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USING WEIGHTED NONLINEAR LEAST SQUARES TO ESTIMATE FISH POPULATION DYNAMICS MODELS

by

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ABSTRACT

Estimation and inference for weighted nonlinear least squares regressions are examined for the case in which the regressors are stochastic, rather than fixed, and where errors may be both heteroscedastic and serially correlated. The usual least squares parameter covariance matrix estimator may be invalid in such cases, and a new covariance matrix estimator is given. General statistics for testing hypotheses about the parameters are provided, as well as new tests for model misspecification. The methodology is applied to the estimation of population dynamics models for the northern anchovy (Engraulis mordax).

KEY WORDS: nonlinear least squares, heteroscedasticity, misspecification, population models, northern anchovy.

The weighted nonlinear least squares (WLS) estimator $\hat{\theta}_n$ solves the problem

$$(2) \qquad \min_{\theta \in \Theta} Q_n(\theta) \equiv n^{-1} \sum_{t=1}^{n} [Y_t - f_t(X_t,\theta)]^2 W_t$$

where Θ is the admissible parameter space, and $\{W_t\}$ is taken to be a sequence of bounded positive weights which may themselves be random variables. In particular, the weights may be functions of the explanatory variables. The properties of $\hat{\theta}_n$ are of important practical interest. For example, $\hat{\theta}_n$ should be a consistent estimator of θ_o, the true parameters of interest to the investigator. The asymptotic distribution of the WLS estimator is required for the purpose of testing hypotheses. Statistical tests of model adequacy also depend on such results.

In many applications, the vector of random variables (X_t,ε_t) may not be identically distributed or independent over time. This engenders both theoretical and practical problems. General conditions which ensure the consistency and asymptotic normality of the WLS estimator are provided in Domowitz and White (1982). The purpose of this paper is to provide the applied researcher with the practical consequences of the theory and to provide an application to fish population models. With heterogeneous, time-dependent explanatory variables and errors, the usual parameter covariance estimator is inapplicable, yielding standard errors of incorrect size and invalidating common hypothesis testing procedures. A covariance matrix estimator which is consistent regardless of the presence of heteroscedasticity and/or serial correlation of unknown form in regressors and errors is presented. Robust forms of common test statistics are given, justifying analogues of the familiar t and F tests. Direct tests for heteroscedasticity

$$(4) \qquad A_n = 2n^{-1} \sum_{t=1}^{n} E[(\nabla f_t' \nabla f_t - \nabla^2 f_t \varepsilon_t) W_t)]$$

where $\nabla^2 f_t$ is the pxp matrix with elements $\partial^2 f_t(X_t,\theta)/\partial\theta_i\partial\theta_j$, $i,j\varepsilon\{1,\ldots,p\}$, evaluated at θ_o.

Also define

$$(5) \qquad B_n = 4n^{-1} \sum_{t=1}^{n} E(W_t^2 \varepsilon_t^2 \nabla f_t' \nabla f_t)$$

$$+ 4n^{-1} \sum_{\tau=1}^{n-1} \sum_{t=\tau+1}^{n} E(W_t W_{t-\tau} \varepsilon_t \varepsilon_{t-\tau} [\nabla f_t' \nabla f_{t-\tau} + \nabla f_{t-\tau}' \nabla f_t])$$

Conditions are provided in Domowitz and White (1982) which guarantee that $\sqrt{n}\, B_n^{-1/2} A_n(\hat{\theta}_n - \theta_o)$ is asymptotically distributed as $N(0,I_p)$, where $B_n^{-1/2}$ denotes the inverse matrix square root of B_n, which is assumed to be positive definite for sufficiently large n. The asymptotic parameter estimator covariance matrix may then be written as $C_n = A_n^{-1} B_n A_n^{-1}$. Note that C_n may be a function of n, and is not required to converge to a limit, a restrictive and unnecessary assumption that is avoided here. The asymptotic normality result provides the basis for hypothesis testing procedures, provided consistent estimators for A_n and B_n can be found. Before examining this issue, insight into the covariance structure may be gained by examining several special cases.

In the linear model with fixed regressors, $A_n/2 = \tilde{X}'\tilde{X}/n$, where $\tilde{X}$ is the nxp matrix with rows $\sqrt{W_t}X_t$. In nonlinear models, the second term $(\nabla^2 f_t \varepsilon_t W_t)$ usually vanishes, because it is assumed that the conditional expectation of ε_t, given the explanatory varibles X_t, is zero, yielding $A_n = 2n^{-1} \sum_{t=1}^{n} E(\nabla f_t' \nabla f_t)$.

