisfy Theorem 1's requirements, then Lemma 4 implies that the associated α-mechanism is individually rational and incentive compatible.

Central to Theorem 1's requirements is inequality (13), the incentive compatibility and individual rationality constraint. For the case of α-mechanisms, substitution of (38) and (39) into (13) yields:

$$G(\alpha) = \int\ldots\int\Big\{\sum_{i=1}^{M}\psi_i^B(x_i,1)p_i^\alpha(x,z) - \sum_{j=1}^{N}\psi_j^S(z_j,1)[1-q_j^\alpha(x,z)]\Big\}g(x,z)dxdz \tag{42}$$

$$\geqslant 0.$$

Let an α-mechanism be called an α^*-mechanism only if (i) $G(\alpha^*) = 0$ for some $\alpha^* \varepsilon [0, 1]$ or (ii) $G(0) > 0$.

Theorem 2 states conditions under which an α^*-mechanism—if such a mechanism exists—is <u>ex ante</u> efficient, incentive compatible, and individually rational. Theorem 3 specifies conditions sufficient for the existence of an α^*-mechanism that is <u>ex ante</u> efficient, incentive compatible, and individually rational. These two theorems, together with Lemma 4, provide a straightforward, recipe of four steps for constructing an <u>ex ante</u> trading mechanism for a trading problem, (F, H). The steps are : (i) Verify that the distributions (F, H) satisfy the requirements of Theorem 3. (ii) Calculate α^* by solving the equation $G(\alpha) = 0$. (iii) Construct the probability schedules $(p^{\alpha^*}, q^{\alpha^*})$. (iv) Construct the expected payment schedules (r, s) using Lemma 4's formulae.

<u>Theorem 2.</u> Let (F, H) describe a trading problem for which an α^*-mechanism exists. Suppose that the distributions (F, H) have the properties that, for every buyer i and seller j , $\bar{p}_i^{\alpha^*}(x_i)$ and $\bar{q}_j^{\alpha^*}(z_j)$ are nondecreasing over the intervals $[a_i, b_i]$ and $[c_j, d_j]$ respectively. The α^*-trading mechanism $(p^{\alpha^*}, q^{\alpha^*}, r^{\alpha^*}, s^{\alpha^*})$ is <u>ex ante</u> efficient, individually rational, and incentive compatible for (F, H). Its expected gains from trade are positive.

Lemma 5, which derives monotonicity properties for $G(\alpha)$, lays the groundwork for the Theorem 2's proof.

<u>Lemma 5.</u> For $\alpha \in [0, 1]$, $G(\alpha)$ is nondecreasing and, for $\alpha \geq 1$, $G(\alpha)$ is nonincreasing.

Proof of Lemma 5. Let $\gamma, \beta \in [0, 1)$ and $\beta > \gamma$. Fix the value of (x, z) at any point within $G(\alpha)$'s region of integration. The integrand of G is

$$K(\alpha,x,z) = \Big(\sum_{i=1}^{M} \psi_i^B(x_i,1)p_i^\alpha(x,z) - \sum_{j=1}^{N} \psi_j^S(z_j,1)[1-q_j^\alpha(x,z)]\Big)g(x,z). \quad (43)$$

As α increases the value of $\psi_i^B(x_i, \alpha)$ decreases linearly for all buyers i and the value of $\psi_j^S(z_j, \alpha)$ increases linearly for all sellers j. These changes in values affect the values of the probability schedules $p_i^\alpha(x, z)$ and $q_j^\alpha(x, z)$ and therefore cause $K(\alpha, x, z)$ to change in value.

For example, one possibility is that, for some buyer k, $R_k(x, z, \gamma) \leq N$ and $R_k(x, z, \beta) > N$, which has the implication that $p_k^\gamma(x, z) = 1$ and $p_k^\beta(x, z) = 0$. This can happen because, as α increased, $\psi_k^B(x_k, \alpha)$ decreases in value, may fall in rank, and be replaced in the top N by some buyer i or seller j. Suppose, as the first of the three possible cases we must consider, buyer k is replaced by buyer ℓ. Suppose further, without any loss of generality, as α increases from γ to β this is the only change that occurs in $K(\alpha, x, z)$.

