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THE ALLAIS PARADOX, DUTCH AUCTIONS,
AND ALPHA-UTILITY THEORY

by

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Abstract

Numerous experimental studies have shown that the expressed preferences of individuals commonly violate the substitution axiom of classical utility theory. The Allais paradox is a well-known example of one type of violation. Another example is provided by recent auction experiments which indicate that Dutch auctions systematically yield lower revenues than sealed-bid auctions, despite the fact that these two auction procedures are strategically equivalent from the perspective of expected utility maximization. A striking feature of these recent experimental results is that the direction of violation of the substitution axiom is opposite from that associated with the Allais paradox.

In this paper, we examine the "alpha-utility" model of preferences which arises when the substitution axiom is weakened. We provide an interpretation of the alpha-component of the model as a measure of the "conceivability" of an outcome. This interpretation suggests that the alpha-component will typically take a particular form. Finally, we show that the suggested form yields a model of decision-making which is consistent with both the Allais paradox and the auction phenomenon.

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Introduction.

The assumption that individuals act as expected utility maximizers lies at the foundations of both microeconomics and the theory of noncooperative games. Yet experiments have repeatedly shown that individuals in fact fail to act in a manner consistent with expected utility maximization (i.e., fail to act "rationally"); moreover, these experiments have uncovered systematic deviations from the predictions of classical utility theory.

It is, of course, possible to defend analyses based on the assumption of rationality, by claiming that "most" individuals "usually" act "almost" as if they were expected utility maximizers. Still, it is desirable to have available a positive theory of decision-making which accords well with experimental results. We present such a theory here.

Among the most well-known cases in which irrational behavior has been observed are the "paradoxes" of Allais [1953], one of which is discussed in detail in the next section. Various attempts have been made to present a model of decision-making which is consistent with the observations of Allais (see, for example, Kahneman and Tversky [1979]). However, recent experiments conducted by Cox, Roberson, and Smith [1981], involving Dutch and sealed-bid auctions, have revealed new

systematic deviations from rational behavior which (in a sense to be made precise below) are polar to the Allais paradoxes. Most of the attempts to explain the Allais paradoxes lead to predictions directly opposite to the phenomenon observed by Cox, Roberson, and Smith (hereinafter CRS). Our goal is to provide an alternative model of decision-making which accords well with both of the observed deviations from rationality noted above.

Recently, Chew and MacCrimmon [1979a,b] proposed a weakening of the standard substitution axiom of utility theory, and developed a new model of decision-making which they named "alpha-utility theory." The development presented by them, as well as by Fishburn [1981], is primarily algebraic. We present here a development along geometric lines, which we feel contributes to the empirical issue of alpha-utility measurement. In addition, we present an interpretation of the alpha-utility function which leads to a natural conjecture concerning the shape of the alpha component of the utility function. Finally, we show that both the Allais and the CRS observations are consistent with this conjecture. In a final section of this paper, we discuss several of the broader implications of alpha-utility theory, to time-varying preferences and experimental design.

The Allais Paradox.

Actually, Allais presented several related phenomena which are inconsistent with rational behavior. We shall focus on the phenomenon referred to by Chew and MacCrimmon as the "Allais ratio paradox," and

shall discuss this phenomenon in the form it was presented by Kahneman and Tversky. Consider four lotteries:

A: (3000,1.00) or B: (4000,.80) ,
C: (3000,.25) or D: (4000,.20) .

(Each of the four pairs (M,p) represents a lottery which awards the amount M with probability p , and 0 otherwise.) The Kahneman and Tversky experiments were conducted in Israel, and the payoffs were in Israeli pounds; at the time, 3000 pounds was the median monthly net family income. Lottery A was chosen over lottery B by 80 percent of the experimental subjects, and D was chosen over C by 65 percent. This despite the fact that C and D, respectively, are formally equivalent to compound lotteries which yield A or B with probability 0.25, and 0 with probability 0.75.

An implication of expected utility maximization, often stated as the "substitution" or "independence" axiom, is the following: If one is indifferent between (lotteries) A and B, then for any $0 < p < 1$ and any C, one is indifferent between $pA + (1-p)C$ and $pB + (1-p)C$. (Equivalently, if A is preferred to B, then $pA + (1-p)C$ is preferred to $pB + (1-p)C$.) The experimental results reported by Kahneman and Tversky directly contradict this axiom. Therefore, an axiomatic theory of preference which seeks to explain the experimental results must involve a weakening of the substitution axiom.

Dutch and Sealed-Bid Auctions.