The form of B_n depends on the joint stochastic structure of the

$\hat{C}_n = \hat{A}_n^{-1} \hat{B}_n \hat{A}_n^{-1}$ provides the time-series generalization of White's (1980) heteroscedasticity-consistent covariance matrix estimator. Along with the asymptotic normality result, it may be used to develop asymptotically valid hypothesis testing procedures. Suppose it is desired to test

$$H_o: s(\theta_o) = 0$$

against the general alternative

$$H_1: s(\theta_o) \neq 0$$

where $s(\theta)$ is a continuously differentiable function, such that its Jacobian at θ_o, $\nabla s(\theta_o)$, is finite and has full row rank k. For example, k linear constraints would yield $s(\theta_o) = R\theta_o - r$, where R is a kxp matrix and r is a k-vector of constants. Under similar conditions to those used to demonstrate the asympototic normality of the WLS estimator, it can be shown that

$$n\, \hat{s}'[\nabla\hat{s}\hat{C}_n\nabla\hat{s}']^{-1}\hat{s} \tag{9}$$

has the χ^2 distribution asymptotically with k degrees of freedom, where $\hat{s} = s(\hat{\theta}_n)$

Equation (9) is the familar Wald test [cf. Rao (1973), p. 417] with a robust parameter covariance estimator. It is sometimes the case that the alternative is substantially more complicated than the model under the null hypothesis. In such cases, the Lagrange Multiplier (LM) principle can be used to test H_o against H_1. Let $\tilde{\theta}_n$ be the WLS estimator under the null hypothesis; i.e., the minimizer of the weighted sum of squares subject to the constraint inference. If the tests reject the null hypothesis that $B_n = 2\sigma_o^2 A_n$, further calculations are necessary in order to construct the appropriate estimate of C_n, given by (6) and (7). These tests may be applied to unweighted regressions, or to weighted or otherwise transformed regressions in order to verify constant error variances after transformation. The tests will also be sensitive to situations in which the model is misspecified in such a way as to produce inconsistent estimates of θ_o.

Without heteroscedasticity, the first term of B_n in equation (5) equals $2\sigma_o^2 A_n$, and without serial correlation in $\nabla f_t' \varepsilon_t W_t$, the remaining terms vanish. Heteroscedasticity in isolation may be examined by comparing the first term of $\hat{B}_n$ to $2\hat{\sigma}_n^2 \hat{A}_n$. Serial correlation in the gradient-error crossproduct may then be examined by comparing each of the remaining terms of $\hat{B}_n$ to zero.[3]

The statistic for the heteroscedasticity/misspecification test is the same as that given in White (1982) for the case of independent observations. Let $\nabla f_t(\theta)$ be the 1xp vector with typical element $f_{ti}(\theta) = \partial f_t(X_t,\theta)/\partial\theta_i$. Under the conditions given in White and Domowitz (1981), the appropriate test statistic is asymptotically distributed as $\chi^2_{k_o}$ under the null hypothesis, $B_n = 2\sigma_o^2 A_n$, and is given by n times the constant-adjusted R^2 from the artificial regression

$$(11) \qquad \hat{\varepsilon}_t^2 = \alpha_o + \hat{\psi}_{to}\alpha \qquad (t = 1,\ldots,n)$$

where $\hat{\psi}_{t0}$ is the $1xk_0$ vector with elements $f_{ti}(\hat{\theta}_n)f_{tj}(\hat{\theta}_n)$, $i,j \in \{1,\ldots,p\}$. An analogous test for the presence of autoregressive conditional heteroscedasticity [ARCH, see Engle (1982)] can be constructed under similar conditions, replacing the vector $\hat{\psi}_{t0}$ in (3.1) by a q-vector of lagged squared

This fact has been exploited previously in the context of maximum likelihood estimation and asymptotically efficient estimators [e.g., Hausman (1978) and White (1982)]. The next test does not rely on asymptotic efficiency, since this would generally require a knowledge of the joint distribution of the errors and covariance stationarity.

