

Discussion Paper No. 531
ON THE OPTIMALITY OF THE DEMAND REVEALING
MECHANISM IN LARGE ECONOMIES*

by

Ramon Marimon

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*The author acknowledges John O. Ledyard for helpful comments, but claims full responsibility for any errors. Grant #SES-8106896 from the National Science Foundation supported this research.

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ABSTRACT

This paper studies the optimality of the Vickery-Clarke-Groves mechanism (DRM) when the number of consumers in the economy tends to infinity. The impossibility results for an incentive compatible and efficient mechanism are well known for finite economies. It is also known that the price mechanism is efficient and limiting incentive compatible. For public goods economies, however, there is an apparent contradiction between the general results showing that problems of incentive compatibility tend to worsen as the number of consumers increases, and the conjecture that in the V-C-G mechanism Clarke taxes go to zero.

We study only the class of economies in which the DRM is incentive compatible. For a finite economy this mechanism, if compatible, is not efficient since it generates a surplus and may lead to bankruptcy of some consumers. It is shown here that, under very weak assumptions, if the economy is replicated and if the allocation of public goods is convergent (either to finite or infinite bundles), then the mechanism is also limiting efficient in the sense that Clarke taxes and the total surplus per capita go to zero. This result is proved for economies with n public goods and where there is a crowding effect.

INTRODUCTION

There seems to be some confusion regarding the optimality of the demand revealing mechanism when the number of agents in the economy tends to infinity. The impossibility results for an incentive compatible and efficient mechanism are well known for finite economies (Hurwicz (1975), Green-Kohlberg-Laffont (1976), Walker (1980)), either for private goods or public goods with quasi-linear profiles. Roberts and Postlewaite (1976) have proved that in a private goods environment this impossibility result does not hold in the limit of a sequence of expanding economies, that is, the price mechanism is limiting incentive compatible and efficient. Roberts (1976) has also shown that this limit result is no longer true in the allocation of public goods if the implicit tax does not go to zero. On the other hand Tiedeman-Tullok (1977) conjecture that Clarke taxes go to zero in a replicate economy. The proof of this conjecture would support the applicability of the DRM in large economies as long as the non-income effect assumption is accepted.

In general, for this study of the existence of a limiting incentive compatible and efficient mechanism, we proceed as follows. First, a sequence of economies is defined in which the differences in the equilibrium allocations are due to differences in size. In order to obtain such a sequence, a given economy is replicated introducing one new consumer of each type at each step of the sequence. Second, which properties are common to all economies in the sequence and which are obtained as an asymptotic result are chosen with a degree of freedom. In addition to feasibility, there seems to be a consensus in rating efficiency and incentive compatibility as the most desirable properties in the design of resource allocation mechanisms. One approach, followed by Roberts (1976), is to consider sequences of efficient

economies and to study the incentive problem in the limit. While this approach is appropriate when studying private ownership economies, it is not the most fruitful when analyzing public goods environments.

An alternative approach is to study the class of economies in which the DRM is incentive compatible. For a finite economy, this mechanism, if feasible, is not efficient since it generates surplus and may lead to bankruptcy of some consumers. It is shown here that, for an expanding sequence of economies of this class, in which the allocation of public goods is convergent, the mechanism is also limiting efficient in the sense that Clarke taxes and the total surplus per capita go to zero. This result is proved for economies with n public goods and $m+1$ private goods where the cost of producing a public good may be dependent on the number of people in the economy. It follows that for the usual case of one pure public good (service) this holds for any economy in which the DRM is incentive compatible. (For n public goods it is required only that the hessian matrix of the aggregate utility function have a diagonal with negative elements for all n public goods.) In this sense the result applies to a "large subset" of the class of economies in which the DRM is IC and the sequence of allocations of public goods is convergent. Our result depends on the replication procedure. If we expand the economy in a predetermined manner (for example, by not increasing the number of consumers of one particular type) it is possible to construct examples where our assumptions are satisfied but the Clarke taxes do not converge to zero.¹

Finally, the generalization of the DRM to n public goods is an

¹ W. Thomson called my attention in this point. In Thomson (1979 (Appendix)) there are some counterexamples of this type. On the other hand it is possible to show that for any economy with a continuum of consumers there is in some sense an equivalent economy that can be approximated with a sequence of replicated economies. We will not develop this argument further in this paper.

additional support for its implementation since, as will be shown, the assumptions in the DRM usually imply strict concavity of the utility functions, which in turn implies single-peakedness when there is only one public good. In this context Groves and Ledyard (1977) have shown that the Majority Rule mechanism is not only incentive compatible but may also be pareto superior to the DRM in a finite economy. The equilibrium conditions for MR in a n-dimensional space (Kramer (1976)), however, are far more restrictive than the assumptions for the DRM.⁽²⁾