Many procedures exist for selling a single object at auction (see, for example, Milgrom and Weber [1982]). We focus here on two particular procedures. In a Dutch auction, a price clock is initially set at a very high level (above the value of the object to any of the bidders), and is then continuously decreased. At any moment, a bidder may claim the object; he is required to pay the amount registered by the clock at the instant he makes his claim. In a (first-price) sealed-bid auction, the bidders all submit sealed bids; the highest bidder obtains the object for the amount of his bid.

Consider a bidder participating in a Dutch auction. If he is rational (and if the auction is conducted in such a manner that he can observe only the price clock, and not the idiosyncratic actions of the other bidders during the course of the auction), then he will be content to let an agent act on his behalf in accordance with instructions of the form: "Claim the object at the price level b , if no other bidder has claimed it at a higher level." Consequently, his decision problem is equivalent to the choice of a value for b . If each bidder gives such instructions to his agent, then the bidder who chooses the largest value for b will obtain the object, and will be required to pay b . This is precisely the way a sealed-bid auction is resolved; hence, the two auction procedures are strategically equivalent. The assumption of rationality therefore leads to the prediction that an object will sell for the same price in both auctions.

CRS conducted a series of experiments to test this prediction, and found that there was a significant difference between the prices generated by these two auction procedures. Specifically, in every one of nine paired experimental sequences, they obtained lower average prices from Dutch auctions than from sealed-bid auctions.

What could explain this phenomenon? If we carefully examine the agent-based argument given above, there seems to be only one possibility: the price b at which a bidder would claim the object if he were directly involved in a Dutch auction is not the bid he would write down in the instructions to his agent (i.e., the bid he would submit in a sealed-bid auction). In choosing whether to claim the object when the price clock has dropped to b , or to wait until the clock drops to a lower level b^- , a bidder chooses between two lotteries:

$$A: (v-b, 1) \quad \text{or} \quad B: (v-b^-, l^-) ,$$

where v represents the value of the object to the bidder (this value was privately known by each bidder in the CRS experiments), and l^- represents the probability that the object will still be unclaimed at the lower price level. In choosing whether to submit a sealed bid of b , or the smaller amount b^- , a bidder chooses between two lotteries:

$$C: (v-b, p) \quad \text{or} \quad D: (v-b^-, p^-) ,$$

where p and $p^{\bar{}}$ represent, respectively, the probabilities that no other bidder bids above b or above $b^{\bar{}}$. If all other bidders are assumed to behave in the same fashion (i.e., associate the same bid with their valuation of the object) in both auctions, then $p^{\bar{}} = p \cdot l^{\bar{}}$. Therefore C and D , respectively, are compound lotteries which yield A or B with probability p , and 0 otherwise. The CRS experiments indicate that in some cases subjects tend to choose B over A , and C over D .

Note the contrast with the Allais paradox. In that case, subjects chose the lower, certain reward over a higher, uncertain reward. However, when the reward probabilities were scaled down (through compounding of the original lotteries with a payoff of 0), they chose the lottery offering the prospect of the higher reward. Here, subjects appear to choose the higher, uncertain reward over a lower, certain reward; when the reward probabilities are scaled down, they choose the lottery offering the prospect of the lower reward. It is in this sense that we consider the phenomenon observed by CRS to be polar to that observed by Allais.

The Weak Substitution Axiom.

In an attempt to account for behavior of the type discussed by Allais, Chew and MacCrimmon proposed the following weak substitution axiom: If one is indifferent between A and B , then for every p there exists a q such that, for all C , one is indifferent between $pA + (1-p)C$ and $qB + (1-q)C$. (An alternative version of this axiom,

called the "impartiality axiom" by Fishburn, is: If one is indifferent between A and B, if one is not indifferent between B and C, and if one is indifferent between $pA + (1-p)C$ and $qB + (1-q)C$, then for every D, one is indifferent between $pA + (1-p)D$ and $qB + (1-q)D$.) We shall not present a rationale for this axiom, and experimental investigation of its validity has not yet been carried out. Instead, we will examine its implications, noting for the present only that anyone who accepts the original substitution axiom cannot object to this one.

Let \succ be an asymmetric weak order on a convex space of lotteries, and assume that the order is open, i.e., has neither maximal nor minimal elements. Let \sim be the indifference (equivalence) relation induced by \succ . We say that \succ is continuous, if for every $A \sim B \succ C$, there is some $0 < p < 1$ such that $B \sim pA + (1-p)C$. We say that \succ is monotonic, if for every $A \succ B$ and $0 \leq p < q \leq 1$, $qA + (1-q)B \succ pA + (1-p)B$. The following theorem is due to Fishburn (an equivalent theorem was proved independently by this author, using the approach outlined below); it represents a strengthening of the original characterization theorem proved by Chew and MacCrimmon.