Let $\{\hat{\theta}_{1n}\}$ and $\{\hat{\theta}_{2n}\}$ be two sequences of WLS estimators using weights $\{W_{1t}\}$ and $\{W_{2t}\}$. The sequence $\{W_{1t}\}$ may simply be a sequence of ones, making $\hat{\theta}_{1n}$ the unweighted nonlinear least squares estimator. Under the conditions in Domowitz and White (1982), any misspecification implying $E(\varepsilon_t|X_t) \neq 0$ is eventually measured by $\hat{\theta}_{1n} - \hat{\theta}_{2n}$. If the model is correct, $\sqrt{n}(\hat{\theta}_{1n} - \hat{\theta}_{2n})$ should have mean zero and be asymptotically normally distributed. Let $\nabla\hat{f}_t = \nabla f_t(\hat{\theta}_{1n})$ and $\hat{e}_t = Y_t - f_t(X_t, \hat{\theta}_{1n})$. Define

$$(13) \qquad \hat{R}_n = 4n^{-1} \sum_{t=1}^{n} W_{1t} W_{2t} \hat{\varepsilon}_t^2 \, \nabla\hat{f}_t' \, \nabla\tilde{f}_t$$

$$+ 4n^{-1} \sum_{\tau=1}^{\ell} \sum_{t=\tau+1}^{n} \hat{\varepsilon}_t \, \hat{\varepsilon}_{t-\tau} (\nabla\hat{f}_t' \, \nabla\hat{f}_{t-\tau} W_{1t} W_{2t-\tau} + \nabla\hat{f}_{t-\tau}' \nabla\hat{f}_t W_{1t-\tau} W_{2t})$$

The covariance estimator required for the test statistic is given by

$$(14) \qquad \hat{S}_n = \hat{A}_{1n}^{-1} \hat{B}_{1n} \hat{A}_{1n}^{-1} + \hat{A}_{2n}^{-1} \hat{B}_{2n} \hat{A}_{2n}^{-1}$$

$$- \hat{A}_{1n}^{-1} \hat{R}_n \hat{A}_{2n}^{-1} - \hat{A}_{2n}^{-1} \hat{R}_n A_{1n}^{-1}$$

where $\hat{A}_{in}$, $\hat{B}_{in}$ are evaluated at $\hat{\theta}_{in}$.

Under conditions given in Domowitz and White (1982), the statistic model are apparently incompatible. Comparisons of the test results with the theory generating the model may suggest ways of respecifying the regression function. Finally, the direct test for parameter inconsistency given by (15) may be applied to determine whether the resulting specification is correct. If the latter hypothesis cannot be rejected, the WLS estimator may be taken to be consistent for parameters of interest, allowing proper interpretation of the results. The heteroscedasticity-robust Wald or LM statistics may then be used to test hypotheses of interest, even if the covariance structure of the errors is incorrectly specified. Rejection is an indication of the inconsistency of the WLS estimator for parameters of interest, and signals the need for another careful reexamination of the model. These tests are obviously dependent, and a treatment of the complicated pre-test problem inherent in such a procedure is beyond the scope of this study. In any case, such a procedure should provide some insurance against improper use of a misspecified model.

ESTIMATION OF ANCHOVY POPULATION DYNAMICS MODELS

The fundamental basis of the current fishery management plan for the northern anchovy (Engraulis mordax Girard) is a stochastic model of anchovy population dynamics (see footnote 1). The results of the preceding sections are applied here in the estimation of alternative anchovy population models.

The northern anchovy is abundant off the coast of California and Baja California. The central stock extends from 30°N to 38°N and has been estimated at 3 to 4 million short tons of spawning biomass in recent years [Vrooman and Smith (1972), Smith (1972), MacCall (1980)]. The reader is referred to MacCall (1980) for a discussion of historical growth patterns and of the application of growth models to the management of the anchovy fishery.

where u_t is a random disturbance resulting from stochastic variability in the recruitment relation. For example, recruitment might be approximated by a Ricker curve, giving

$$(17) \qquad B_t = S_{t-1}B_{t-1} + aB_{t-1}\, e^{-bB_{t-1}} - C_{t-1} + u_t.$$

More generally, surviving biomass and recruitment may be combined, and the biomass dynamics approximated by a general parametric specification of the form

$$(18) \qquad B_t = f(B_{t-1}, C_{t-1}, \theta) + u_t$$

where θ denotes a vector of unknown parameters to be estimated.