Buyer ℓ replacing buyer k in the top N implies four additional facts: $p_\ell^\gamma(x, z) = 0$, $p_\ell^\beta(x,z) = 1$, $\psi_k^B(x_k, \gamma) > \psi_\ell^B(x_\ell, \gamma)$, and $\psi_k^B(x_k, \beta) < \psi_\ell^B(x_\ell, \beta)$. Therefore $\psi_k^B(x_k, 1) < \psi_\ell^B(x_\ell, 1)$ because $\gamma < \beta < 1$ and the ψ functions are linear in α. By assumption the only change that occurs in $K(\alpha, x, z)$ at point (x, z) as α increases from γ to β involves the buyer k term and the buyer ℓ term. Specifically, when $\alpha = \gamma$,

$$\psi_k^B(x_k,1)p_k^\gamma(x,z) + \psi_\ell^B(x_\ell,1)p_\ell^\gamma(x,z)$$
$$= \psi_k^B(x_k,\ 1) \cdot 1 + \psi_\ell^B(x_\ell,\ 1) \cdot 0 = \psi_k^B(x_k,\ 1) \tag{44}$$

and, when $\alpha = \beta$,

$$\psi_k^B(x_k,1)p_k^\beta(x,z) + \psi_\ell^B(x_\ell,1)p_\ell^\beta(x,z)$$
$$= \psi_k^B(x_k,\ 1) \cdot 0 + \psi_\ell^B(x_\ell,\ 1) \cdot 1 = \psi_\ell^B(x_\ell,1). \tag{45}$$

Therefore, at the point $(x,\ z)$, $K(\gamma) < K(\beta)$, which means that K is increasing, provided that one buyer replacing another buyer in the top N is the cause of the change in value.

In addition to one buyer replacing another buyer in the top N of virtual reservation values, $K(\alpha)$ can change in value two other ways as α increases. They are: one seller can replace another seller in the top N and one seller can replace one buyer in the top N. A buyer cannot replace a seller because, with respect to α, ψ_i^B is decreasing and ψ_j^S is increasing. Our demonstration for the buyer replaces buyer case can be repeated for the two additional cases to show that K necessarily increases as α increases, provided $\alpha < 1$. Since K is the integrand of G and since K can only increase as α increases, $G(\alpha)$ is necessarily nondecreasing for $\alpha < 1$.

For $\alpha > 1$ the arguments reverse. Inspection of our demonstration for the buyer replaces buyer case shows that the linearity of the ψ functions implies that $K(\gamma) > K(\beta)$ whenever $1 < \gamma < \beta$. The same is true for the other two cases. Therefore, for $\alpha > 1$, $G(\alpha)$ is nonincreasing.●

Proof of Theorem 2. Our optimization problem is:

$$\max_{p,q} \int\cdots\int [\sum_{i=1}^{M} x_i p_i(x,z) - \sum_{j=1}^{N} z_j(1 - q_j(x,z))] g(x,z)dxdz \tag{46}$$

$$\text{s.t.} \int\cdots\int [\sum_{i=1}^{M} \psi_i^B(x_i,1)p_i(x,z) - \sum_{j=1}^{N} \psi_j^S(z_j,1)(1 - q_j(x,z))] g(x,z)dxdz \geq 0 \tag{47}$$

$$\sum_{i=1}^{M} p_i(x,z) + \sum_{j=1}^{N} q_j(x,z) = N. \tag{48}$$

We originally derived expression (46), the objective function, as equation (20) in Lemma 2's proof. It represents the sum of the traders' <u>ex ante</u> expected utilities. The first constraint, inequality (47), is Theorem 1's individual rationality and incentive compatibility constraint, inequality (13), rewritten in the ψ notation. It and the assumptions Theorem 2's hypothesis makes concerning the $\bar{p}^{\alpha *}$ and $\bar{q}^{\alpha *}$ probability schedules guarantees that the solution to (46-48) satisfies Theorem 1's necessary and sufficient conditions for a mechanism to be individually rational and incentive compatible. The second constraint is equation (1), the balance of goods constraint. Problem (46-48) does not show the balance of payments constraint, equation (2), explicitly because it was incorporated into the derivation of (46) and is therefore always satisfied.

The problem's Lagrangian is:

$$L = \int\cdots\int \{ \sum_{i=1}^{M} [x_i + \lambda\psi_i^B(x_i,1)]p_i(x,z) - \sum_{j=1}^{N} [z + \lambda\psi_j^S(z_j,1)](1 - q_j(x,z))\} g(x,z)dxdz$$
$$+ \mu\{ \sum_{i=1}^{M} p_i(x,z) + \sum_{j=1}^{N} q_j(x,z) - N\}$$

where λ and μ are the multipliers and similar terms in the objective function and the first constraint have been collected. Additional simplification is possible if we notice that

$$x_i + \lambda\psi_i^B(x_i,1) = x_i + \lambda x_i + \lambda\left(\frac{F_i(x_i) - 1}{f_i(x_i)}\right)$$

$$= (1 + \lambda)\left[x_i + \frac{\lambda}{1+\lambda}\left(\frac{F_i(x_i) - 1}{f_i(x_i)}\right)\right]$$

$$= (1 + \lambda)\psi_i^B\left(x_i, \frac{\lambda}{1+\lambda}\right).$$

Analogously,

$$z_j + \lambda\psi_j^S(z_j,1) = (1 + \lambda)\psi_j^S\left(z_j, \frac{\lambda}{1+\lambda}\right).$$