Theorem 1: An open, asymmetric weak order \succ on a convex space of lotteries is continuous, monotonic, and satisfies the weak substitution axiom, if and only if there are linear functionals w and α on the lottery space, with α strictly positive, such that

$$A \succ B \text{ if and only if } \frac{w(A)}{\alpha(A)} > \frac{w(B)}{\alpha(B)} .$$

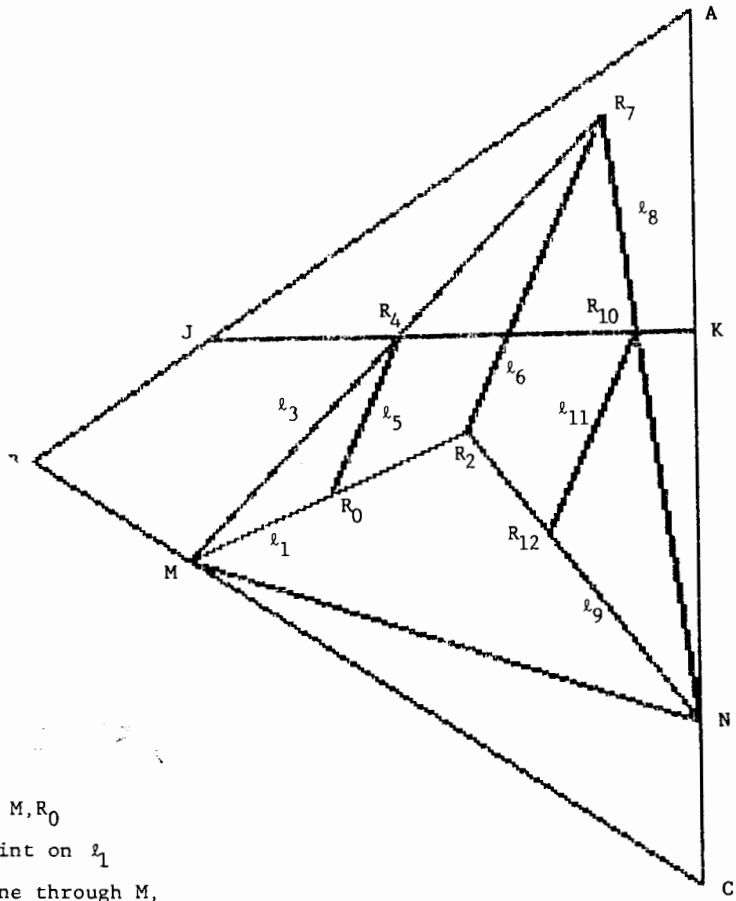
Proof: Rather than present a formal proof of the theorem, we will sketch a geometric derivation of the central step in the proof of the forward implication (the characterization result).

Let $A \succ B \succ C$ be any three lotteries, and consider the simplex of (compound) lotteries involving these three. The weak substitution axiom implies that for any lotteries $X \sim Y$ in the simplex, there is a substitution function $q: [0,1] \rightarrow [0,1]$ (depending on X and Y) such that for all p and Z , $pX + (1-p)Z \sim q(p)Y + (1-q(p))Z$.

If \succ satisfies the traditional substitution axiom, then it is well-known that the isopreference sets in the simplex are parallel lines. In the current setting, consider lotteries $X \sim Y$. For any $0 < p < 1$, $pX + (1-p)Y \sim q(p)Y + (1-q(p))Y = Y$. Consequently, the isopreference sets are convex; the continuity and monotonicity of \succ then imply that the isopreference sets are lines.

We next show that any two isopreference lines determine all the others. Consider the diagram in Figure 1. Assume that JK and MN are isopreference lines, and consider the lottery R_0 . For some p and q , $R_4 \sim pM + (1-p)R_7$ and $R_{10} \sim qN + (1-q)R_7$. Let $q(\cdot)$ be the substitution function for the (ordered pair of) lotteries M and N . Since $R_4 \sim R_{10}$, it must be that $q = q(p)$. But $R_0 \sim pM + (1-p)R_2$, and $R_{12} \sim qN + (1-q)R_2$. Consequently, $R_0 \sim R_{12}$. This final indifference determines the isopreference line through R_0 .