Two alternative specifications for anchovy biomass dynamics were investigated. The first is a simple power function model, given by

$$(19) \qquad B_t = \alpha B^{\beta}_{t-1} + \gamma\, C_{t-1} + u_t$$

in which the effect of the fishery enters additively. The second model is a stock-stock logistic specification, derived from the deterministic logistic function

$$B_t = B_\infty/[1 + \exp\,(-\,(M + rt))]$$

[see Huppert, MacCall, and Stauffer (1980)] where B_∞ is the maximum equilibrium population size and r is the intrinsic rate of growth. The annual biomass transitions are given by correlation and found to be serially uncorrelated. The squared residuals were also tested for up to sixth order autoregressive heteroscedasticity using the ARCH test, and no such effects were observed. The direct test for heteroscedasticity given by (11) resulted in χ^2 statistics of approximately 16 for the unweighted power and logistic functions, indicating a strong rejection of the null hypothesis of homoscedastic errors at the 5% level of significance. The heteroscedasticity-robust standard errors are thus appropriate, and the covariance matrix estimator $A_n(\hat{\theta}_n)^{-1}B_n(\hat{\theta}_n)A_n(\hat{\theta}_n)^{-1}$ is the proper covariance estimator for use in further diagnostic testing procedures.

Examination of residual plots revealed that the variance of the series appears to increase as a function of the biomass level. The models were reestimated using several alternative data-based weighting schemes. Since none of them appeared to produce errors with constant variance, only the simplest is presented here for purposes of comparison. The rows of Tables 1 and 2 labelled $1/B_t$ contain parameter estimates obtained by weighting the squared residuals by the inverse of the biomass level at each time period. Weighting the regression function appears to produce a higher equilibrium biomass in the logistic model, but the rate of growth turns negative, indicating an unstable model. The standard error on that parameter is quite large, however. The weighting decreases equilibrium biomass in the power function specification, from 2.81 million tons to .27 million tons, although forecasts of B_t from the weighted and unweighted regressions using $B_{t-1} = 2$ million tons are 1.6 and 2.2 million tons, for example, a much less drastic difference.

The effect of catch on the biomass transitions decreases for the weighted regressions, and the standard errors are large. The bottom rows of Tables 1 and 2 contain parameter estimates using only the raw biomass data, without rejected. It might be noted that the logistic model is not strongly rejected based on this particular test of model adequacy. The instability of the model estimated by weighted least squares casts additional doubt on its usefulness as a forecasting and management tool, however.

SUMMARY AND CONCLUDING REMARKS

The practical implications of a unified theory of nonlinear estimation methods and associated statistical inference is presented in this study for the case of weighted nonlinear least squares estimation. Observations may come from a stratified cross-section, stochastic time-series, pooled cross-section/times-series, or experiment. The data may be heterogeneous with respect to their distributional properties. Regression errors may be serially correlated and/or heteroscedastic.

The asymptotic distribution of the weighted least squares estimator is normal, with mean zero and a covariance matrix which differs from that in the classical theory. A consistent estimator for this parameter covariance matrix is provided. These results are used to provide statistics valid for testing hypotheses of interest in spite of heterogeneity in the data and errors.

Direct tests for heteroscedasticity and serial correlation in the gradient-error crossproduct are given. These tests are useful in assessing the validity of standard inferential procedures, as well as providing a check on the efficacy of certain types of weighting schemes and data transformations. A test for overall model misspecification is also provided.

The methodology is applied to the estimation of alternative population models for the northern anchovy. Diagnostic testing procedures reveal severe weaknesses in the logistic population model, which has been used previously in assessing fishery management plans for the northern anchovy. A simple power

TABLE 1

POWER FUNCTION ESTIMATES

WEIGHTS	α	β	γ
1	6.03 (4.49) (3.83)	.774 (.093) (.082)	-1.48 (1.05) (1.21)
$1/B_t$	1.72 (1.36) (1.25)	.903 (.105) (.094)	-.263 (.732) (.723)
1	3.72 (3.04) (2.65)	.823 (.106) (.094)	——
$1/B_t$	1.52 (1.19) (.979)	.916 (.106) (.088)	——

Number of observations = 32

FOOTNOTES

1 See, for example, Huppert, MacCall, and Stauffer (1980) and MacCall (1980) for a discussion of the issues in the anchovy management plan. This plan is currently under review, partially due to changes in the anchovy biomass data base.