The model for the economy E_m

A sequence of economies E_m , whose commodity space is R^{N+M+1} is considered, where the first N coordinates correspond to public goods, the following M to private goods and the last 'one' to a private good that may be viewed as money-income. In each economy E_m there are J types of consumers and m consumers of each type. The preferences of each consumer are defined over the non-negative orthant of R^{N+M+1} and are quasi-linear with respect to the last coordinate (no income effects). The endowment vectors are of the form $(0, w_i)$ where $w_i (>0)$ in R^{M+1} is the endowment of one consumer of type i in E_m . An allocation for such an economy is represented by a vector $((x_{1q}, \dots, x_{Jq})_{q=1, \dots, m}, y)$ in $R^{m \cdot J(M+1) + N}$.

² The reader not familiar with this literature can find more detailed definitions and explanations of the concepts used in this paper in Groves (1979), Groves and Ledyard (1977), or Roberts (1976).

We assume:

If i is of type h ($h=1, \dots, J$), then

i) $U^i(y, z, x) = V^h(y, z^i) + x^i$ with $w^i = w^h$, x^i is in R_+ and z^i is in R^M .

ii) For all i , $V^i: R^{N+M}$ to R , is twice differentiable and concave.

It is usually assumed that the cost of producing a pure public good is independent of the number of people to whom it is provided. Since we are considering economies of radically different sizes, however, for some components of the public good vector the cost of production may also be a function of the size of the economy. Furthermore, it will be possible to consider as particular cases the pure public goods in which the cost is independent of the population and the pure public service in which the cost is proportional to the number of people in the economy.

iii) For all m , $C_m: R^N \rightarrow R_+$ is nonnegative, twice differentiable and convex, where $C_m(y) \equiv C(y, m \cdot J)$.

Although both private and public goods may be taken into consideration, this paper is concerned only with a mechanism for the allocation of public goods. It is assumed with respect to the private goods that the price mechanism defines the final allocation, and the usual conditions for the existence of equilibrium exist. Private goods are introduced in order to define the cost function. In fact, in comparing economies of different size all the differences in the final allocation and prices are due to the presence

of public goods. In a strict sense this analysis is only partial equilibrium.

The mechanism for the public sector in the economy E_m is characterized by

The message space $M_m = \{ \Psi' : R^N \rightarrow R^N \}$, where Ψ' is the revealed V' and the Ψ' function is in C^1 .

The outcome rule $\Phi_m^i(\Psi') = [y_m^*, w_m^i - t_m^{*i}]$.

In other words, consumers report their marginal valuation functions and the center then defines the final outcome. In order to characterize the outcome rule it is convenient to introduce some notation. Furthermore, as it will be shown, there is no loss of generality in assuming that consumers of the same type send the same message.

Let $H_m : R^N \rightarrow R$ and $G_{i,m} : R^N \rightarrow R$, be defined by :

$$H_m(y) = C(y) - \sum_{i=1}^{m \cdot J} \Psi^i(y) ; G_{i,m}(y) = H_m(y) + \Psi^i$$

and $h_m^k(y) = \frac{\partial C}{\partial y^k}(y) - m \cdot \frac{\partial \psi}{\partial y^k}(y)$, where $\psi = \sum_{j=1}^J \Psi^j$

$$g_{i,m}^k(y) = h_m^k(y) + \frac{\partial \Psi^i}{\partial y^k}$$

$$H'_m(y) = [h_m^1(y), \dots, h_m^N(y)] \quad \text{and} \quad G'_{i,m}(y) = [g_{i,m}^1(y), \dots, g_{i,m}^N(y)]$$

The H_m function defines the social cost of implementing the bundle y .

Correspondingly, the h^k 's functions are the social marginal evaluation of the k th component of the public bundle and the g_i^k 's functions the social marginal evaluation excluding consumer i .

Then we have:

$$(1) \quad y_m^* = \underset{y \geq 0}{\operatorname{argmin}} H_m(y)$$

$$(2) \quad t_{i,m}^*(y) = \int_{y_{i,m}}^y \sum_{k=1}^N g_{i,m}^k(s) ds^k = \sum_{k=1}^N \int_{y_{i,m}}^{y^k} g_{i,m}^k(s) ds^k$$

where

$$(3) \quad \bar{y}_{i,m} = \max\{0, \tilde{y}_{i,m}\} \quad \text{and} \quad G'_{i,m}(\tilde{y}_{i,m}) = 0$$