Thus,

$$L = (1+\lambda)\int\ldots\int\left\{\sum_{i=1}^{M}\psi_i^B\left(x_i,\frac{\lambda}{1+\lambda}\right)p_i(x,z) - \sum_{j=1}^{N}\psi_j^S\left(z_j,\frac{\lambda}{1+\lambda}\right)(1 - q_j(x,z))\right\}$$
$$\cdot\ g(x,z)dxdz + \mu\left\{\sum_{i=1}^{M} p_i(x,z) + \sum_{j=1}^{N} q_j(x,z) - N\right\}. \tag{49}$$

Precise interpretation of (49) is crucial. If for some $\lambda \geq 0$ and some μ a pair of probability schedules (p, q) is found that (i) maximizes (49) and (ii) satisfies constraints (47) and (48), then (p, q) are the schedules that maximize the gains from trade. Moreover, if (p, q) are such that constraint (47) is slack, then λ must be set equal to zero.

For any given $\lambda \geq 0$ the probability schedules (p^α, q^α) maximize (49) if α is set equal to $\lambda/(1+\lambda)$. This is seen in two steps. First, for any α, $p_i(x,z) = p_i^\alpha(x,z)$ and $q_j(x,z) = q_j^\alpha(x,z)$ guarantee that the balance of goods constraint is satisfied. Dropping that constraint, substituting in (p^α, q^α), and rearranging the remaining terms gives:

$$L = (1+\lambda)\int\ldots\int\{\sum_{i=1}^{M}\psi_i^B(x_i,\frac{\lambda}{1+\lambda})p_i^\alpha(x,z) + \sum_{j=1}^{N}\psi_j^S(z_j,\frac{\lambda}{1+\lambda})q_j^\alpha(x,z)\}g(x,z)dxdz$$

$$- (1+\lambda)\int\ldots\int\{\sum_{j=1}^{N}\psi_j^S(x,\frac{\lambda}{1+\lambda})g(x,z)\}dxdz. \tag{50}$$

Second, for any fixed, nonnegative λ and at every point (x, z) within the region of integration, the integrand of (49) is maximized when $\alpha = \lambda/(1+\lambda)$. This is because setting $\alpha = \lambda/(1+\lambda)$ means that the N available objects are allocated to those N traders who have the N highest virtual reservation values for the object. Therefore the N largest ψ values receive weight one and all other ψ values receive a weight of zero on the first line of (50). This clearly maximizes the first line of L; the second line of L is of no consequence because with λ fixed, its value is fixed. Since, conditional on the value of λ, the integrand is maximized at every point, the integral itself is maximized.[11]

By assumption an $\alpha^* \in [0, 1]$ exists such that $G(\alpha^*) = 0$ or $G(0) > 0$. Set $\lambda = \lambda^* = \alpha^*/(1-\alpha^*)$ and note that λ^* so defined satisfies the equation $\alpha^* = \lambda^*/(1+\lambda^*)$. Since the probability schedules $(p^{\alpha^*}, q^{\alpha^*})$ maximize the value of the Lagrangian (conditional on $\lambda = \lambda^*$) and satisfy the constraint $G(\alpha^*) \geq 0$, they are a solution to our optimization problem provided that no other solutions to the equation $G(\alpha) = 0$ exist that dominate the α^*-mechanism in terms of expected gains from trade.

Other solutions to G do exist, but none of them dominate. Lemma 5 implies that on the nonnegative line $G(\alpha)$ is a unimodal function with its mode at one. Since the result is stated in terms of G being

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[11]This argument is valid both for $\lambda > 0$ when the constraint is binding and $\lambda = 0$ when the constraint is nonbinding.

nondecreasing and nonincreasing, intervals may occur over which G is constant. These flats have an important property: If an interval $[\alpha_1, \alpha_2]$ exists such that, for all $\alpha \in [\alpha_1, \alpha_2]$, $G(\alpha) = 0$, then each of these α-mechanisms has the same expected gains from trade. The reason is this. Lemma 5's proof makes clear that G increases as α increases because, at particular points (x, y) within the region of integration, the probability schedules (p^α, q^α) switch the traders to which they assign the objects. Let Γ denote that subset of G's region of integration where, as α increases from α_1 to α_2, changes occur in how the mechanism assigns objects to traders. Given, as the proof of Lemma 5 showed, the monotonicity of G with respect to any reassignment of objects, the only way in which G can be constant as α increases from α_1 to α_2 is for Γ to have measure zero with respect to the density g(x,y). But this means that, except over a region that has zero probability of being realized, the α_1-mechanism and the α_2-mechanism, assign the objects identically. Therefore, as asserted, they must have identical expected gains from trade.

