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Monopolistic Competition, Aggregation of Competitive Information, and the Amount of Product Differentiation

by

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Introduction

The graphical technique that Chamberlin (1962, pp. 81-100) developed to analyze equilibrium within a monopolistically competitive industry incorporates the assumption that only aggregate statistics of competing firms' strategic choices affect an individual firm. This assumption crops up in his discussion of his Figure 14,[1] which we have reproduced as our Figure 1: "The curve dd', then, explains why each seller is led to reduce his price; the curve DD' shows his actual sales as the general downward movement takes place. The former curve 'slides' downwards along the latter as prices are

[1]Chamberlin (1962, p. 91).

lowered, and the movement comes to a stop at the price of AR."[2] In other words, the position of the dd' curve depends on the general, or average, level of competitors' prices.

Our purpose in this paper is to show that this aggregation of competitive information is a strong assumption that restricts on the variety of equilibria that can be observed for a monopolistically competitive model. We do this within a model whose key features are that the number of firms is large, firms are prohibited from employing mixed strategies, and each firm's profit function is parameterized by one or more statistics of the other firms' strategies. Thus each firm's profit depends on its own strategy and, for instance, the average price that competing firms are charging. With an elementary example we show that each firm's profit function in such a model may violate quasi-concavity and cause nonexistence of an equilibrium that is symmetric in the sense that in equilibrium every firm chooses an identical strategy.[3] In such cases, however, an equilibrium generally exists that is asymmetric.

Let M be the number of statistics of competing firms' strategies that parameterizes each firm's profit function. Our main result is that the industry equilibrium generically involves the firms employing at most M+1 distinct strategies. Consider as an example an industry where price is each firm's strategic variable and the average price charged by competitors parameterizes the profit function. Then generically only two types of equilibria can exist: a symmetric equilibrium where each firm charges the

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[2]Chamberlin (1962, pp. 91-92).

[3]Roberts and Sonnenschein (1977) showed that a pure strategy equilibrium may fail to exist for a the Cournot model even if its demand and cost conditions are completely nonpathological. Our example is thus an illustration of their point within a different context.

identical price and an asymmetric equilibrium where firms divide into a group of low price sellers and a group of high price sellers. If, by coincidence, an equilibrium should exist where firms divide into more than two groups, then almost any perturbation of the firms' profit functions collapses the equilibrium to either of the two generic possibilities. For the purposes of positive analysis such structurally unstable equilibria are uninteresting since they are unlikely to be ever observed.

Our result shows the extent to which asymmetric equilibria may exist within monopolistically competitive models. That asymmetric equilibria do exist in industries that we normally think of as being monopolistically competitive is supported by casual observation. For example, some firms advertise much more than other firms that are in the same line of business and apparently have the same strategic capababilities. Also, discount retail establishments seem to coexist with nondiscount retail establishments. Since these strategic differences persist beyond the length of time that firms require to copy a competing, more successful firm's strategy, the underlying reason for the asymmetry must be that the equilibrium allows identical firms pursuing widely divergent strategies to earn identical or almost identical profits.

We divide the paper into three sections beyond this introduction. In Section 2 we discuss in detail a simple example of a model where firms have two strategic variables, price and advertising, and where other firms' average price parameterizes each firm's profit function. The only equilibrium in this model is asymmetric. It illustrates all the main points of the paper. Section 3 presents our formal model and results. Section 4 contains a few concluding remarks.

An Example

Description. Our example is a model of a monopolistically competitive industry composed of a large number of firms, each supplying a single good that is highly, but not perfectly, substitutible for every other firm's good. Through their search activities consumers are informed of the prices of a sample of firms. The rate at which consumers substitute between the products of two particular firms about which they have knowledge varies across consumers because each product has certain idiosyncratic characteristics (subtle aspects of style, location, etc.) that different consumers react to differently and are very hard or impossible for firms to fully control. Therefore if each firm sets the same price, then among those firms about which a consumer has knowledge he or she selects that good whose idiosyncratic characteristics most closely fit his or her preferences. As a result each firm has a spectrum of customers that range from those who just marginally prefer its product over some other firm's product to those who really like its product and are very loyal. Consequently as a firm raises or reduces its price slightly the quantity demanded from it varies smoothly.[4]

Each firm controls two variables: the price it charges and the quantity of advertising it purchases. Demand and cost conditions are perfectly symmetric among the firms, which means that they all have identical profit functions $\pi(p_i, A_i; \bar{p}_i)$ where $p_i \geq 0$ is the price firm i charges, $A_i \geq 0$ is the amount of advertising firm i purchases, and $\bar{p}_i$ is the average of the

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[4]A model with demand of this type is developed in Satterthwaite (1979). The smoothness of demand that this particular model exhibits is not essential to our main argument.

prices firm i's competitors charge. Its form is

$$\pi(p_i, A_i; \bar{p}_i) = p_i f(p_i, A_i, \bar{p}_i) - A_i \tag{2.01}$$

where f is the demand function,

$$f(p_i, A_i, \bar{p}_i) = (a + b\bar{p}_i) + (p_i - \bar{p}_i)(\alpha + \frac{1}{\gamma + \delta A_i}), \tag{2.02}$$

and $a > 0$, $b < 0$, $\alpha > 0$, $\delta > 0$, and $\alpha + (1/\gamma) < 0$. As is evident from inspection of (2.01), the marginal cost of producing an additional unit of output is zero and the marginal cost of purchasing an additional unit of advertising is one.

The structure that the demand function f has with respect to price is exactly the same as Chamberlin's basic model and is illustrated by Figure 1. Suppose the price that all other firms in the market area are charging is OE. Firm i takes this as exogenous since it is only one firm among many and has a negligible effect on the demand of other firms; consequently no other firm has a reason to react to its price changes. The dd' demand curve in the figure represents firm i's demand curve when it changes price while all other firms keep their prices constant. It, which we call the firm demand curve, is the graph of f as p_i varies and $\bar{p}_i$ and A_i is held constant. Consequently it is the demand curve that is relevant to firm i in its maximizing decision with respect to price. The DD' demand curve traces out the quantity demanded from firm i when all firms in the market area, including i, change their prices together. This curve, which we call the fractional industry demand curve, is the graph of $f(\bar{p}_i, A_i, \bar{p}_i)$ as $\bar{p}_i$ is varied and A_i is held constant. It is less elastic than the firm demand curve because it only incorporates changes in the quantities consumers purchase; unlike the firm demand curve it does not include the effect of consumers switching from high to low price firms.

Figure 2 shows the effect advertising has under our specification (2.02) of the demand function f. Curve $f(p, 0; \bar{p})$ represents i's firm demand curve when A_i is zero, curve $f(p, A_1; \bar{p})$ represents the altered firm demand curve when A_i is increased to some positive amount, and $f(p, A_2; \bar{p})$ represents i's firm demand curve when A_i is increased further.[5] The idea behind this specification is that firm i's advertising increases the likelihood that a randomly drawn consumer will know how firm i's price compares with the prices that other firms are charging. If firm i's price is high relative to other prices, then a consumer who is a regular customer of firm i may decide it is worthwhile to experiment with the other firms' products provided that his or her preference for the idiosyncratic characteristics of i's product is not too strong. Similarly, if firm i's price is low relative to other firm's prices, then a consumer who is a regular, but not particularly loyal, customer of some other firm's product may decide to give firm i's product a try.