Finally, we show that the isopreference lines are either parallel, or all meet in the same point. This is the crucial step in the characterization, since it implies that the position of the lottery B



- Legend:
- l_1 . line through M, R_0
 - R_2 . arbitrary point on l_1
 - l_3 . arbitrary line through M ,
crossing JK
 - R_4 . intersection of JK, l_3
 - l_5 . line through R_0, R_4
 - l_6 . line through R_2 , parallel to l_5
 - R_7 . intersection of l_3, l_6
 - l_8 . line through R_7, N
 - l_9 . line through R_2, N
 - R_{10} . intersection of l_8, JK
 - l_{11} . line through R_{10} , parallel to l_5
 - R_{12} . intersection of l_9, l_{11}

Figure 1.

in the preference structure (relative to A and C) can be characterized by two parameters: the p such that $B \sim pA + (1-p)C$, and the barycentric B-coordinate of the point of concurrence of the isopreference lines. View the central portion of the diagram as the projection of a tetrahedron with vertices M, N, and R_7 in the ABC-plane, and vertex R_2 above the plane. The lines R_4R_{10} and MN are coplanar; so are the lines R_0R_{12} and MN. Finally, the lines R_4R_{10} and R_0R_{12} are coplanar, since R_0R_4 and $R_{12}R_{10}$ are parallel. Therefore, if any two of the lines intersect, the point of intersection must lie in all three planes. The remaining line is the line of intersection of two of the planes, and hence must also pass through the (unique) point of intersection of the three planes. The only other possibility is that none of the three lines intersect, in which case they must be parallel. Q.E.D.

From the geometric argument come two empirical predictions. If a decision-maker's preferences satisfy the continuity, monotonicity, and weak substitution axioms, his isopreference sets in any three-lottery simplex must be linear (as they certainly must be if he is an expected utility maximizer), and furthermore, they must be either parallel or coincident. Estimation of the α -component of the position of any lottery in his preference structure is equivalent to the estimation of the point of coincidence.

The following result (Fishburn [1981]) indicates the relationship between alternative representations of \succsim .

Theorem 2: If (w, α) and (w', α') both represent \succ , then there are constants $a, b, c,$ and d satisfying $ad > bc$, such that

$$w' = aw + b\alpha \quad ,$$

$$\alpha' = cw + d\alpha \quad .$$

Alpha-Utility Theory.

Chew and MacCrimmon formulate the characterization result of Theorem 1 slightly differently. For any lottery A , define $u(A) = w(A)/\alpha(A)$. Then $A \succ B$ if and only if $u(A) > u(B)$. This appears quite similar to the standard utility representation of preferences. However, each lottery A has two parameters associated with it: $u(A)$ and $\alpha(A)$. The second parameter (from which the theory takes its name) is used to evaluate compound lotteries. If $C = pA + (1-p)B$, then $u(C) = [p\alpha(A)u(A) + (1-p)\alpha(B)u(B)]/[p\alpha(A) + (1-p)\alpha(B)]$, and $\alpha(C) = p\alpha(A) + (1-p)\alpha(B)$. If we consider $u(\cdot)$ as the "utility" function and $\alpha(\cdot)$ as the "weighting" function, it is apparent that alpha-utility theory is a type of weighted utility theory. If u is unbounded, it follows from Theorem 2 that (u', α') will represent the same preferences if and only if u' is a positive affine translate of u (i.e., $u' = au + b$, with $a > 0$) and α' is a positive multiple of α . Classical expected utility theory corresponds to the special case in which the weighting function is constant.

What interpretation can we give to the weights? Consider Figure 2, and view u as a classical utility function. However, assume that the

decision-maker is skeptical about the actual outcome of the lottery $C = pA + (1-p)B$. The decision-maker believes that when Nature chooses A, he will actually receive A with probability $\alpha(A)$; he believes that Nature will, with probability $1-\alpha(A)$, return to the beginning of the lottery and choose again between A and B. Similarly, when Nature chooses B, he believes that he will actually only receive B with probability $\alpha(B)$; with probability $1-\alpha(B)$, Nature will renege and choose again. Then $(p\alpha(A)u(A) + (1-p)\alpha(B)u(B))/(p\alpha(A) + (1-p)\alpha(B))$ is the classical expected utility of the lottery C.

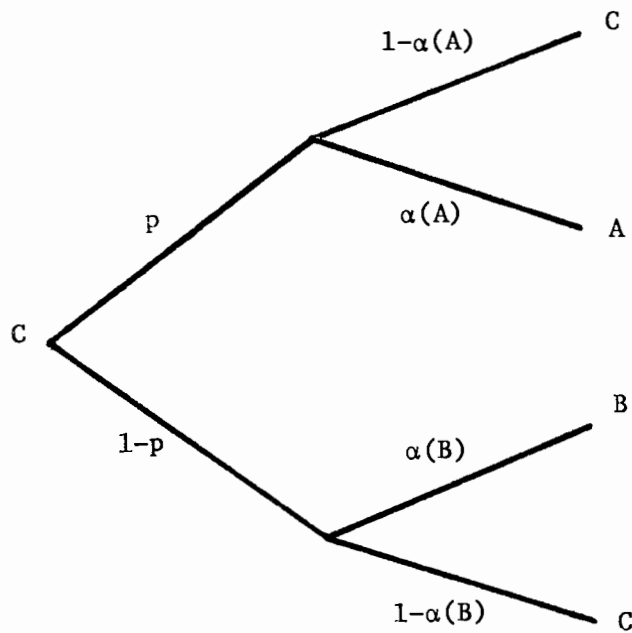


Figure 2.