If α is made sufficiently large, then $p_i^\alpha(x, z) = 0$ and $q_j^\alpha(x, z) = 1$ for all (x, z) in G's region of integration, $G(\alpha) = 0$ necessarily, and no trade ever takes place. This happens because a very large α guarantees that buyers' virtual reservation values are less than sellers' virtual reservation values. Thus some $\bar{\alpha} > 1$ exists such that $G(\alpha) = 0$ for all $\alpha > \bar{\alpha}$. These $\bar{\alpha}$-mechanisms are incentive compatible and individually rational, but uninteresting because their expected gains from trade are zero.

The unimodality of the G function implies that only two intervals can exist for which $G(\alpha) = 0$: one to the left of unity on the nonnegative line and one to the right of unity. The interval to the right of unity is without interest since it involves no trade with certainty. By assumption the interval to the left of unity exists and contains at least the point $\alpha^* \varepsilon [0, 1]$. Any additional points it contains are equivalent because, as shown above, they result in identical gains from trade. Consequently, all that remains to be shown is that the α^*-mechanism has positive expected gains from trade and therefore dominates the no-trade mechanisms.

Myerson and Satterthwaite (1983, Theorem 2) show for the two trader case ($M = N = 1$) that if (F, H) is a trading problem and if an α^* exists such that $G(\alpha^*) = 0$, then the α^*-mechanism's ex ante expected gains from trade is strictly positive.[12] This implies that the α^*-mechanism's ex ante expected gains for the general case of many traders must also be positive. This can be seen by picking a buyer i and seller j pair for whom $b_i > c_j$, constructing the Myerson and Satterthwaite two trader optimal mechanism for them alone, and not letting any other buyers and sellers trade. This special mechanism, which probably fails to maximize

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[12]They assume that the interiors of $[a_1, b_1]$ and $[c_1, d_1]$ overlap. This rules out solutions of the form $G(0) > 0$. Our definition of what constitutes a trading problem (F, H) includes the weaker assumption that, for some i and some j, $b_i > c_j$. This latter assumption is consistent with buyers' reservation values being greater than sellers' reservation values with certainty. In such cases the α^*-mechanism is ex post efficient as well as ex ante efficient, the individual rationality and incentive compatibility constraint is not binding, and the sum of traders' ex ante expected utilities is positive.

the total gains from trade, is individually rational, incentive compatible, and has positive total expected gains from trade since it gives that one pair positive _ex ante_ gains from trade. The α^*-mechanism must do at least as well; it therefore must also have positive expected gains from trade.∎

Theorem 3. Let (F, H) be a trading problem. If, for all buyers i and sellers j, the functions $\psi_i^B(\cdot, 1)$ and $\psi_j^S(\cdot, 1)$ are nondecreasing in x_i and z_j, then an α^*-mechanism exists that is _ex ante_ efficient, individually rational, and incentive compatible for (F, H). Moreover, the sum of the traders' _ex ante_ expected utilities is positive.

If the conditions on the functions ψ_i^B and ψ_j^S are not satisfied, then possibly, for some i or j, $\bar{p}_i^{\alpha^*}(\cdot)$ or $\bar{q}_j^{\alpha^*}(\cdot)$ is decreasing. If so, Theorem 1 no longer applies and the α^*-mechanism is not incentive compatible. Therefore, for trading problems that do not satisfy Theorem 3's conditions, we do not know (i) if incentive compatible and individually rational mechanisms exist that result in some trades being realized and (ii), if they do exist, what form the _ex ante_ efficient mechanisms then assume.

Proof of Theorem 3. By hypothesis, for all buyers i and sellers j, $\psi_i^B(\cdot, 1)$ and $\psi_j^S(\cdot, 1)$ are nondecreasing. This implies that, for every α in the unit interval, $\psi_i^B(\cdot, \alpha)$ and $\psi_j^S(\cdot, \alpha)$ are nondecreasing. This then implies that, for every i, $p_i^\alpha(x, z)$ is nondecreasing in x_i

and, for every j, $q_j^{\alpha}(x, z)$ is nondecreasing in z_j. Therefore $\bar{p}_i^{\alpha}(\cdot)$ and $\bar{q}_j^{\alpha}(\cdot)$ are nondecreasing as Theorem 1 requires.

If $G(0) \geq 0$, then the conditions of Theorem 2 are met and the proof is done. If $G(0) < 0$, then we must show that an $\alpha^* \varepsilon (0, 1)$ exists such that $G(\alpha^*) = 0$. We do this by showing that $G(\cdot)$ is continuous and $G(1) > 0$. These two facts, together with $G(0) < 0$, imply that an $\alpha^* \varepsilon [0, 1]$ exists such that $G(\alpha^*) = 0$. Theorem 2 then applies.