The outcome rule is completely characterized by (1) to (3). We will suppose for a moment that these equations are well defined. Note that (2) is the natural generalization of the Clarke taxes when y is a vector. The line integral is well defined since $G_{i,m}$ is a C^2 function and since there are no income effects. It is easy to see that the mechanism is individually incentive compatible in E_m . A consumer of type i will choose to report the message that satisfies:

$$(4) \quad \tilde{y} = \underset{y}{\operatorname{argmax}} V^i(y) - t_{i,m}^*(y) . \quad \text{The first order conditions (which are sufficient) are:}$$

$$\frac{\partial V^i}{\partial y^k}(y) - g_{i,m}^k(y) = \frac{\partial V^i}{\partial y^k}(y) - h_m^k(y) - \frac{\partial \Psi^i}{\partial y^k} = 0, \quad k=1, \dots, N$$

which are the FOC in (1). The optimal strategy for consumer i will be to report $\Psi' = V'$, obtaining $\tilde{y} = y_m^*$.

It should be noted that the Clarke taxes define a pivot mechanism since only the pivotal consumers, whose own demand affects the final allocation either by increasing or decreasing the amount demanded by all the others, are positively taxed, and that there are no transfers.

In order to have a well-defined mechanism, however, the limits of integration have to be uniquely defined, which means that the solution to (3) and (4) (and correspondingly (1)), must be unique, which in turn means that the outcome rule must be single valued. In order to have this result our assumptions must be strengthened:

- iv) For all m , for all i , the demand functions for public goods are downward sloping, i.e., $\frac{\partial^2 v}{\partial y^k} < 0$, $k=1, \dots, N$.

Given iii), this implies that the functions $g_{i,m}$, h_m are strictly increasing for all m .

Note that this is a stronger assumption than the strict convexity of the H_m and $G_{i,m}$ functions, which is a sufficient condition for single valuedness of the outcome rule. The assumption that all evaluation functions are one-to-one is required in order to have a unique relation between social evaluations and allocations.

Given these assumptions, for each economy E_m , there exists a unique equilibrium allocation of public goods, the DRM is individually incentive compatible, and the Samuelson-Lindhal conditions are satisfied (from (1)).

There are two more properties to be fulfilled in this mechanism:

) A balanced budget, i.e., $m \cdot \sum_{j=1}^J t_{j,m}^(y_m^*) = C(y_m^*, m \cdot J)$

***) Utility-respectfulness, i.e.,

$U^i(y_m^*, x_m^{*i}) > U^i(0, w^i)$, where x_m^{*i} is the equilibrium allocation of private goods for a consumer of type i in E_m .

A mechanism that satisfies property *) and that fulfills the Samuelson-Lindhal conditions is said to be Nonwasteful. It is easy to see that, in general, properties *) and ***) do not hold for finite economies.

The sequence of expanding economies

Now we study a sequence of expanding economies in equilibrium. In this section we will study the asymptotic behavior of the Clarke taxes for sequences of economies that define a convergent sequence of optimal allocations (convergent possibly to infinity). In the next section we will study this convergence property. Two more assumptions are required in order to obtain our results:

$$v) \frac{\partial C_m}{\partial y^k}(0) < +\infty, \frac{\partial V}{\partial y^k}(0) > 0, \text{ for all } \epsilon > 0, \frac{\partial V}{\partial y^k}(y+\epsilon) < +\infty$$

$$\text{and } \lim_{y \rightarrow \infty} \frac{\partial V^i}{\partial y^k}(y) < 0 \text{ for all } i, \text{ all } m, \text{ and all } k.$$

vi) For all y , $C(m,y)$ is a nondecreasing, concave function in m .

Furthermore, the function $\frac{\partial C_{m+1}}{\partial y^k}(\cdot) - \frac{\partial C_m}{\partial y^k}(\cdot)$ is nondecreasing.

Assumption v) is introduced for mathematical convenience but does not represent any important restriction on the class of preferences under consideration. In particular, the last part of assumption v) says that no consumer will demand an infinite amount of public good if the price is strictly positive.

The first part of assumption vi) is a monotonicity assumption which restricts the possible crowding effects under consideration. In lemma 2 it is proved that under this assumption the sequence of optimal allocations of public goods is nondecreasing. Except for concavity, assumption vi) is a necessary condition for our result in the sense that if one of the conditions is not satisfied we can find an economy in which all the other assumptions are satisfied but Clarke taxes do not converge to zero.³ Concavity in m is not necessary and one can show that with strict convexity the optimal allocation of public goods converges to zero and our result is satisfied in a trivial way. We could also consider functions (lm, y) which are neither globally concave nor globally convex and we would still have the same result under the assumption that the allocation of optimal public goods is convergent (perhaps to infinity). The concavity assumption is satisfied for the two most interesting cases (pure public goods and pure public services) and we will maintain it in order to avoid unnecessary complications. The second part of assumption vi) is the extension of our uniqueness condition for E_m to the sequence of expanding economies. The results that we obtain are:

³I am grateful to T. Grove for a counterexample to my result when the nondecreasingness assumption is violated. The example is very simple:

$$C'_m(y) = \frac{1}{m}; \quad V'_i(y) = \frac{\beta_i}{y} \quad \text{and} \quad \sum_{j=1}^J \beta_j = 1, \quad \text{then} \quad y_m^* = m^2 \quad \text{but} \quad y_m^* - \bar{y}_{i,m} = \\ = m \cdot \beta_i \rightarrow \infty \quad \text{as} \quad m \rightarrow \infty.$$

Lemma 1 If the assumptions i) to vi) are satisfied, then the following properties are true:

1.1) If $y_m^* \rightarrow 0$, as $m \rightarrow \infty$, then for all i ($i=1, \dots, J$) $\bar{y}_{i,m} \rightarrow 0$.

1.2) If $y_m^* \rightarrow y^*$, as $m \rightarrow \infty$, and $y^* > 0$, then there exists $\epsilon > 0$ and m_0 such that for all i , for all $m \geq m_0$: $\bar{y}_{i,m} > \epsilon$.

Lemma 1 says that even if pivotal consumers change the final allocation they cannot change finite allocations for infinitesimal ones (and vice versa) for large enough economies. This gives the technical support for proving:

Lemma 2 If assumption vi) is satisfied, then $\{y_m^*\}$ is monotone nondecreasing sequence. Under assumptions i) to vi) $\{\bar{y}_{i,m}\}$ is a nondecreasing sequence, for all i . Furthermore, $\bar{y}_{i,m} \rightarrow y^*$ provided that

$$\lim_{y \rightarrow \infty} \frac{\partial V^i}{\partial y^k}(y) < 0 \text{ for all } i.$$

Once Lemma 2 has been proved then our main result is straightforward:

Proposition 1: If assumptions i) to vi) are satisfied, then for any type of consumers, the Clarke taxes converge to zero (i.e., $t_{i,m}^* \rightarrow 0$).

We have defined a set of sufficient conditions in which Clarke taxes go to zero as the number of consumers tends to infinity. In order to have

feasibility when the cost of producing the public goods vector is positive, however, a fixed tax share must be added. It is said that a set of taxes is weakly feasible if the aggregate tax covers the cost of producing the public good and strongly feasible if every consumer can satisfy his budget constraint (i.e., if they can avoid bankruptcy). A set of taxes defined in a sequence of expanding economies is said to be limiting weakly (strongly) feasible if for large enough economies it is weakly (strongly) feasible.

Let $\alpha_{i,m} = \frac{1}{m} \cdot \tilde{\alpha}_i$, where $\tilde{\alpha}_i > 0$ and $\sum_{i=1}^J \tilde{\alpha}_i = 1$. Then $\lim_m \alpha_{i,m} = 0$

We define:

a) $t_{i,m}^*$ the set of taxes defined in (2)

b) $\tilde{t}_{i,m}(y) = t_{i,m}^*(y) + \alpha_{i,m} \cdot C_m(\bar{y}_{i,m})$

The set of taxes a) is not strongly feasible (for positive cost) or weakly feasible and it is limiting strongly feasible but not limiting weakly feasible (if $C(y^*) > 0$, where $C = \lim C_m$).

The set of taxes b) is not utility respecting since one type of consumer can be indifferent to the public good. It does not guarantee weak or strong feasibility for finite economies. It is limiting - weakly and strongly - feasible, and thus limiting utility respecting, if the assumptions of our previous proposition are satisfied. This follows from the fact that:

$$\begin{aligned} \lim_m \sum_{i=1}^{m \cdot J} \tilde{t}_{i,m}(y_m^*) &= \lim_m m \cdot \sum_{i=1}^J t_{i,m}^*(y_{i,m}^*) + \lim_m \frac{m}{m} \cdot \sum_{i=1}^J \tilde{\alpha}_i \cdot C_m(\bar{y}_{i,m}) = \\ \lim_m m \cdot \sum_{i=1}^J t_{i,m}^* + \lim_m C_m(\bar{y}_{i,m}) &> C_\infty(y^*) \quad . \end{aligned}$$