Curves $f(p, 0; \bar{p})$, $f(p, A_1; \bar{p})$, and $f(p, A_2; \bar{p})$ all cross the fractional industry demand curve at the price $\bar{p}$ because if firm i advertises that it charges the same price as other competing firms do on average, then this information concerning its own price should have little if any effect on the quantity demanded. This is because the improved, comparative information that the advertising provides consumers reveals only that the firm is providing an

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[5]Chamberlin (1962, p. 131) in his discussion of the effects of advertising assumes that advertising shifts the firm demand curve outward in a parallel fashion rather than rotating it. His assumption concerning the effects of advertising seems neither more nor less plausible than ours. In fact our hypothesis is that if we had reliable means to measure the effects of advertising, both effects would be observed in strengths that would vary with the nature of the industry.

average value; by itself this information gives consumers no reason to switch firms. Consequently the overall effect of advertising is to twist the firm demand curve counterclockwise about point A. The fractional industry demand curve remains unchanged in Figure 2 because no reason exists why better price information about competing firms should affect the total quantity demanded.

The justification for parameterizing firm i's profit function by $\bar{p}_i$ rather than by the full vector of the other firms' prices has two aspects. First, the large number of firms means that firms are anonymous both to each other and to consumers. Therefore both firms and consumers in making their decisions may summarize their information concerning strategies firms are using into a set of summary statistics. For example, consumers who are calculating their optimal search strategies may have estimates of the first two moments of the distribution of prices across firms. If this is the case for all consumers, then the demand each firm faces depends only on the firm's own price and the first two moments of the price distribution.

Second, parameterizing the profit function with a statistic or vector of statistics has analytical utility. As Chamberlin (1962, ch. IV, sec. 3) points out, a monopolistically competitive industry can be modeled as a large number of purely monopolistic firms that are linked by the presence of each other's individual prices in their demand functions. Equilibrium in the industry can then be studied through the general equilibrium approach of calculating a consistent and maximizing set of prices for all the monopolists.[6] This is hard to do on two levels. At the conceptual level, if

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[6]Taking the general equilibrium approach of solving for all the monopolists' prices simultaneously is essential. Chamberlin's criticism (1962, p. 69) of the purely monopolistic approach is that the general equilibrium approach is normally not taken: "Within any group of closely related products . . . the demand and cost conditions (and hence the price) of any one are defined only if the demand and cost conditions with respect to the

the industry consists of I firms where I is a large number, difficulties occur with this approach in trying to specify how each of the I-1 competitors' prices enter into a given monopolist's demand function. At the computational level, solving I equations simultaneously and studying their comparative static properties is hard. It is much easier to follow Chamberlin's lead and do as we are doing in the example of this section: solve and analyze the three equations that arise as a result of the profit function being parameterized by a single statistic.

Because of their numbers firms behave as noncooperative Nash competitors who take $\bar{p}_i$ as given and choose p_i and A_i to maximize $\pi(p_i, A_i; \bar{p}_i)$. Firms are not permitted to employ mixed strategies.[7] We define an equilibrium configuration for the industry to be a positive integer K, a $\bar{p} = \bar{p}_1 = \ldots = \bar{p}_i = \ldots$ and a 3K-vector $(q_1, p_1, A_1), \ldots, (q_K, p_K, A_K)$ of proportions, prices, and advertising levels such that $\Sigma_{k=1}^K q_k = 1$, $\bar{p} = \Sigma_{k=1}^K q_k p_k$, and

$$(p_k, A_k) \in \underset{p,\, A}{\operatorname{argmax}}\, \pi(p, A; \bar{p}). \tag{2.03}$$

for k = 1, . . .,K. The interpretation of this equilibrium definition is that, for each k, q_k proportion of the firms play the strategy (p_k, A_k). The

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others are taken as given. Partial solutions of this sort, yielded by the theory of monopoly, contribute nothing towards a solution of the whole problem, for each rests upon assumptions with respect to the others."

[7]We reject mixed strategies for the same reasons that Roberts and Sonnenschein (1977, p. 111) object to them, "The problem is to reconcile such systematic randomization with standard economic theory and observed behavior." This is not to say that randomization does not make sense in some economic models; Butters' (1977) and Varian's (1980) models are cases where mixed strategies do make sense. Nevertheless our perception is that most firms do not and, in many cases, can not reasonably carry out mixed strategies for pricing and product design decisions.

number of firms is large enough that each firm's impact on the average price is negligible; therefore the approximation $\bar{p} = \bar{p}_1 = . . . = \bar{p}_i = . . .$ is acceptable.[8] The K strategies that are played in an equilibrium are called the active strategies. If K = 1, then the equilibrium is symmetric; if K > 1, then the equilibrium is asymmetric.

Analysis. The demand function f is linear in both the firm's own price, p_i, nonlinear in its advertising level, A_i, and linear in the average price of its competitors, $\bar{p}$. It is the nonlinearity with respect to A_i that makes each firms profit function not quasi-concave. As Figure 3 shows A_i has no effect on the quantity demanded when p_i equals $\bar{p}_i$. When p_i is either greater or less than $\bar{p}$ then A_i decreases or increases respectively the quantity demanded at a rate that diminishes toward zero as A_i becomes substantial.

The first observation we make about this example is that if the parameter δ, which is the coefficient of A_i within the demand function, is large enough, then (a) no symmetric equilibrium exists and (b) an asymmetric equilibrium exists where the two strategies that are active are $(p_0^*, 0)$ in proportion q^* and (p_A^*, A^*) in proportion $1-q^*$ where $p_A^* < p_0^*$ and $\bar{p}^* = q^* p_0^* + (1 - q^*)p_0^*$. In other words, in the asymmetric equilibrium q^* proportion of the firms do not advertise and charge the high price p_0^* and $1-q^*$ proportion of the firms advertise at the positive level A^* and charge the low price p_A^*.

Part (a) of the observation is argued as follows. Suppose a symmetric equilibrium exists. Then in equilibrium no firm advertises because advertisng has no effect on the quantity demanded from a firm whenever that firm, as must be the case in the symmetric equilibrium, charges the same price as every

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[8]See Section 4 for additional discussion of this assumption.

other firm. But any firm can reduce its price a little below the average price, advertise this fact, increase its quantity sold, and break the equilibrium provided the marginal revenue earned by this increased quantity is larger than the marginal cost of the reduced price and increased advertising. As δ is made larger the marginal revenue of this strategy increases; therefore for large enough δ the symmetric equilibrium can not exist.