$G(\alpha)$ may be rewritten as follows:

$$
\begin{aligned}
G(\alpha) &= \int \ldots \int \Big\{ \sum_{i=1}^{M} \psi^B(x_i, 1) p_i^{\alpha}(x,z) - \sum_{j=1}^{N} \psi_j^S(z_j, 1)[1 - q_j^{\alpha}(x,z)] \Big\} g(x,z)\,dx\,dz \\
&= \sum_{i=1}^{M} \int_{a_i}^{b_i} \psi_i^B(x_i, 1) \bar{p}_i^{\alpha}(x_i) f(x_i)\,dx_i - \sum_{j=1}^{N} \int_{c_j}^{d_j} \psi_j^S(z_j, 1)[1 - \bar{q}_j^{\alpha}(z_j)] h(z_j)\,dz_j
\end{aligned}
\tag{51}
$$

where each term has been integrated M+N-1 times. G is continuous if each of its terms is continuous. Each of them are continuous if every $\bar{p}_i^{\alpha}$ and $\bar{q}_j^{\alpha}$ is continuous. Define the function Λ to be:

$$\Lambda_i(x_{-i}, z, \alpha) = \{\text{minimal } x_i \ \varepsilon\ [a_i, b_i] \mid R_i(x, z, \alpha) \leq N\}.$$

Thus, given x_{-i}, z, and α, Λ is the smallest value of x_i that results in i's virtual reservation value ranking in the top N. Define $\Lambda_j(x, z_{-j}, \alpha)$ analogously. All the Λ functions are continuous because the underlying ψ_i^B and ψ_j^S are continuous. Without loss of generality, consider $\bar{p}_1^{\alpha}(x_1)$:

$$\bar{p}_1^{\alpha}(x_i) = \int_{a_2}^{b_2} \cdots \int_{c_{N-1}}^{d_{N-1}} \int_{c_N}^{d_N} p_1^{\alpha}(x,z)g(x_{-1},z)dx_{-1}dz$$
$$= \int_{a_2}^{b_2} \cdots \int_{c_{N-1}}^{d_{N-1}} \{\int_{\Lambda_N}^{d_N} h_N(z_N)dz_N\}\ g(x_{-1},z_{-N})dx_{-1}dz_{-N} \quad (52)$$

where $\Lambda_N = \Lambda_N(x,z_{-N},\alpha)$. It is continuous because of the continuity of both Λ and the density functions h_N and g. Exactly parallel arguments follow for every other i and j.

The argument that $G(1) > 0$ is this. Suppose that $G(1) = 0$. Lemma 5 showed that $G(\alpha)$ is unimodal with its mode at one. Incidental to its main thrust, the proof of Theorem 2 established the existence of an $\bar{\alpha}$ such that, for all $\alpha > \bar{\alpha}$, $G(\alpha) = 0$. The trading mechanisms associated with these roots of G are mechanisms that never permit any trades to be realized and thus have zero expected gains from trade. The unimodality of G therefore implies that $G(1) > 0$. The unimodality of G then further implies that $\bar{\alpha} \leq 1$ and no $\alpha < \bar{\alpha}$ exists such that $G(\alpha) = 0$. This means $\alpha^* = \bar{a}$ and the no trade mechanism maximizes the gains from trade. Therefore no mechanism exists that gives positive expected gains from trade. But Myerson and Satterthwaite's theorem applies; therefore for at least one buyer-seller pair a trading mechanism exists that gives positive expected gains from trade. This contradicts the conclusion that no mechanism exists that has positive expected gains from trade.●

An Example

In this section we construct *ex ante* efficient, incentive compatible, and individually rational trading mechanisms for the special class of trading problems (F, H) where the number of buyers equals the number of sellers (N = M) and all traders' reservation values are identically and uniformly distributed on the unit interval. The distribution from which traders' reservation values are drawn satisfy Theorem 3's requirements for existence of *ex ante* efficient α^*-mechanisms. We use numerical methods to calculate efficient mechanisms for varying numbers of traders and observe that, relative to the *ex post* efficient mechanism, the expected gains from trade the *ex ante* efficient mechanism fails to realize decreases in a quadratic manner.