Note that the aggregate Clarke taxes does not necessarily converge to

zero even if each individual tax converges to zero. If we consider, however, the total surplus generated per capita then this converges to zero:

$$\begin{aligned} \lim_m \frac{1}{m} \cdot \left(\sum_{i=1}^{m \cdot J} \tilde{t}_{i,m}(y_m^*) - C_m(y_m^*) \right) &= \lim_m \frac{m}{m} \cdot \sum_{i=1}^J t_{i,m}^* + \\ + \lim_m \frac{m}{m} \cdot \sum_{i=1}^J \tilde{\alpha}_i \cdot C_m(\bar{y}_m) - \lim_m \frac{1}{m} \cdot C_m(y_m^*) &= 0 \text{ if } C_\infty(y^*) < 0. \end{aligned}$$

In any case, $\lim_m \frac{m}{m} \cdot \sum_{i=1}^J t_{i,m}^* = 0$

The following result has been proved:

Theorem 1: Let (E_m) be an expanding sequence of public goods economies such that assumptions i) to vi) are satisfied. If there is a system of Clarke taxes with fixed tax shares (type b) and where the sequence of equilibrium allocation of public goods converges then the Vickery-Clark-Groves mechanism is individually incentive compatible, limiting utility-respecting and limiting strongly feasible. Furthermore, the aggregate Clarke tax per capita goes to zero.

Proofs:

In order not to be overwhelmed by notation we will prove our results for the one single good case. From the definition of the model it is clear that the extension to the N public goods is straightforward.

One fact that will be used extensively is :

$h_m(y) = C'_m(y) - m \cdot \psi'(y)$; $g_{i,m}(y) = h_m(y) + \Psi'_i(y)$, then we have

$$g_{i,m+1}(y) - g_{i,m}(y) = h_{m+1}(y) - h_m(y) = C'_{m+1}(y) - C'_m(y) - \psi'(y)$$

Notation: $\Delta C'_m(\cdot) \equiv C'_{m+1}(\cdot) - C'_m(\cdot)$ and $\Delta_n C'_m(\cdot) \equiv C'_{m+n}(\cdot) - C'_m(\cdot)$.

Lemma 1

1.1: The sequence $\{\bar{y}_{i,m}^*\}$ is bounded below by zero. We have to prove that if $y^* \rightarrow 0$, then $\limsup \bar{y}_{i,m} = 0$. Suppose that for some i , $\limsup \bar{y}_{i,m} > 0$. By continuity of the h functions, for any $\epsilon > 0$ there exists a n_m such that for all $n \geq n_m$

$h_m(y_n^*) - h_m(0) < \epsilon$. Let $\epsilon = \Psi'_i(0)/K$ for some positive constant k . Since $\lim_m h_m(0) = 0$ by assumption, there is some m large enough for which $m \geq n_m$, which leads us to a contradiction :

$$\epsilon > h_m(y_m^*) - h_m(0) = h_m(0) = g_{i,m}(0) - \Psi'_i(0) > \Psi'_i(0) = \epsilon \cdot K$$

The last inequality follows from the fact that $g_{i,m}(0) \geq 0$ if and only if $\bar{y}_{i,m} = 0$ (assumption iv and (3)). Note that for corner solutions the first equality is not true, but in this case $g_{i,m}(0) > 0$.

1.2 Let $y_m^* \rightarrow y^*$, $y^* > 0$, then we have to prove that $\liminf \bar{y}_{i,m} > \epsilon$ for some $\epsilon > 0$. We consider the three possible cases:

i) Corner solutions. $\bar{y}_{i,m} = 0$ can not be a corner solution for m large enough. This follows from assumption v) and the definition

$$\text{of } g \text{ , i.e., } g_{i,m}(0) = C'_m(0) - m \cdot \sum_{j \neq i}^J \Psi'_j(0) - (m-1) \cdot \Psi'_i(0) .$$

ii) Let $y_m^* \rightarrow +\infty$ and suppose that for some i $\liminf y < \epsilon$, for a given ϵ . This means that for any $M > 0$ there exists a m such that $g_{i,m}(y_m^*) > M$ but $g_{i,m}(y_m^*) = 0 + \Psi'_i(y_m^*)$ and by assumption

for any $\epsilon > 0$, $\Psi'_i(y+\epsilon)$ is bounded. A contradiction.

iii) Let $y_m^* \rightarrow y^*$, $0 < y^* < +\infty$, and given $\epsilon > 0$, $\epsilon < y^*/3$, suppose that for some i , $\liminf \bar{y}_{i,m} < \epsilon'$. (ϵ' is defined below).

Define $\alpha_m = \min_{y \in A_\epsilon} g'_{i,m}(y)$, $\beta_m = \max_{y \in A_\epsilon} h'_m(y)$ where $A_\epsilon = [0, y^* + \epsilon]$

Let $\epsilon_m = \alpha_m \cdot \epsilon$. By the Mean Value Theorem if either $|g_{i,m}(y') - g_{i,m}(y'')| < \epsilon_m$ or $|h_m(y') - h_m(y'')| < \epsilon_m$, then $|y' - y''| < \epsilon$; alternatively, if $|y' - y''| < \frac{\alpha_m}{\beta_m} \cdot \epsilon$ then both above expressions are no greater than ϵ_m .