2 More precisely, the condition states that $\ell \to \infty$ as $n \to \infty$ such that $\ell = O(n^{\gamma})$, $o < \gamma < \delta/(r + \delta) \leq 1/2$, where $\delta > o$ and $r > 1$. The parameter r indexes the amount of dependence in the regressors and errors.

3 Treating B_n term by term leaves open the possibility of detecting some departure from the null hypothesis appropriate for each term, which nevertheless yields $B_n = 2\sigma_o^2 A_n$. Such cases should be rare, however.

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Figure 1

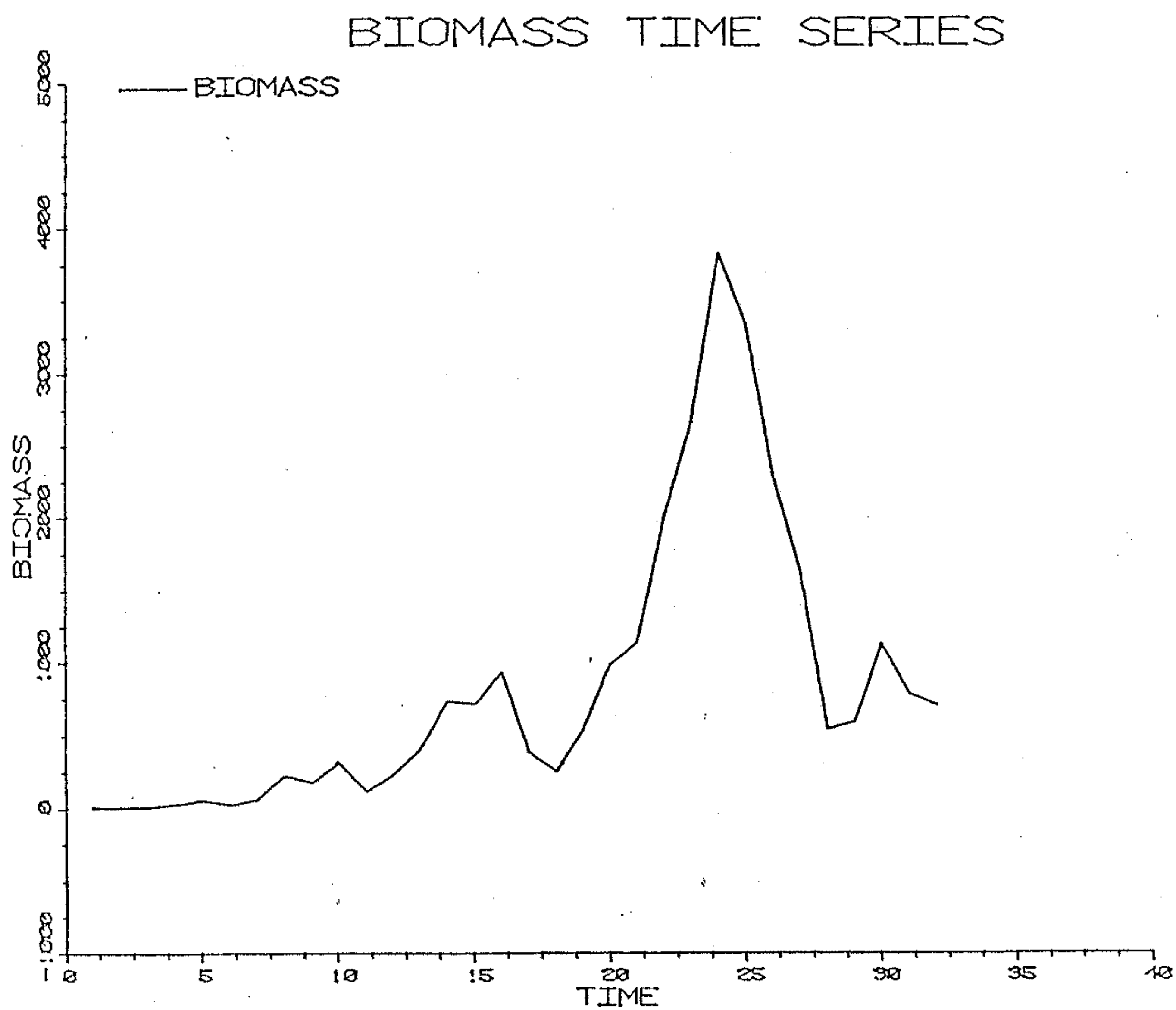

BIOMASS TIME SERIES
BIOMASS
BIOMASS
-1000
0
1000
2000
3000
4000
5000
TIME
0
5
10
15
20
25
30