Theorems 2 and 3 establish that *ex ante* efficient mechanisms are α^*-mechanisms. Therefore the key problem in constructing an optimal mechanism for a given number of traders is to calculate the solution to $G(\alpha) = 0$ that lies within the unit interval. Given that N = M and traders' reservation values are uniformly distributed over [0, 1],

$$\psi_i^B(x_i,\alpha) = (1 + \alpha)x_i - \alpha$$

and

$$\psi_j^S(z_j,\alpha) = (1 + \alpha)z_j.$$

Since N = M the equation G = 0 reduces to

$$G(\alpha) = N \left\{ \int_0^1 \psi^B(x,1)\bar{p}^{\alpha}(x)f(x)dx - \int_0^1 \psi^S(z,1)[1-\bar{q}^{\alpha}(z)]h(z)dz \right\}$$

$$= N \left\{ \int_0^1(2x - 1)\bar{p}^{\alpha}(x)dx - \int_0^1 2z(1 - \bar{q}^{\alpha}(z))dz \right\} = 0.$$

where the first line is the form of G found in equation (51) and all i and j subscripts have been supressed because all traders are symmetric with each other. It may be rewritten as:

$$\int_0^1 \{[2x - 1]\bar{p}^{\alpha}(x) - 2x[1 - \bar{q}^{\alpha}(x)]\} \, dx = 0. \tag{53}$$

This is the key equation that we must solve for α in order to construct <u>ex ante</u> efficient mechanisms.

Calculation of the marginal probabilities $\bar{p}^{\alpha}(x)$ and $\bar{q}^{\alpha}(z)$ is messy and requires some new notation and preliminary calculations. Let

$$y_i = \psi_i^B(x_i, \alpha)$$

and

$$w_j = \psi_j^S(z_j, \alpha).$$

That y_i and w_j are functions of α and, respectively, x_i and z_j is important and should not be forgotten, even though we suppress their arguments. Given that every x_i and z_j is uniformly distributed over $[0,1]$, y_i and w_j have densities:

$$f(y_i) = \begin{cases} 1/1+\alpha & \text{if } y_i \varepsilon [-\alpha, 1] \\ 0 & \text{if } y_i \notin [-\alpha, 1] \end{cases}$$

and

$$h(w_j) = \begin{cases} 1/1+\alpha & \text{if } w_j \varepsilon [0, 1+\alpha] \\ 0 & \text{if } w_j \notin [0, 1+\alpha]. \end{cases}$$

Whether f_i and h_j denote the densities of y_i and w_j or x_i and z_j is generally obvious from the context and should not create confusion. If s is a given scalar within the interval $[-\alpha, 1]$, then

$$P(y_i < s) = \int_{-\alpha}^{s} 1/(1+\alpha)dt = (s+\alpha)/(1+\alpha)$$

and

$$P(y_i > s) = 1 - (s+\alpha)/(1+\alpha) = (1-s)/(1+\alpha)$$

where $P(y_i < s)$ is the probability (conditional on the value of α) that y_i is less than s. Similarly, if $s \in [0, 1+\alpha]$, then

$$P(w_j < s) = \int_0^s \frac{1}{1+\alpha} dt = s/(1+\alpha)$$

and

$$P(w_j > s) = 1 - s/(1+\alpha) = (1+\alpha-s)/(1+\alpha).$$

Given this notation and results, calculation of $\bar{p}^{\alpha}(\cdot)$ and $\bar{q}^{\alpha}(\cdot)$ divides into four cases.

<u>Case 1</u>: $\bar{p}^{\alpha}(x_i)$ when $x_i \in [0, \alpha/(1+\alpha))$. If buyer i's reservation value, x_i, falls in this interval, then its associated virtual reservation value, $y_i(x_i)$, falls in the interval $[-\alpha, 0)$. A necessary condition for $p_i^{\alpha}(x, z) = 1$ is that y_i must have rank no greater than N relative to all buyers' and sellers' virtual reservation values. This is impossible because the N sellers' virtual reservation values are distributed over the interval $[0, 1+\alpha]$; therefore all N of the sellers' virtual reservation values outrank y_i and $p_i^{\alpha}(x, z) = 0$. This means $\bar{p}^{\alpha}(x_i) = 0$.

<u>Case 2</u>: $\bar{p}^{\alpha}(x_i)$ when $x_i \in [\alpha/(1+\alpha), 1]$. If buyer i's reservation value, x_i, falls in this interval, then its associated virtual reservation value, $y_i(x_i)$ falls in the interval $[0, 1]$. The marginal probability $\bar{p}^{\alpha}(x_i)$ is the probability that the virtual reservation value $y_i(x_i)$ has rank no greater than N relative to all buyers' and sellers' virtual reservation values. Equivalently, since there are N buyers and N sellers, it is the probability that at least N buyers and sellers have virtual reservation values that have greater rank than y_i. Thus:

$$\bar{p}^{\alpha}(x_i) = P\{R[y_i(x_i)] \leq N \mid y_i = s \;\&\; s \in [0, 1]\}$$
$$= \sum_{k=0}^{N-1} P\{\#(y_\ell < s) = k \;\&\; \#(w_j < s) \geq (N-k)\}$$