Part (b) of the observation follows from solving the Kuhn-Tucker, first order conditions for the firm i's optimal price, p_i^*, in terms of $\bar{p}$ and A_i, i.e., $p_i^* = g(\bar{p}, A_i)$. If this function is substituted into π, to define $\tilde{\pi}(\bar{p}, A_i) = \pi[g(\bar{p}, A_i), A_i; \bar{p}]$, then Figure 4 shows the graph of $\tilde{\pi}$ as a function of A_i for different values of $\bar{p}$. The value $\bar{p}_{min}$ is the maximum value of $\bar{p}$ for which the only relative maximum of $\tilde{\pi}$ occurs at $A_i=0$.[9] The value $\bar{p}_{max}$ is the minimum value of $\bar{p}$ for which the only relative maximum of $\tilde{\pi}$ occurs for a positive value of A_i. For the interval between $\bar{p}_{min}$ and $\bar{p}_{max}$ two relative maxima occur: one for $A_i = 0$ and one for $A_i > 0$. Within this interval a $\bar{p}^*$ exists such that $\tilde{\pi}$ attains the same value at both its relative maxima. This $\bar{p}^*$ is the asymmetric equilibrium value of $\bar{p}$. The proportion q^* is picked such that $\bar{p}^* = q^*g(\bar{p}^*, 0) + (1-q^*)g(\bar{p}^*, A^*)$ where A^* is the level of $A_i > 0$ that maximizes $\tilde{\pi}(\bar{p}^*, A_i)$. Figure 5 shows contours of $\pi(p_i, A_i; \bar{p}^*)$ as a function of p_i and A_i when $\bar{p}$ is set at the value that produces an asymmetric equilibrium.[10]

Our second observation is that the asymmetric equilibrium in which the firms divide into two distinct classes is not pathological in the sense of being structural unstable. By this is meant that if we vary the profit

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[9]Recall that A_i is restricted to be non-negative.

[10]Andrew Melczer originally calculated this equilibrium and plotted its contours.

function with sufficiently small C^2 perturbations, then the qualitative nature of the asymmetric equilibrium remains intact in the sense that an asymmetric equilibrium with two active strategies that are close to the original two active strategies, $(q, p_0^*, 0)$ and $(1-q, p_A^*, A^*)$, continues to exist. To show that this is true, three conditions must hold for small perturbations of the profit function: (a) the two local maxima must persist, (b) a $\bar{p}'$ must exist that is close to $\bar{p}^*$ and equalizes the profit levels of the two local maxima, and (c) a q' must exist that is close to q^* and satisfies $\bar{p}' = q'g(\bar{p}', 0) + (1-q')g(\bar{p}', A')$ where A' is the level of advertising that maximizes profits when $\bar{p} = \bar{p}'$.

Assume that, as is the case in Figure 4, $\bar{p}^* \varepsilon (p_{min}, p_{max})$. Then the Hessian matrix evaluated at the local, positive advertising (interior) maximum is negative definite and, as a consequence of the implicit function theorem, that local maximum persists under perturbation. Similarly, at the local, zero advertising (boundary) maximum both the first partial of π with respect to A_i and the second partial of π with respect to p_i are strictly negative, which means that maximum also persists under perturbation. Therefore condition (a) holds.

It follows that for some neighborhood of $\bar{p}^*$ the profit levels at the two local maxima are well defined, continuous, and differentiable as a function of $\bar{p}$. Let

$$P(\bar{p}) = \pi(p_0^*(\bar{p}), 0, \bar{p}) - \pi(p_A^*(\bar{p}), A^*(\bar{p}), \bar{p}) \tag{2.04}$$

where p_0^*, p_A^*, and A^* are implicit functions of $\bar{p}$. The function P is the difference in profit levels at the zero advertising local maxima and the positive advertising local maxima. Since it is clearly differentiable, a small perturbation of the profit function will induce a small perturbation in it also. The question of structural stability therefore is: does a $\bar{p}$ exist such that the perturbed profit function equals zero.

Figure 6 shows that such a $\bar{p}$ does not necessarily exist. If the unperturbed difference function's graph is the dotted line with $\bar{p}^*$ being the equilibrium value of $\bar{p}$, then a small perturbation could push the graph upward as the solid line shows resulting in the non-existence of an equilibrium value of $\bar{p}$. Consequently a necessary and sufficient condition for the existence of an equilibrium $\bar{p}$ for any small perturbation of the profit function, π, is that the unperturbed difference function, P, be transversal to the $\bar{p}$ axis; i.e., its slope at $\bar{p}^*$ must be nonzero. Figure 6 illustrates this case with the solid line. This, as a small amount of analysis shows, is the case in which our example falls.[11]

The third observation we make about this example is that no structurally stable equilibrium can exist that involves three or more active strategies. A structurally stable equilibrium with at least three active strategies could only exist if at least two statistics of competing firms' strategies parameterized each firm's profit function. To see this suppose that an asymmetric equilibrium involving three active strategies exists for our example. We show that this hypothesized equilibrium can not be structurally stable.

Let (p_1^*,A_1^*), (p_2^*,A_2^*) and (p_3^*,A_3^*) be the three local maxima of the profit function that are active in the equilibrium. Each is implicitly a function of $\bar{p}$. For $\bar{p}$'s equilibrium value, $\bar{p}^*$, the profit levels at each of the three local maxima are equal. Let the the profit difference function be the vector-

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[11]The necessary analysis is in Kumar (1981, ch. II).

valued function of $\bar{p}$:

$$P(\bar{p}) = \left\{ \begin{matrix} \pi(p_1^*(\bar{p}), A_1^*(\bar{p}), \bar{p}) - \pi(p_2^*(\bar{p}), A_2^*(\bar{p}), \bar{p}) \\ \pi(p_2^*(\bar{p}), A_2^*(\bar{p}), \bar{p}) - \pi(p_3^*(\bar{p}), A_3^*(\bar{p}), \bar{p}) \end{matrix} \right\}. \tag{2.05}$$

Since $\bar{p}^*$ is the equilibrium value of $\bar{p}$, $P(\bar{p}^*) = 0$.

Figure 7 graphs the values of P as a function of $\bar{p}$. The origin, where the value of P is the null vector, is the image of $\bar{p} = \bar{p}^*$ and point A is the image of some $\bar{p} = \bar{p}' \neq \bar{p}^*$. Thus as $\bar{p}$ varies the image moves along the solid line that is plotted on the figure. This map must pass through the origin for an equilibrium to exist. But almost any small perturbation of the profit function causes a small pertubation of this map (as shown by the dotted line). Almost certainly, the perturbed map misses the origin, which means no equilibrium near to the original equilibrium exists. In other words, our hypothesized, asymmetric equilibrium with three active strategies is not structurally stable.

More formally, the requirement for structural stability is that the derivative map,

$$DP(\bar{p}): \quad \mathbf{R}^1 \to \mathbf{R}^2, \tag{2.06}$$

evaluated at $\bar{p}^*$, the equilibrium value of $\bar{p}$, must span the range space. This, however, is dimensionally impossible. Therefore a three-strategy, asymmetric equilibrium can never be structurally stable given that only one statistic parameterizes firms' profit functions. If the profit function were parameterized by the average of competing firms' advertising levels, $\bar{A}$, as well as by $\bar{p}$, then P's derivative map would be from R^2 to R^2 and three-strategy equilibria generally would be structurally stable. Underlying this is the general result, which is proved in the next section, that if K statistics of other firms' strategies parameterize each firms' profit function, then only equilibria involving at most K+1 active strategies can be structurally stable.