Define $\epsilon' = \min_m \frac{\alpha_m}{\beta_m} \cdot \epsilon$. For m large enough $|y_m^* - y^*| < \epsilon$, suppose that $\bar{y}_{i,m} < \epsilon'$ and for some $n \geq 1$, $\bar{y}_{i,m+n} < \epsilon'$, then

$$\begin{aligned} g_{i,m+n}(\bar{y}_{i,m}) - g_{i,m}(\bar{y}_{i,m}) &= g_{i,m+n}(\bar{y}_{i,m}) = \\ &= \Delta_n C'_m(\bar{y}_{i,m}) - n \cdot \psi'(\bar{y}_{i,m}) < \epsilon_m, \text{ similarly,} \\ h_{m+n}(y_m^*) &= \Delta_n C'_m(y_m^*) - n \cdot \psi'(y_m^*) < \epsilon_m \end{aligned}$$

Consider the set $C_m = \{ y : -\epsilon_m \leq \Delta_n C'_m(y) - \psi'(y) \leq \epsilon_m \}$

since the function $\Delta_n C'_m(\cdot) - n \cdot \psi'(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and by assumption v) one-to-one, the set C_m is a closed interval, i.e., is connected, which contradicts the fact that we started with two disjoint sets.

Lemma 2

I first prove that, under assumption vi) the sequence $\{y_m^*\}$ is nondecreasing. For corner solutions $y_m^* = 0$. So we only have to consider interior solutions. Let $y_m^* > 0$, then (using the fact described on page 13-14)

$$\begin{aligned} h_{m+1}(y_m^*) - h_m(y_m^*) &= h_{m+1}(y_m^*) - 0 = \Delta C'_m(y_m^*) - \psi'(y_m^*) \\ &= \Delta C'_m(y_m^*) - \frac{1}{m} C'_m(y_m^*) < 0. \end{aligned}$$

The set of equalities are equilibrium conditions and the inequality follows from assumption vi) (by vi, Marginal Cost is never above Average Cost).

In particular, this means that the sequence of allocations of optimal public goods is either convergent to a finite allocation or convergent to $+\infty$. Now I prove the second part of the lemma:

$$\begin{aligned} \text{Let } \Psi'_x(y) &= \max \{ \Psi'_1(y), \dots, \Psi'_J(y), 0 \} \\ \Psi'_n(y) &= \min \{ \Psi'_1(y), \dots, \Psi'_J(y), 0 \} \quad , \text{ then for all } i \text{ and for all } m : \\ h_m(\cdot) + \Psi'_n(\cdot) &\leq g_{i,m}(\cdot) \leq h_m(\cdot) + \Psi'_x(\cdot) \end{aligned}$$

$$\begin{aligned} \text{Consider the set } A_m &= \{ y : -\Psi'_x(y) \leq h_m(y) \leq -\Psi'_n(y) \} = \\ &= \{ y : -\frac{1}{m} \cdot \Psi'_x(y) \leq \frac{1}{m} \cdot C'_m(y) - \psi'(y) \leq -\frac{1}{m} \cdot \Psi'_n(y) \} \end{aligned}$$

A_m is a closed set in R . Furthermore, y_m^* , $\bar{y}_{i,m} \quad i=1, \dots, J$ are in A_m

(i.e., A_m is not empty).

Suppose that both ψ'_x and ψ'_m are bounded, i.e., there exist positive numbers M_x and M_n such that $\psi'_x(y) \leq M_x$ and $-\psi'_n(y) \leq M_n$ for all y .

$$\text{Let } D_m = \{ y : -M_x \leq h_m(y) \leq M_n \} = \{ y : -M_x \leq C'_m(y) - m\psi'(y) \leq M_n \} .$$

For all m , $A_m \subset D_m$ and D_m is a closed nonempty set in R .

Now we want to prove that $\{D_m\}$ is a monotone decreasing sequence of sets (by decreasing we mean nonincreasing). Using an induction argument, suppose that $D_{m+1} \subset D_m$. We have:

$$D_m = \{y: -M_x \leq C'_m(y) - m\psi'(y) \leq M_n\}$$

$$D_{m+1} = \{y: -M_x \leq C'_m(y) - m\psi'(y) + (\Delta C'_m(y) - \psi'(y)) \leq M_n\}$$

$$D_{m+2} = \{y: -M_x \leq C'_m(y) - m\psi'(y) + (\Delta C'_m(y) - \psi'(y)) + (\Delta C'_{m+1}(y) - \psi'(y)) \leq M_n\}$$

Suppose that there exists

\tilde{y} such that \tilde{y} is in $D_{m+2} \setminus D_{m+1}$ (i.e., \tilde{y} is not in D_m).