where (i) $R(y_i(x_i))$ is read as the rank of the virtual reservation value y_i relative to all other buyers' and sellers' virtual reservation values and (ii) $\#(y_\ell < s) = k$ is read as the number of buyers ℓ ($\ell = 1, \ldots, N; \ell \neq i$) whose virtual reservation values, y_ℓ, are less than s is equal to k. Given these probabilities and the independence of the traders' reservation values, calculation of the probability is straightforward using the binomial formula:

$$\bar{p}^{\alpha}(x_i) = \sum_{k=0}^{N-1} P\{\#(y_\ell < s) = k \;\&\; \#(w_j < s) \geq (N-k)\}$$

$$= \sum_{k=0}^{N-1} \binom{N-1}{k} \left(\frac{s+\alpha}{1+\alpha}\right)^k \left(\frac{1-s}{1+\alpha}\right)^{N-1-k} \cdot \sum_{m=N-k}^{N} \binom{N}{m} \left(\frac{s}{1+\alpha}\right)^m \left(\frac{1+\alpha-s}{1+\alpha}\right)^{N-m} \tag{54}$$

$$= \left(\frac{1}{1+\alpha}\right)^{2N-1} \sum_{k=0}^{N-1} \sum_{m=N-k}^{N} \binom{N-1}{k}\binom{N}{m} (s+\alpha)^k (1-s)^{N-k-1} s^m (1+\alpha-s)^{N-m}.$$

<u>Case 3:</u> $\bar{q}^{\alpha}(z_j)$ when $z_j = (1/(1+\alpha)]$. This case parallels case 1. Given the interval from which seller's reservation value z_j is drawn, the virtual reservation value $w_j(z_j)$ must lie in the interval $(1, 1+\alpha]$. Consequently, it is greater than every buyer's virtual reservation value and has a rank no greater than N. Therefore $\bar{q}^{\alpha}(z_j) = 1$.

<u>Case 4:</u> $\bar{q}^{\alpha}(z_j)$ when $z_j \in [0, 1/(1+\alpha)]$. This case parallels case 2.

$$\bar{q}^{\alpha}(z_j) = P\{R[w_j(z_j)] \leq N \mid w_j = s \;\&\; s \in [0, 1]\}$$

$$= \sum_{k=0}^{N-1} P\{\#(w_\ell < s) = k \;\&\; \#(y_i < s) \geq (N-k)\}$$

$$= \sum_{k=0}^{N-1} \binom{N-1}{k} \left(\frac{s}{1+\alpha}\right)^k \left(\frac{1+\alpha-s}{1+\alpha}\right)^{N-1-k} \cdot \sum_{m=N-k}^{N} \binom{N}{m} \left(\frac{s+\alpha}{1+\alpha}\right)^m \left(\frac{1-s}{1+\alpha}\right)^{N-m}$$

$$= \left(\frac{1}{1+\alpha}\right)^{2N-1} \sum_{k=0}^{N-1} \sum_{m=N-k}^{N} \binom{N-1}{k}\binom{N}{m} s^k (1+\alpha-s)^{N-k-1} (s+\alpha)^m (1-s)^{N-m}.$$

When these parameterizations for $\bar{p}^\alpha$ and $\bar{q}^\alpha$ are substituted into form (53) of the equation $G(\alpha) = 0$, the result is intractible. We therefore utilized a numerical integration routine and a numerical nonlinear equation solver in tandem to find its zeros.

Table 1 presents the results of the calculations when the number of traders on each side of the market varies from one to twelve.[13] For this special case of uniformly distributed reservation values, the calculated values of α^* have the following interpretation. If buyer i has reservation value x_i and seller j has reservation value z_j, then a necessary condition for both i and j to trade ($p_i^{\alpha^*}(x, z) = 0$ and $q_j^{\alpha^*}(x, z) = 1$) is that i's virtual reservation value be greater than j's virtual reservation value, i.e., $\psi_i^B(x_i, \alpha^*) = y_i(x_i) > w_j(z_j) = \psi_j^S(z_j, \alpha^*)$. Manipulation of this inequality shows that it can only be satisfied if the buyer's reservation value, x_i, exceeds the seller's reservation value, z_j, by at least $\alpha^*/(1+\alpha^*)$. In other words, a necessary condition for both buyer i and seller j to trade is

$$x_i - z_j > \frac{\alpha^*}{1 + \alpha^*}.$$

This required, positive difference in reservation values is the <u>wedge</u> that results from imperfect information whenever the number of traders

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[13]The Table's α^* entry for N = M = 1 agrees with the value that Myerson and Satterthwaite (1981) calculated analytically.