Model and Results

Let the industry contain I firms and let X be the strategy space of each firm within the industry. Firms are limited to playing pure strategies. I is assumed to be a large, finite number and X is a compact, convex subset of R^L. The notation $x_i = (x_{i1}, \ldots, x_{iL}) \varepsilon X$ represents a feasible strategy for firm i, $x = (x_1, \ldots, x_I) \varepsilon X^I$ represents a vector of strategies for the entire industry, and $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_I)$ represents the strategies for all firms in the industry except firm i. Let

$$z = ((q_k, x_k)_{k=1,K}) = \{(x_1, \ldots, x_K), (q_1, \ldots, q_K)\} \qquad (3.01)$$

represent a configuration of the industry where K is the finite number of strategies that are active, $q_k > 0$ is the proportion of firms playing the the strategy x_k, and $\Sigma_{k=1}^{K} q_k = 1$. A strategy is active if and only if some positive proportion of firms are employing it. Finally, let Z denote the set of configurations that are feasible for the industry.

Competing firms' strategies parameterize the profit function of each firm through an M-vector of statistics, $y_i = f(x_{-i})$, where f: $X^{I-1} \rightarrow R^M$ is a twice continuously differentiable function that is symmetric in all its arguments. Let Y be f's range. Since I is large, we make the approximation that $f(x_{-i})$

$\approx f(x) = f(z)$ where the x and z notations for describing the industry's configuration are used interchangably. In the example of the previous section the function f took the form

$$y = f(z) = \Sigma_{k=1}^{K} q_k p_k. \tag{3.02}$$

More generally f might calculate several moments of the distribution of competing firms' strategies. The linear structure of (3.02) is not important for the paper's main result; only for one of the paper's secondary results is the linearity important.

Let π: $X \times Y \to R^1$ be the twice continuously differentiable profit function that is common to all the firms. Thus if firm i follows strategy x_i and the industry has configuration z, then firm i's payoff is $\pi[x_i; f(z)]$. Given this basic structure, an industry is the triplet $\langle X, \pi, f \rangle$.

Firms are profit maximizers that take the strategies of other firms as given. Equilibrium is therefore of the Nash-Cournot variety and is defined to be any configuration $z \in Z$ such that, for all i,

$$x_i \in \underset{\zeta \in X}{\operatorname{argmax}}\, \pi[\zeta; f(z)]. \tag{3.03}$$

Let firms' best response set when faced with the configuration z be

$$R[f(z)] = R(y) \equiv \{ x_i \mid x_i \in \underset{\zeta \in X}{\operatorname{argmax}}\, \pi[\zeta; f(z)]\}, \tag{3.04}$$

In equilibrium only elements of the best response set, $R(y^*)$, can be active strategies.

Let $C^2_{XY} = \{h: X \times Y \to R \mid h \varepsilon C^2\}$ be the set of continuously twice differentiable real-valued functions defined on $X \times Y$. The function $h_\lambda \varepsilon C^2_{XY}$ is a homotopic perturbation of a function $h \varepsilon C^2_{XY}$ if and only if $h_\lambda = h + \lambda g$ for some $g \varepsilon C^2_{XY}$ and scalar λ. These perturbations have the property that, for λ small enough, not only is h_λ close to h, but the first and second derivatives of h_λ are also close to the first and second derivatives of f.

An industry configuration z^* is a structurally stable equilibrium if, for all homotopic perturbations π_λ of π, a $\bar{\lambda} > 0$ exists such that, for all $\lambda \varepsilon (0, \bar{\lambda})$, an equilibrium $z^*(\lambda)$ exists for π_λ that is close to the original equilibrium z^*. The distance between two equilibria, z^* and $z^*(\lambda)$, that have the same number of active strategies is

$$\delta[z^*, z^*(\lambda)] = \max_k \| [x^*_k , q^*_k] - [x^*_k(\lambda), q^*_k(\lambda)]\| \tag{3.05}$$

where $\| . \|$ is the Euclidean metric.[12] If, for a particular perturbation π_λ, a continuous differentiable function $z(\lambda)$ exists such that, for all $\lambda \varepsilon (0, \bar{\lambda})$, $z(\lambda)$ is an equilibrium configuration, then the condition for closeness is satisfied. This is because the continuity of $z(\lambda)$ implies that, for any $\alpha > 0$, a $\lambda' > 0$ exists such that, for all $\lambda \varepsilon (0, \lambda')$, $\delta[z^*, z(\lambda)] < \alpha$.

Suppose that $R(y^*)$ has K elements where K is finite. Since π is continuously differentiable and $R(y)$ has a finite number of elements,

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[12] In our model two equilibria that have different numbers of active strategies are defined as being not close. One could, however, define closeness in sensible ways that would allow equilibria with different numbers of active strategies to be close. For example, if the M-vector of statistics y^* adequately describes the equilibrium, then two equilibria could be defined as close if their respective y^* vectors are close.

each $x_k^* \varepsilon R(y^*)$, (k = 1, . . ., K) is a nondegenerate local maximum of $\pi(. ; y^*)$. Pick a particular maximum x_k^*. If it is an interior point of X, then it solves the first order conditions for a maximum of $\pi(. ; y^*)$:

$$\frac{\partial \pi(x_k^*; y^*)}{\partial x_k} = 0. \tag{3.06}$$

The nondegeneracy of the maximum implies that the Hessian matrix is nonsingular, which means that the implicit function theorem may be applied. Therefore a neighborhood $N_k(y^*)$ of y^* and a differentiable function $x_k(y)$ exist such that, for all $y \varepsilon N_k(y^*)$,

$$\frac{\partial \pi[x_k(y), y]}{\partial x_k} = 0, \tag{3.07}$$

i.e., $x_k(y)$ traces out the movement of π's kth relative maximum as y varies. If x_k^* is not an interior point, then two cases are possible. First, if a binding constraint keeps x_k^* on X's frontier rather than straying outside, then the implicit function theorem applies to the resulting constrained maximum problem and again a differentiable $x_k(y)$ exists that traces the movement of the maximum. If x_k^* is on the frontier without any constraint being binding, then, with some extra work, the implicit function theorem still applies piecemeal: a $x_k(y)$ exists that is continuous, but not necessarily differentiable.

If $N(y^*)$ is restricted to be a subset of the intersection of the K neighborhoods $N_k(y^*)$, then on $N(y^*)$ the K functions, $x_k(y)$, all exist. Make $N(y^*)$ small enough so that no other local maximum of $\pi(. ; y)$ exists whose value is as great as the values $\pi(. ; y)$ attains at the K local maxima the functions $x_k(y)$ describe. This can be done because $|R(y^*)| = K$,[13] which means that the value $\pi(. ; y^*)$ takes on at any local maxima other than the K local maxima that are elements of $R(y^*)$ is strictly less that the value $\pi(. ; y^*)$ takes on at those K global maxima. This strict inequality is preserved within $N(y^*)$ provided it is made small enough. The implication of defining $N(y^*)$ in this way is that, for any $y \varepsilon N(y^*)$, only the points $x_k(y)$ are candidates for being elements of $R(y)$. Therefore, to summarize this and the previous paragraph, within some neighborhood $N(y^*)$ of y^* continuous functions $x(y) = \{x_1(y), . . ., x_K(y)\}$ exist that trace K of π's relative maxima, provided $|R(y^*)|$ is finite.