Consider first the case

$$a) M_n < C'_m(\tilde{y}) - m\psi'(\tilde{y})$$

$$b) M_n < C'_m(\tilde{y}) - m\psi'(\tilde{y}) + \Delta C'_m(\tilde{y}) - \psi'(\tilde{y})$$

$$c) M_n > C'_m(\tilde{y}) - m\psi'(\tilde{y}) + \Delta C'_m(\tilde{y}) - \psi'(\tilde{y}) + \Delta C'_{m+1}(\tilde{y}) - \psi'(\tilde{y})$$

Multiplying b) by -2 and adding it to c) we get

$$\begin{aligned} M_n &> C'_m(\tilde{y}) - m\psi'(\tilde{y}) + \Delta C'_{m+1}(\tilde{y}) - \Delta C'_m(\tilde{y}) = \\ &= C'_m(\tilde{y}) - m\psi'(\tilde{y}) + C'_{m+2}(\tilde{y}) + C'_m(\tilde{y}) - 2C'_{m+1}(\tilde{y}) \\ &> C'_m(\tilde{y}) - m\psi'(\tilde{y}) \end{aligned}$$

The last inequality follows from assumption vi) and contradicts a) so

$\tilde{y} \in D_{m+2} \setminus D_{m+1}$ if and only if

d) $0 > -M_x > C'_{m+1}(\tilde{y}) - (m+1) \psi'(\tilde{y})$

and e) $-M_x \leq C'_{m+1}(\tilde{y}) - (m+1)\psi'(\tilde{y}) + \Delta C'_{m+1}(\tilde{y}) - \psi'(\tilde{y})$

subtracting d) from e) we get

$$\Delta C'_{m+1}(\tilde{y}) > \psi'(\tilde{y}) > \frac{1}{m+1} \cdot M + \frac{1}{m+1} \cdot C'_{m+1}(\tilde{y}) > \frac{1}{m+1} \cdot C'_{m+1}(\tilde{y})$$

but by assumption vi)

$$\Delta C'_{m+1}(y) \leq \frac{1}{m+1} C'_m(y) \text{ for all } y. \text{ A contradiction.}$$

We have proved that $\{D_m\}$ is a monotone decreasing sequence of nonempty closed sets in R and consequently that $\{\bar{y}_{i,m}\}$ is a nondecreasing sequence.

Finally, suppose that $\lim_{y \rightarrow \infty} V'(y) < 0$. Then h_m is one-to-one in $[0, +\infty]$, since

$$D_m = \left\{ y: -\frac{1}{m} M_x \leq \frac{1}{m} C'_m(y) - \psi'(y) \leq \frac{1}{m} M_m \right\}$$

$d(D_m) \rightarrow 0$ as $m \rightarrow \infty$, where d is the diameter of D_m (or the Lebesgue measure in R). It is well known that, in a complete metric space, as in the case of (M,d) , a monotone decreasing sequence of nonempty closed sets with diameters converging to zero has one and only one point that belongs to $D = \bigcap_{m=1}^{\infty} D_m$. (See for example A. Friedman, theorem 3.4.1, p. 105).

Since for all m , $A_m \subset D_m$ and y_m^* , $\bar{y}_{i,m}$ (i.e., $1, \dots, J$) are in A_m , we

have that $y^* = \lim_m y_m^*$ is well defined and for all $i, i = 1, \dots, J$,

$\lim_m \bar{y}_{i,m} = y^*$ if our boundness and limiting negative demand assumptions are

satisfied. We now turn to this boundness assumption.

By lemma 1.1 we only have to consider the case $y_m^* \rightarrow y^*, y^* > 0$, but then by lemma 1.2 for m large enough, for all $i, \bar{y}_{i,m} > \epsilon$. By assumption v) $\Psi'_x(y)$ is bounded for $y > \epsilon$, and we can restrict our attention to the sequence of sets $A_m [0, \epsilon)$.

Suppose that there exists y_0 such that $\Psi'_n(y_0) = -\infty$. This will happen if and only if $\psi'(y_0) = -\infty$ for all $y > y_0$.

y_m^* is the solution to $\frac{1}{m} \cdot C'(y) - \psi'(y) = 0$, since $C' > 0$, this implies that $y_m^* < y_0$.

Similarly, $\bar{y}_{i,m}$ is the solution to:

$$\frac{1}{m} \cdot C'_m(y) - \sum_{j \neq i}^J \Psi'_j(y) - \frac{m-1}{m} \cdot \Psi'_i(y) = 0$$

by the same argument $\bar{y}_{i,m} < y_0$ for all i .