is small. Its presence is what makes the achievement of ex post efficiency impossible. Note that as α^* becomes small the size of this wedge becomes essentially equal to the value of α^* itself. The fourth column displays $1/\alpha^*$ and shows that α^* is apparently bounded from below by $1/2N$. Therefore in the limit as the number of traders becomes large the wedge vanishes at the same rate $1/2N$ approaches zero.[14]

The column labeled "Gains(α^*)" contains for each size market the expected gains from trade for the ex ante efficient, α^*-mechanism. As was stated in the introduction, by expected gains from trade we mean the average gains from trade that the N+M traders would realize if (i) they trade repeatedly using the α^*-mechanism a large number of times and (ii) for each repitition their reservation values are freshly and independently drawn. The column labeled "Gains(0)" contains the expected gains from trade that an ex post efficient mechanism would generate, if such a mechanism were to exist.[15] The "Efficiency" column is column five divided by column six. It represents the proportion of the expected gains from trade that the ex ante efficient mechanism achieves relative to the expected gains from trade that an ex post efficient mechanism would achieve.

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[14]This statement is conditional on the validity of extrapolating to large N on the basis of numerical results contained in Table 1.

[15]The gains from trade for both the ex ante and the ex post efficient mechanisms are calculated using the probability schedules (p^α, q^α) substituted into equation (20). As the labeling of the columns suggests, α is set equal to α^* for the ex ante case and to zero for the ex post case.

Table 1: Properties of the α^*-Mechanism as the Number of Buyers and Sellers Varies.

N=M	α^*	$\alpha^*/(1+\alpha^*)$	$1/\alpha^*$	Gains(α^*)	Gains(0)	Efficiency
1	.3333	.2500	3.00	.14060	.16667	.8436
2	.2256	.1841	4.43	.37746	.39999	.9437
3	.1603	.1382	6.24	.62572	.64286	.9733
4	.1225	.1091	8.17	.87527	.88887	.9847
6	.0827	.0764	12.09	1.37507	1.38462	.9931
8	.0622	.0586	16.08	1.87504	1.88235	.9961
10	.0499	.0475	20.04	2.37501	2.38095	.9975
12	.0416	.0399	24.04	2.87501	2.88000	.9983

These calculations demonstrate that--for this particular example of a simple market--the inefficiency of imperfectly competitive trade disappears in an approximately quadratic manner as the number of buyers and sellers increases. Thus when the number of buyers and sellers total twelve the inefficiency of the α^*-mechanism is $1.0000 - 0.9931 = 0.0069$. When the number of traders doubles to twenty-four the inefficiency is cut to $1.0000 - 0.9983 = 0.0017$, almost exactly one-quarter that of the six buyer and six seller case. Moreover, by the time the market reaches ten or twelve traders, the inefficiency is down to the negligible level of about 1%.

That the inefficiency disappears quadratically with the number of traders is not surprising given that α^* decreases at the rate of $1/2N$. This latter fact means that as the number of traders doubles, the wedge is approximately cut in half. Therefore, in expected value terms, the proportion of trades that (i) would be realized if the mechanism were *ex* *post* efficient and (ii) are excluded by the *ex ante* efficient mechanism is also cut in half as the number of traders doubles. Additionally, those trades that the *ex ante* efficient mechanism with the doubled number of traders excludes are trades that possess only half the expected gains as do the trades the *ex ante* efficient mechanism excludes when the number of traders is undoubled. This is because excluded trades have at most gains from trade equal to the size of the wedge. Therefore doubling the number of traders has two, sequential effects on the inefficiency of the α^*-mechanism: (i) it cuts in half the proportion of desirable trades that are excluded from being realized and (ii) it cuts in half the average size of the gains from trade lost from each excluded trade. The two effects are multiplicative; therefore a doubling of the number of traders cuts the α^*-mechanism's inefficiency by a factor of four.

Conclusions

In this paper we have developed a general technique for computing the ex ante efficient trading mechanism when the number of traders on each side of the market is arbitrary and each trader's reservation value is independent of the other traders' reservation values. Using our technique we computed the ex ante efficient mechanism for markets where (i) traders' reservation values are uniformly, independently, and identically distributed and (ii) the number of traders on each side of the market ranged from one to twelve. These calculations showed that the efficiency of the ex ante optimal mechanism approaches ex post efficiency in a quadratic manner. Thus by the time each side of the market contains six traders the ex ante efficient mechanism is essentially ex post efficient. Our conjecture is that these numerical results are robust asymptotically with respect to how the reservation values are distributed. Specifically, we conjecture that if reservation values are drawn independently and identically from a differentiable and positive density function defined over a closed interval, then asymptotically the ex ante efficient mechanism converges to ex post efficiency in a quadratic manner.

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