This suggests that $\alpha(A)$ can be viewed as a measure of the "conceivability" of the outcome A. When we purchase a lottery ticket

which offers \$1 million as the prize, we commonly say to ourselves, "Really, I can't imagine myself winning." It being difficult to conceive of such a great change in our wealth, we subconsciously lower our perception of the likelihood of winning. This down-grading of perceived likelihood corresponds to the assignment of a lower α -weight to the outcome "\$1 million" than to the outcome "\$0" (no change in current wealth).

If we accept this interpretation, then we should expect the "typical" weighting function (when outcomes correspond to changes in wealth) to be maximal at 0, and decreasing as one moves away from zero in either direction. Since the weighting function must be positive, it is not unreasonable for us therefore to expect it to be bell-shaped (concave near the origin and convex further away, although not necessarily asymptotically zero). In the next sections, we explore the consequences of assuming that decision-makers are alpha-utility maximizers, and that their weighting functions are bell-shaped and maximal at 0. We will also assume that u is unbounded. Since the weighting functions which appear in the (α, u) -pairs representing a decision-maker's preferences are determined up to a positive scalar factor, the assumed properties of the weighting function will be invariant under the choice of representation.

The Allais Paradox Revisited.

Let $L < M < H$ denote three monetary outcomes, and consider the lotteries

$$\begin{array}{ll} A: (M, q; L, 1-q) & B: (H, p; L, 1-p) \\ C: (M, q; L, 1-q) & D: (H, pq; L, 1-pq) \end{array} .$$

We shall call preferences for A over B, and D over C, Allais-type preferences. We assume below that u is increasing, and without loss of generality, we further assume that u has been chosen so that $u(L) = 0$.

Theorem 3: The pair (α, u) corresponds to Allais-type preferences only if $\alpha(M)u(M) < p \cdot \alpha(H)u(H) < (p\alpha(H) + (1-p)\alpha(L))u(M)$. If these inequalities are satisfied, then Allais-type preferences will be exhibited for all sufficiently small q .

Proof: The two inequalities, $u(A) > u(B)$ and $u(C) < u(D)$, yield lower and upper bounds on $u(H)/u(M)$. The first inequality of the theorem, and the final statement, follow upon noting that the lower bound on $u(H)/u(M)$ is increasing in q , and equals $\alpha(M)/(p \cdot \alpha(H))$ when $q = 0$. The second inequality is direct. Q.E.D.

An immediate consequence of the two inequalities of Theorem 1 is that Allais-type preferences can only be observed when the points $(u(L), \alpha(L))$, $(u(M), \alpha(M))$, and $(u(H), \alpha(H))$ lie on the frontier of the graph of a convex function. If we further assume that u is concave on the interval (L, H) , this in turn implies that the points $(L, \alpha(L))$, $(M, \alpha(M))$, and $(H, \alpha(H))$ lie on the frontier of a convex function. We

expect this to be the case when $L = 0$ and M and H are relatively large, and indeed, the commonly-reported examples of the Allais paradox involve quantities M and H of substantial magnitude.

MacCrimmon and Larson [1979] present the results of a number of experiments investigating Allais-type preferences. One interesting feature of the experiments was that the scale of M and H was varied over several orders of magnitude. They observed Allais-type behavior most frequently when the scale was at its maximum level, less frequently when the scale was reduced, and they observed a number of reversals of the Allais phenomenon when the scale was at its minimum level. These results are fully consistent with our hypothesis that the typical weighting function is bell-shaped.

Dutch and Sealed-Bid Auctions Revisited.

Assume that n bidders compete in the auction of a single object. The object is subjectively valued by each bidder in monetary terms; the valuations X_1, \dots, X_n are independent, identically-distributed, nonnegative random variables, and the common distribution G has a density $G' = g$. (This model accords well with the CRS experiments.) Further assume that the utility component of the bidders' alpha-utility functions is linear (in money), and the weighting component α is the same for all bidders (up to scalar multiples). (In essence, we mean to study the behavior of risk-neutral bidders with nontrivial weighting functions.)