35
40

Figure 3

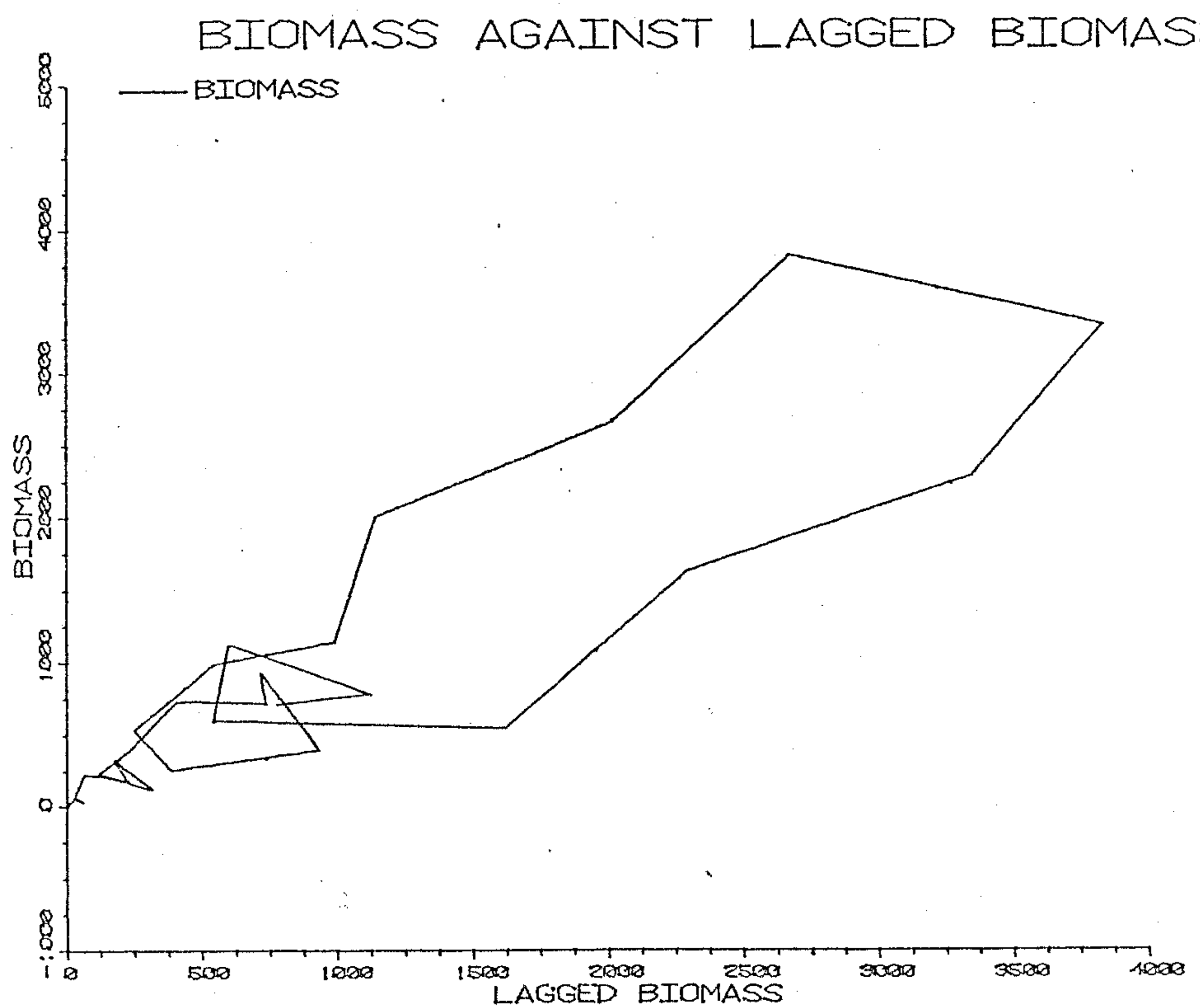
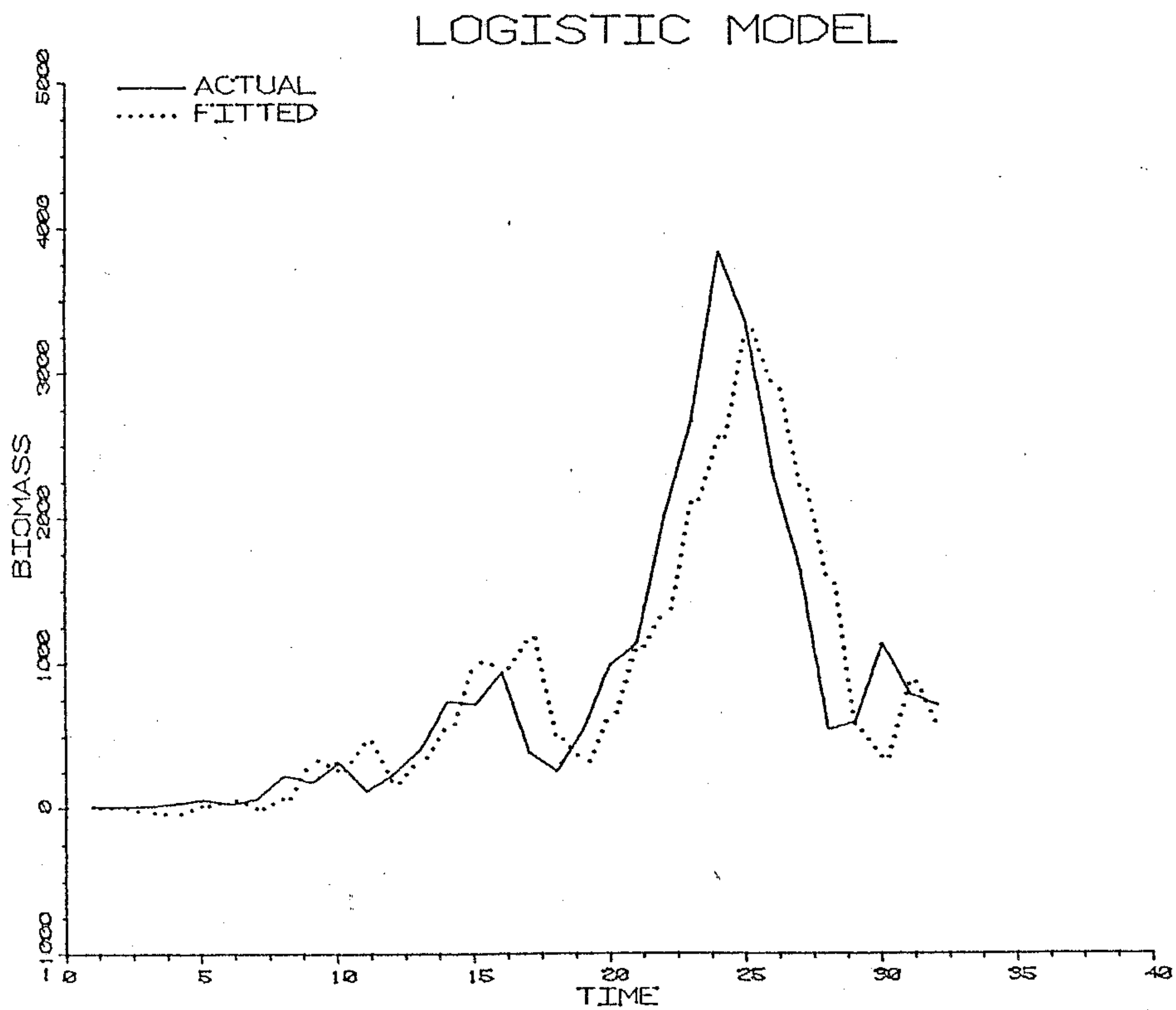

BIOMASS AGAINST LAGGED BIOMASS
BIOMASS
BIOMASS
5000
4000
3000
2000
1000
0
-1000
0
500
1000
1500
2000
2500
3000
3500
4000
LAGGED BIOMASS

Figure 5


LOGISTIC MODEL
ACTUAL
FITTED
BIOMASS
5000
4000
3000
2000
1000
0
-1000
0
5
10
15
20
25
30
35
40
TIME