Given the existence of the continuous functions $x(y)$, we define the function $P : N(y^*) \to R^{K-1}$ to be

$$P(y) = \left\{ \begin{array}{c} \pi[x_1(y);\ y] - \pi[x_2(y);\ y] \\ . . . \\ \pi[x_{K-1}(y);\ y] - \pi[x_K(y);\ y] \end{array} \right\}. \qquad (3.08)$$

Note that $P(y) = 0$ is a necessary condition for every one of the K strategies $(x_1(y), . . ., x_K(y))$ to be elements of the best response set $R(y)$.

Our main result is:

Theorem 1. Suppose an industry, $\langle X, \pi, f \rangle$, has an equilibrium configuration $z^* = \{(x_1^*, . . ., x_K^*), (q_1^*, . . ., q_K^*)\}$. If $K > M+1$, then the equilibrium is structurally unstable.

Thus, if the number of active strategies in the equilibrium configuration

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[13] $|R(y)|$ denotes the cardinality of the best response set.

exceeds one plus the number of statistics that parameterize the profit function, then the equilibrium is unstable.

Proof. Suppose $z^* = \{(x_1^*, \ldots, x_K^*), (q_1^*, \ldots, q_K^*)\}$ is an equilibrium configuration where $K > M+1$. We will show that if π is perturbed, then almost surely no configuration z close to z* exists that can be an equilibrium configuration for the industry, which is sufficient to establish that z* is not structurally stable. Let $y^* = f(z^*)$. Consider first the case where $K = |R(y_{-i}^*)|$ and is finite; the other cases are treated separately at the end of the proof.

The domain space of P, which is N(y*), has dimension M since it is a subset of Y. By assumption this is less than K-1, the dimension of the range space. This implies that the range of P is a closed set of measure zero in R^{k-1}. Let P_λ be the perturbation of P that the pertubation, π_λ, of π induces;[14] the range of P_λ is also a set of measure zero in R^{K-1}. Since the ranges of both P and $P_{i\lambda}$ have measure zero and are not identical, almost every point in P's range is not a point in P_λ's range. Consequently, almost surely no $y \in N(y^*)$ exists such that $P_\lambda(y) = 0$.

Therefore, for every $y \in N(y^*)$, a k exists such that $x_k(y) \notin R(y)$. A necessary condition for two configurations, z* and z to be close together is that they have the same number of active strategies. Thus every configuration z close to z* violates a necessary condition for z being an equilibrium strategy. This means the equilibrium z* is not structurally stable and completes the proof for the case where $K = |R(y^*)|$.

Consider next the case where R(y) is finite and $K \neq |R(y)|$. The subcase $K > |R(y^*)|$ is impossible because every active strategy in equilibrium must be

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[14]The perturbation of π induces perturbations in all the $x_k(y)$ functions.

an element of the best response set. In the event of the other subcase, K $< |R(y^*)|$, elements of $R(y^*)$ other than those that are active in z^* might become active as y varies within $N(y^*)$ The resulting equilibrium configuration, even if it had K active strategies, could not be close to z^* because every element of $R(y^*)$ is isolated from every other of its elements. Therefore this subcase does not offer any possibilities beyond those that the $K = |R(y^*)|$ case offered and the proof presented for it suffices.

Suppose, as the last case, that $|R(y^*)|$ is infinite.[15] Pick any perturbing function g; then $\pi_\lambda = \pi + \lambda g$. Let $R_\lambda(y)$ be the best response set of the perturbed profit function π_λ. Assume that z^* is structurally stable and that $K-1 > M$. Since z^* is structurally stable, for $\alpha > 0$ small enough, a $\bar{\lambda} > 0$ exists such that, for all $\lambda \in (0, \bar{\lambda})$, an equilibrium configuration $z(\lambda)$ exists such that $\delta(z_\lambda, z^*) < \alpha$. Sard's Theorem implies that, for any $\lambda \in (0, \bar{\lambda})$, $|R_\lambda(y)|$ is almost certainly finite even though $|R_0(y^*)|$ is infinite. In other words, no matter what shape π has, almost any pertubation of it has a finite number of relative maxima.[16]

Define $\pi' \equiv \pi + \lambda' g$ and $\pi'_\nu \equiv \pi' + \nu g = \pi + (\lambda' + \nu)g$ where $\lambda' \in (0, \bar{\lambda})$. Since $|R_\nu(y^*)|$ is almost certainly finite, the cases proved above for finite best response sets apply to π'. Specifically, $z(\lambda')$, which is an equilibrium configuration of π', can not be a structurally stable equilibrium configuration of π'. Define $z'(\nu) = z(\lambda'+\nu)$. By assumption $z'(\nu)$ is an equilibrium configuration of $\pi_{\lambda'+\nu}$ provided ν is small enough. By the construction of π'_ν, $z'(\nu)$ is also an equilibrium configuration of π'_ν. This

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[15]Even if $|R(y^*)|$ is infinite, K must be finite since the number of firms is large but finite and firms can not employ mixed strategies.

[16]See the theorem on p. 43 of Guillemin and Pollack (1974).

last fact, however, almost surely contradicts the fact that z' is a structurally unstable equilibrium. Therefore z* can not be structurally stable as was assumed.∎

The generality of Theorem 1 can be appreciated by considering its validity if the assumptions on the continuity and differentiability of π and f are dropped and replaced by the assumptions that both are functions of bounded variation. The proof that the result does remain valid, which we sketch here, remains essentially unchanged from above except that the K functions $x_k(y)$ need redefinition. Suppose $z^* = \{(x_1^*, \ldots, x_K^*), (q_1^*, \ldots, q_K^*)\}$ is an equilibrium configuration such that $K > M+1$. Let $y^* = f(z^*)$. Assume that $K = |R(y)|$ and is finite. The arguments presented in Theorem 1's proof continue to apply; therefore the assumption entails no loss of generality. Pick an $\alpha > 0$. We show that, for almost every small pertubation of π (where the perturbing function g has bounded variation), no equilibrium configuration z can exist such that $\delta(z, z^*) < \alpha$, i.e., z* can not be structurally stable.

Given α, around each point x_k^* $(k = 1, \ldots, K)$ neighborhoods $N(x_k^*)$ can be established that have the property: if z is configuration such that $x_k \notin N(x_k^*)$, then $\delta(z, z^*) > \alpha$, i.e., z and z* are not close unless $x_k \in N(x_k^*)$ for all k. If these K neighborhoods are not disjoint, then redefine α to be small enough for them to be disjoint. Define K functions $x_k(y)\colon\ Y \to N(x_k^*)$ each having the property:

$$x_k(y) = \underset{\xi \,\in\, N(x_k^*)}{\operatorname{argmax}} \ \pi(\xi; y) \tag{3.09}$$

if such maximal point exists and, otherwise, $x_k(y) = x_k'$ where x_k' is a fixed, arbitrary point within $N(x_k^*)$.