We shall determine the (unique) symmetric Nash equilibrium point of the Dutch and sealed-bid auction procedures. The equilibrium notion we use for the sealed-bid auction is a direct extension of the standard notion: We require that each bidder's bid be a best response, given the strategies of the others. The equilibrium notion we use for the Dutch auction is a bit more subtle: We assume that each bidder will claim the object as soon as his payoff from the claim is greater than the alpha-utility payoff he expects to receive from waiting to make a later claim, again given the strategies of the others.

Let b_D and b_S denote the symmetric equilibrium strategies in the Dutch and sealed-bid auctions.

Theorem 4: If α is decreasing and concave over the range of potential payoffs (i.e., over the range of $X_i - b_D(X_i)$), then $b_D < b_S$. Consequently, the sealed-bid auction will yield uniformly greater revenues than the Dutch auction.

Proof: The proof proceeds in several stages. First, we will establish necessary conditions for b_D and b_S to be symmetric equilibrium strategies. Then we will show that the solutions of the necessary conditions indeed are equilibrium strategies. Finally, we will establish the asserted ordering between b_D and b_S .

Consider first the Dutch auction. Assume that bidders 2 through n all adopt the increasing strategy $b(\cdot)$, and consider bidder 1's decision problem. Let $\pi(x;t,\tau)$ be his expected payoff (in alpha-

utility terms) when his valuation is x , the price level has descended to $b(t)$, and his plan is to claim the object when the price level further drops to $b(\tau)$. (Clearly, this function is only defined for $\tau < t$.) Then

$$\pi(x;t,\tau) = \frac{\alpha(x-b(\tau)) \cdot F(\tau) \cdot (x-b(\tau))}{\alpha(x-b(\tau)) \cdot F(\tau) + \alpha(0) \cdot (F(t) - F(\tau))} ,$$

where $F = G^{n-1}$ is the distribution of the highest opposing valuation, and $f = F'$.

We call b a symmetric equilibrium strategy (for the Dutch auction) if it has the following two properties: (a) When $t > x$, $\pi(x;t,\tau)$ is not maximal at $\tau = t$, and (b) $\pi(x;x,\tau)$ is maximal at $\tau = x$. In essence, (a) asserts that at all price levels greater than $b(x)$, the bidder anticipates greater expected reward from waiting than from immediately claiming the object, and (b) asserts that at the price level $b(x)$, the bidder considers any wait to be no more attractive than an immediate claim.

In any situation corresponding to a particular valuation x and current price level $b(t)$, τ is bidder 1's decision variable. The derivative of π with respect to τ is

$$\begin{aligned} \frac{\partial \pi}{\partial \tau} (x;t,\tau) &= [\alpha(x-b(\tau)) \cdot \alpha(0) \cdot f(\tau) \cdot F(t) \cdot (x-b(\tau)) - b'(\tau) \cdot (\alpha(x-b(\tau)))^2 \cdot (F(\tau))^2 \\ &\quad - b'(\tau) \cdot \alpha'(x-b(\tau)) \cdot \alpha(0) \cdot (x-b(\tau)) \cdot F(\tau) \cdot (F(t)-F(\tau)) \\ &\quad - b'(\tau) \cdot \alpha(x-b(\tau)) \cdot \alpha(0) \cdot F(\tau) \cdot (F(t)-F(\tau))] \\ &\quad / [\alpha(x-b(\tau)) \cdot F(\tau) + \alpha(0) \cdot (F(t)-F(\tau))]^2 , \end{aligned}$$

The associated first-order condition at equilibrium is $\frac{\partial \pi}{\partial \tau} \Big|_{t=x, \tau=x} = 0$,
or

$$(D) \quad (x-b(x)) \cdot \frac{f(x)}{F(x)} = b'(x) \cdot \frac{\alpha(x-b(x))}{\alpha(0)} .$$

A boundary condition which must be satisfied at equilibrium is that $b(\underline{x}) = \underline{x}$, where \underline{x} is the lowest valuation in the support of the distribution G (i.e., at equilibrium a bidder with the lowest possible valuation must have an expected payoff, conditional on winning, of zero.)

We will require the following lemma on several occasions.

Lemma 1: Let g and h be differentiable functions for which (i) $g(\underline{x}) \geq h(\underline{x})$ and (ii) $g(x) < h(x)$ implies $g'(x) \geq h'(x)$. Then $g(x) \geq h(x)$ for all $x \geq \underline{x}$.

Proof: If $g(x) < h(x)$ for some $x > \underline{x}$ then, by the Mean Value Theorem, there is some \hat{x} in (\underline{x}, x) such that $g(\hat{x}) < h(\hat{x})$ and $g'(\hat{x}) < h'(\hat{x})$. This contradicts (ii). Q.E.D.

Let b satisfy the differential equation (D) and the associated boundary condition. A direct application of Lemma 1 (with $g(x) = x$ and $h(x) = b(x)$) shows that b is increasing. Consider $\frac{\partial \pi}{\partial \tau} \Big|_{\tau=t}$, for $t > x$. Then

$$\frac{\partial \pi}{\partial \tau} \Big|_{\tau=t} = \alpha(x-b(t)) \cdot \alpha(0) \cdot (F(t))^2 \cdot \left[(x-b(t)) \cdot \frac{f(t)}{F(t)} - b'(t) \cdot \frac{\alpha(x-b(t))}{\alpha(0)} \right] / [\alpha(x-b(t)) \cdot F(t)]^2 .$$

For $x = t$, the bracketed expression is zero. (This follows from (D).) When x is decreased below t , the first of the bracketed terms decreases and the second increases (since α is decreasing). Therefore the expression is negative, and (a) is satisfied.

In order to verify that (b) is satisfied, we require another lemma.

Lemma 2: Assume that α is positive, decreasing, and concave on $(0, Z)$. Define

$$A(z) = 1 - \frac{\alpha(z)}{\alpha(0)} + z \cdot \frac{\alpha'(z)}{\alpha(z)} .$$

Then for all $0 < z < Z$, $A(z) < 0$.

Proof: Clearly, $A(0) = 0$. Furthermore,

$A'(z) = \alpha'(z) \cdot \left(\frac{1}{\alpha(z)} - \frac{1}{\alpha(0)} \right) + z \cdot \frac{\alpha''(z)}{\alpha(z)} - z \cdot \left(\frac{\alpha'(z)}{\alpha(z)} \right)^2$. The assumptions of the lemma imply that $A'(z) < 0$ for all z ; consequently,

$A(z) < 0$.

Q.E.D.

After a bit of algebraic manipulation it can be shown that

$$\begin{aligned} \frac{\partial \pi}{\partial \tau} \Big|_{t=x} = & \alpha(x-b(\tau)) \cdot \alpha(0) \cdot F(\tau) \cdot \left[F(x) \cdot \left((x-b(\tau)) \cdot \frac{f(\tau)}{F(\tau)} - b'(\tau) \cdot \frac{\alpha(x-b(\tau))}{\alpha(0)} \right) \right. \\ & \left. - b'(\tau) \cdot (F(x) - F(\tau)) \cdot A(x-b(\tau)) \right] / D^2 , \end{aligned}$$

where A is defined as in Lemma 2, and D is a nonzero expression.

For $x > \tau$, it follows from the first-order condition that the first of the expressions within the brackets is positive; from Lemma 2, it follows that the second expression is negative. Therefore, the entire bracketed expression is positive, and (b) is satisfied.

Consider next the sealed-bid auction, and let $\pi(x;t)$ be the expected payoff to bidder 1 if the other bidders follow the increasing strategy b , if bidder 1's valuation is x , and he submits a bid of $b(t)$. Then

$$\pi(x;t) = \frac{\alpha(x-b(t)) \cdot F(t) \cdot (x-b(t))}{\alpha(x-b(t)) \cdot F(t) + \alpha(0) \cdot (1-F(t))} .$$

If b is to be a symmetric equilibrium strategy (i.e., if for every valuation x , $b(x)$ is to be a best response for bidder 1 when the other bidders use the strategy b), then b must satisfy the first-order condition $\frac{\partial \pi}{\partial t} \Big|_{t=x} = 0$, or equivalently,

$$(S) \quad (x-b(x)) \cdot \frac{f(x)}{F(x)} = b'(x) \cdot \frac{\alpha(x-b(x))}{\alpha(0)} + b'(x) \cdot (1-F(x)) \cdot A(x-b(x)) \quad ,$$

and the boundary condition $b(\underline{x}) = \underline{x}$. Again, it can be shown that the solution of these conditions is indeed a symmetric equilibrium strategy.

Finally, let b_D and b_S , respectively, denote the symmetric equilibrium strategies in the Dutch and sealed-bid auctions.

Subtracting the first-order condition for b_S from the first-order condition for b_D yields

$$(b_S(x) - b_D(x)) \cdot \frac{f(x)}{F(x)} = [b'_D(x) \cdot \alpha(x - b_D(x)) - b'_S(x) \cdot \alpha(x - b_S(x))] / \alpha(0) \\ - b'_S(x) \cdot (1 - F(x)) \cdot A(x - b_S(x)) \quad .$$

From Lemma 2, $A(x - b_S(x)) \leq 0$; therefore, if $b_D(x) > b_S(x)$ for any x , it must be that $b'_D(x) \cdot \alpha(x - b_D(x)) < b'_S(x) \cdot \alpha(x - b_S(x))$. But since α is decreasing, it must also be that $\alpha(x - b_D(x)) > \alpha(x - b_S(x))$. Therefore, it follows that $b'_D(x) < b'_S(x)$. Since $b_D(\underline{x}) = b_S(\underline{x})$, it follows from Lemma 1 that for all $x > \underline{x}$, $b_D(x) < b_S(x)$. Q.E.D.