Given the functions $x_k(y)$, the function P(y) is defined as before and, because z^* is an equilibrium configuration, $P(y^*) = 0$. Thus the origin is an element is P's range. Recall that P's domain Y is contained in R^M and its range is contained within R^{K-1}. By assumption $K-1 > M$; therefore the range of P within R^{K-1} is a space of measure zero. Any arbitrary perturbation of π causes P's range to be perturbed. Since that range is of measure zero, after perturbation it almost certainly does not contain the origin. Therefore z^* is not a structurally stable equilibrium configuration, which completes our sketch of the proof for the case where f and π are assumed only to be functions of bounded variation.

Theorem 2 identifies conditions that are sufficient to guarantee structural stability of an equilibrium configuration for which $K \leq M+1$. Its statement is based on the observation that for a given profit function, π, an symmetric equilibrium configuration z^* must satisfy M+K equations:

$$\begin{aligned} &P(y^*) = 0, && \text{K-1 equations;} \\ &f[(x(y^*), q^*)] - y^* = 0, && \text{M equations;} \qquad (3.10) \\ &\textstyle\sum_{k=1}^{K} q_k^* - 1 = 0, && \text{1 equation;} \end{aligned}$$

where the vectors y and q are the M+K unknowns and the functions $P(y)$ and $x(y)$ are as defined above. Let $G\colon R^{M+K} \to R^{M+K}$ be the mapping defined by the left-hand side of the equation system (3.10). Define an interior equilibrium configuration $z^* = (x^*, q^*)$ to be any equilibrium configuration such that, for all k, x_k^* is an element of X's interior.

Theorem 2. Suppose an industry, $\langle X, \pi, f\rangle$, has an interior equilibrium configuration $z^* = (x^*, q^*)$ for which $y^* = f(z^*)$, $K \leq M+1$, and $K = |R[f(y^*)]|$. If the Hessian of the mapping G is nonsingular at (y^*, q^*), then z^* is a structurally stable equilibrium.

Our example in Section 2 shows that these sufficient conditions are not vacuous because an industry exists that satisfies them.

Proof. Let G_λ be the perturbation of G that a perturbation of π induces. Since (a) z^* is an interior equilibrium and (b) the Hessian of G at (y^*, q^*) is nonsingular, the implicit function theorem implies that, for some scalar $\bar{\lambda} > 0$, differentiable functions $y(\lambda)$ and $q(\lambda)$ exist such that $G(x(\lambda), q(\lambda)) = 0$ for all $\lambda \in [0, \bar{\lambda}]$. Consequently the equilibrium z^* is structurally stable.∎

In Theorems 1 and 2 we have assumed the existence of equilibrium. If a particular linear structure is placed on the statistics function, f, then existence of equilibrium is guaranteed. Specifically, the linear structure that is sufficient for existence of equilibrium is

$$y = f(z) \equiv \Sigma_{k=1}^{K} q_k g(x_k) \tag{3.11}$$

where $g: X \to \mathbf{R}^M$ is the continuously differentiable function that evaluates the contribution that firm j's kth active pure strategy x_{jk} makes toward the statistics y_i. In the example of Section 2 the function g took the simple form $g(p_i, A_i) = p_i/n$. More generally g might be used to calculate M moments of the distribution of competing firms' strategies. Nevertheless the important feature of f is its linear structure, not the specific form of g, which may be any continuously differentiable function.

Theorem 3. If an industry, <X, π, f>, , has a statistics function f endowed with the linear structure of (3.11), then an equilibrium z* exists.

Proof of this result is contained in Kumar (1981, theorem 3.2.2). It is not presented here because it is a straightforward application of a theorem by Glicksburg.[17] The only obscure aspect of the proof is showing that Y is compact and convex as a result of f having the linear form (3.11) and X being compact and convex.[18]

Concluding Comments

The main result we have shown is that in the monopolistic competition model if competitors' strategies affect each firm's profits only through M statistics, then in a structurally stable equilibrium firms employ at most M+1 distinct strategies. The essential condition that drives our result is the aggregation of competitive information. If no aggregation takes place, then our result places no binding restriction on the number of distinct strategies that the industry's firms may be observed employing. For example, if every

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[17]Maskin and Dasgupta (1977) quote this result of Glicksburg (1952).

[18]Strictly speaking, the proof shows that for any profit function an equilibrium exists. The proof does not restrict the number of active strategies to be finite. This, however, is an immaterial problem because if the only symmetric equilibrium involves an infinite number of active strategies, then a small perturbation of the profit function results in the number of active strategies in the equilibrium collapsing to a finite number.

competitors' price enters the demand function of every firm, then in effect every firm's profit function is parameterized by I-1 statistics of its I-1 competitors' strategies. Our result states that the number of active strategies in equilibrium is at most one more than this number of statistics; therefore at most I distinct strategies can be employed in equilibrium. But since I is the total number of firms, every firm can potentially employ a strategy distinct from the strategies of all other firms. Hence if no aggregation of competitive information takes place, no restriction is placed on the character of structurally stable equilibria.

A key assumption on which our main result is based is that I, the number of firms, be large enough that every firm faces essentially identical values of the aggregate statistics, i.e., the approximation $f(x_{-i}) \approx f(x) = f(z)$ is good. If I is small, then this approximation is not good and pure strategy equilibria x^* that approximate the equilibrium configuration z^* (which is calculated on the basis of I being very large) may not exist. Kumar (1981, ch. IV) has analysed a specific, economically interesting model in which, for a small number of firms, a pure stategy equilibrium always exists that approximates the asymmetric equilibrium configuration of the large number case. We are uncertain at the present time to what extent the properties of his example generalize to other models.