The CRS experiments involved relatively small potential gains for the bidders, with the difference between the winning bid and the winner's valuation being less than \$100 in all cases. We believe the assumptions of the theorem to be valid for amounts in this range. Consequently, we believe that the theorem offers an explanation for the phenomenon observed by CRS.

The theoretical situation is not as clear when the difference between bids and valuations can be so large as to lie in the range beyond the point where α becomes convex. However, our intuition (backed up by thought experiments discussed with colleagues) is that the CRS effect might in fact be reversed - that Dutch auctions may draw higher bids than first-price auctions. It appears as if this would be predicted (for appropriately-shaped α -functions and concave u -functions) from the theory laid out above.

The Dynamics of the Weighting Function.

If we interpret $\alpha(x)$ as the "conceivability" of the monetary outcome x , then we would expect an individual's weighting function α to change subsequent to a permanent shift in his total wealth. This raises several interesting issues.

Many authors, starting with Friedman and Savage [1948] and Markowitz [1952], have attempted to explain the variety of risk-seeking and risk-avoiding behaviors typically associated with a single individual, by assuming that the individual's (classical) utility function has several undulations located near the origin (i.e., near his current wealth level). In order for this explanation to be accepted, one must believe that a permanent change in wealth leads to a change in the utility curve. A number of the phenomena these authors have attempted to capture can be given an alternative explanation in the context of alpha-utility theory, by assuming that the utility component of the individual's preferences does not vary, but that the shape of the weighting function changes in response to changes in wealth.

If the weighting function truly measures conceivability, we might expect to find that individuals who have in the past experienced marked changes in their wealth will have relatively constant weighting functions over a wide range of monetary outcomes, and will exhibit behavior more consistent with classical utility theory.

The "prospect theory" of Kahneman and Tversky [1979] involves an editing phase in which lotteries are segregated into riskless and risky components and a reference point (from which to view changes in wealth) is chosen. This editing could well correspond to a shift in an individual's weighting function which occurs (rapidly) between the presentation of a lottery and his assessment of it. Their theory also involves a processing phase in which "decision weights," rather than original probabilities, are used to combine the individual's preferences over the various possible outcomes of a lottery. A major difference between their theory and alpha-utility theory is that their decision weights depend on the original probabilities, but not on the outcomes themselves, while alpha-weights depend directly on the outcomes of the lottery, and only subsequently act as weights which (subjectively) modify the lottery probabilities. It appears that a theory in which the weights depend only on the lottery probabilities cannot simultaneously account for both the Allais and CRS phenomena.

An incidental implication of alpha-utility theory warrants comment. A popular method of conducting decision- or game-theoretic experiments with a limited experimental budget is to tell the subjects that a small number of the trials in which they are involved will be selected at random (at the conclusion of the experiment), and that the

payoffs accruing from only those experiments will actually be made. If the subjects are expected-utility maximizers, this experimental procedure will elicit the same behavior as would be elicited if all trials were to yield actual payoffs. However, if the subjects are alpha-utility maximizers, with nontrivial weighting functions, this limited-payoff procedure may elicit behavior somewhat different from that which a full-payoff procedure would yield.

Summary.

In this paper, we have proposed an explanation for several phenomena which contradict the expected-utility-maximization model of "rational" behavior, and which appear at first glance to lie in opposition to one another. We have presented a geometric perspective on alpha-utility theory, and an interpretation of the weighting function as an index of conceivability. We have shown how the game-theoretic notion of equilibrium can be generalized to subsist in the more-general-than-usual framework of alpha-utility preferences.

Finally, we have shown that an acceptance of alpha-utility theory (and our interpretation of it) leads to a number of empirical predictions - that isopreference curves (in lottery simplices) are linear and coincident, that the weighting function of an individual is typically bell-shaped, and maximal at his current wealth level, that the Allais effect will weaken, and finally reverse, as the payoff scale is decreased, and that the CRS (Dutch auction) effect may reverse as potential payoffs are increased. Clearly, these predictions must be subjected to further experimental investigation. That the theory

presented here will fully describe the behavior of all individuals, or even the behavior in all instances of any single individual, is too much to hope for. Still, we offer this theory as an accounting of what may be an important component of the overall individual decision-making process.

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