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Figure 1

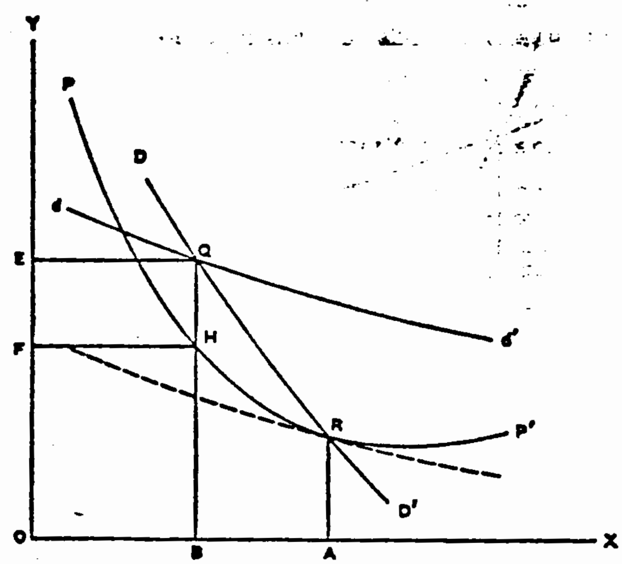


FIGURE 14

Figure 2

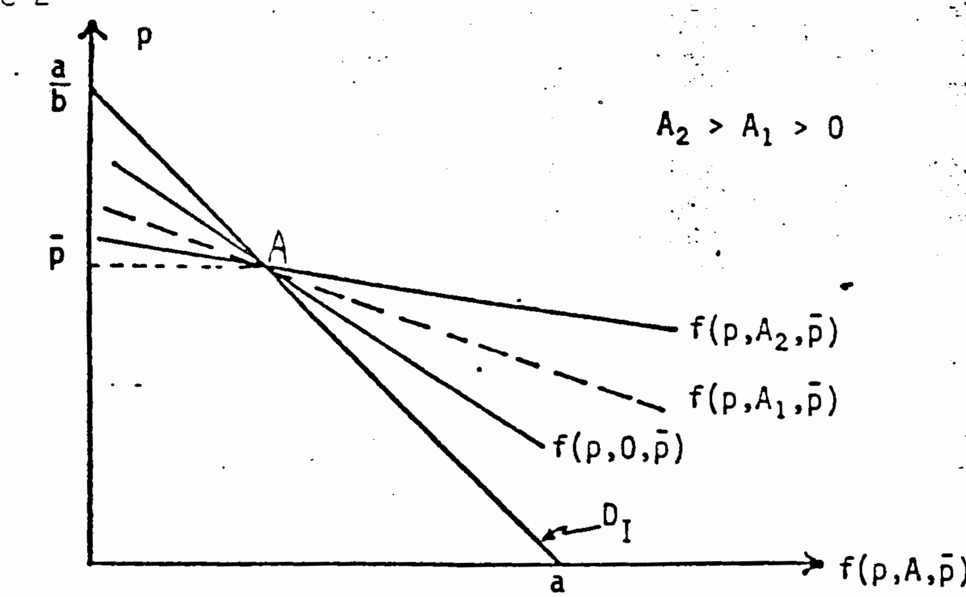


Figure 3.

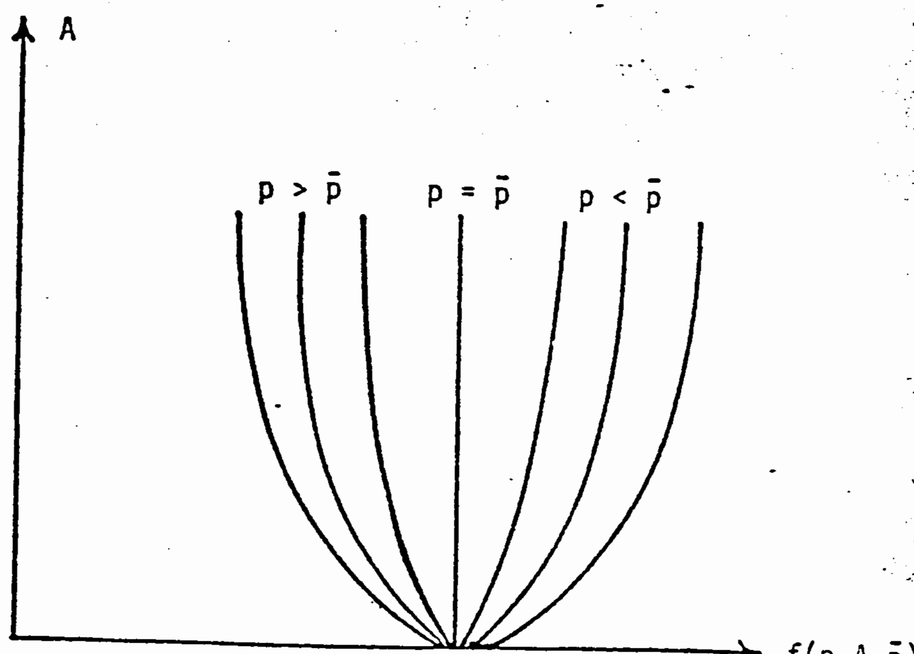
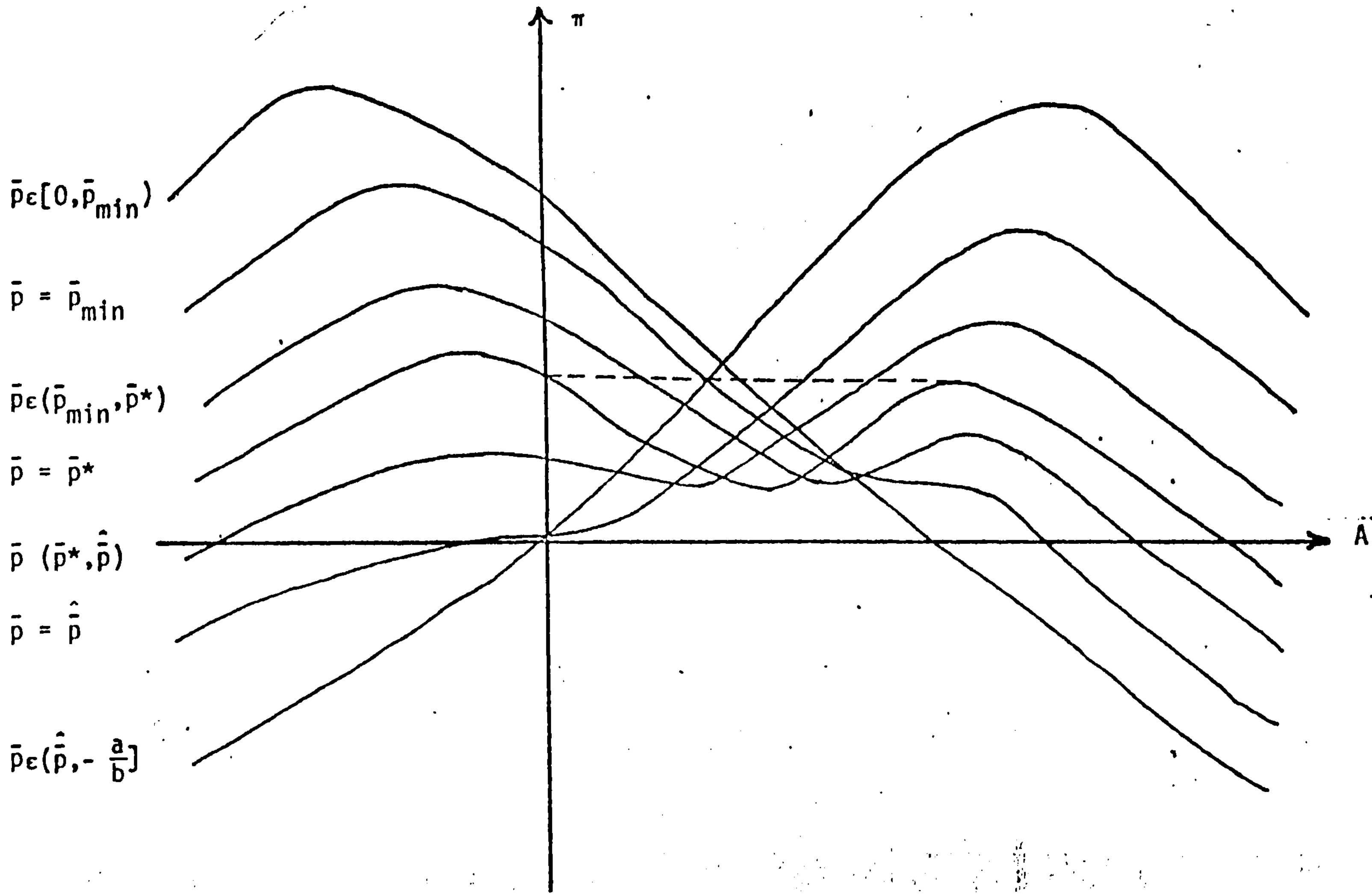


Figure 4



$\bar{p}\epsilon[0,\bar{p}_{min})$
$\bar{p} = \bar{p}_{min}$
$\bar{p}\epsilon(\bar{p}_{min},\bar{p}^*)$
$\bar{p} = \bar{p}^*$
$\bar{p}\ (\bar{p}^*,\hat{\bar{p}})$
$\bar{p} = \hat{\bar{p}}$
$\bar{p}\epsilon(\hat{\bar{p}},-\frac{a}{b}]$
π
$\ddot{A}$

Figure 5

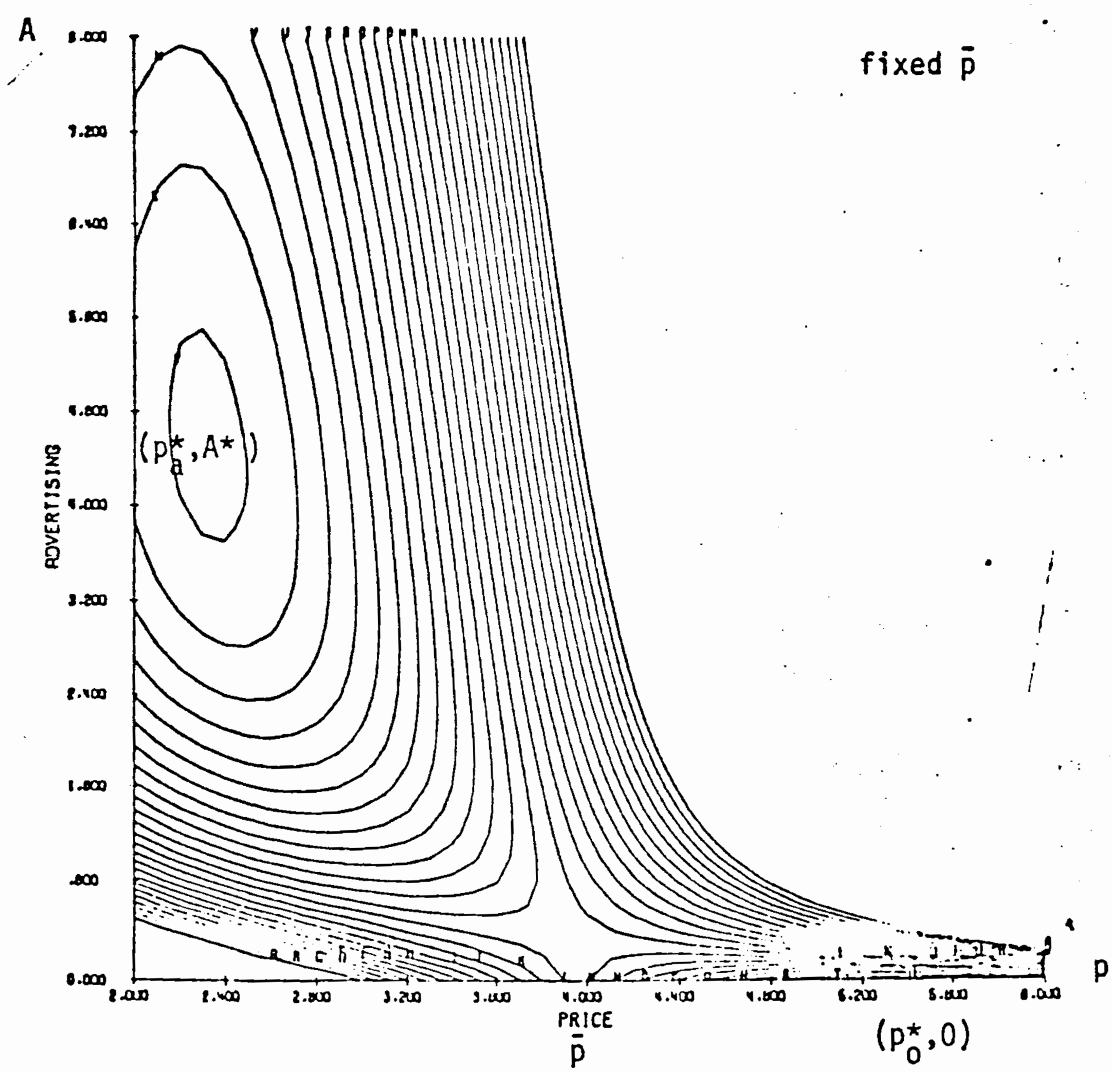


Figure 6

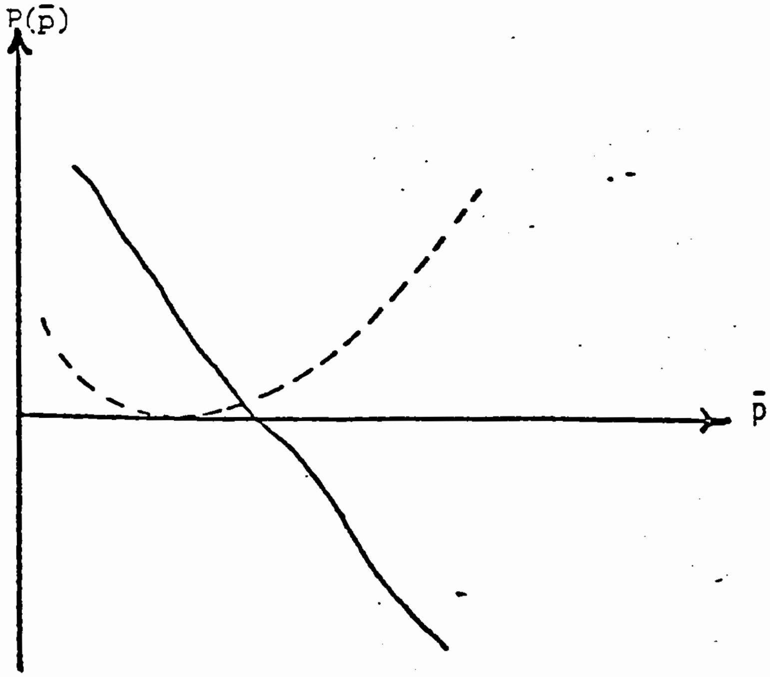


Figure 7