Let $E = \{ y : \psi'(y) \text{ is bounded} \}$, then $y_m^*, \bar{y}_{i,m}$ are in $A_m \cap E$.

Now in the first part of the proof we only have to consider the sequence of sets $B_m = A_m \cap E [0, \epsilon)$ and let $B = \lim_m B_m$. Q.E.D.

Proposition 1

We have to prove that for any i , $\int_{\bar{y}_{i,m}}^{y_m^*} g_{i,m}(s) \cdot ds \rightarrow 0$ as $m \rightarrow \infty$.

If $y_m^* \rightarrow 0$ let $B_m = [0, \epsilon_m]$ $\epsilon_m \rightarrow 0$. If $y^* > 0$ let B_m be as in lemma 2.

for all m , the function $I_{B_m}(y) \cdot g_{i,m}(y)$ is bounded by the Lebesgue integrable function $|\Psi'_i|$.

$[\bar{y}_{i,m}, y_m^*] \subset B_m$ for all i , for any m . By lemmas 1 and 2 if $y_m^* \rightarrow y^*$, then $I_{B_m}(y) \cdot |g_{i,m}(y)| \rightarrow 0$. Where I is the characteristic function.

As we have noticed in Lemma 2 it might be the case that

$$\lim_m d(B_m) \neq 0. \quad 4$$

This will happen only if $\lim_{y \rightarrow \infty} V_i^1(y) = 0$ for some i . In this case $h_m(y)$ is one-to-one for $y \rightarrow \infty$ $0 < y < \infty$ but is not asymptotically one-to-one.

There is, however, no finite number y contained in $\lim_m B_m$, and

⁴A simple example is:
 $C'_m(y) = K$ (pure public good) and $V'_i(y) = \frac{\beta_i}{y}$, for all i , and $\sum_{j=1}^J \beta_j = 1$, then $y_m^* = \frac{m}{K}$ and $y_m^* - \bar{y}_{i,m} = \frac{\beta_i}{K}$, for all i and m , however, $\lim_m I_{B_m}(y) \cdot g_{i,m}(y) \neq 0$.

$I_{B_m}(y) - g_{i,m}(y) \rightarrow 0$. By the Lebesgue Bounded Convergence theorem:

$$t_{i,m}^* = \int_R I_{[\bar{y}_{i,m}, y^*]}(s) \cdot g_{i,m}(s) \cdot ds \leq \int_R I_{B_m}(s) \cdot g_{i,m}(s) \cdot ds \rightarrow 0 \text{ as } m \rightarrow \infty .$$

Convergence conditions

The same technique that we have used to prove our theorem can be applied to the study of strict-convergence of the allocation of public goods (i.e., convergence of finite allocation). The question we now consider is: under what conditions is the sequence of public goods convergent to a finite allocation as we replicate the economy? For simplicity we will study the one dimensional case. The results, however, generalize to the n-dimensional case.

- Facts: 1) $y_m^* \rightarrow y^*$, where $0 < y^* < \infty$ if and only if there exists an y^* ($0 < y^* < \infty$) such that $\lim_m \Delta C'_m(y^*) = \psi'(y^*)$.
- 2) $y_m^* \rightarrow 0$ if and only if $\lim_m \Delta C'_m(0) > \psi'(0)$.

Proof: 1) Follows from the fact that,

$$\lim_m h_{m+1}(y_m^*) = \lim_m (\Delta C'_m(y_m^*) - \psi'(y_m^*))$$

and the h 's functions are one-to-one.

2) From 1) and the fact that for all m $y_m^* > 0$.

A public good will be defined as a pure public good if the cost of production is independent of the number of people in the economy, as a pure public service if the cost is proportional to the number of people in the economy. The above facts prove the following lemma:

Lemma 3: a) The allocation of pure public goods converges to a finite allocation y^* , if and only if all types of consumers have a saturation point in their preferences for public goods.

b) The allocation of pure public services converges to a finite allocation if and only if there exists a $y^* < \infty$ such that :

$$J \cdot \frac{\partial C}{\partial y^k}(y^*) > \psi_k(y^*) \text{ for all } k .$$

(J is the number of types and ψ the type's aggregate utility function. The inequality is equality for $y^* > 0$).

Remark: a) holds also for any sequence of economies in which the increase in the marginal cost converges to zero. (For example if $C(y, m \cdot J) = \frac{m}{m+1} \cdot C(y)$).

These results show that there is a trade-off between the crowding effect and the class of preferences under which the allocation of public goods converges to a finite allocation. The problem that concerns us is whether the Clarke taxes converge to zero, and from proposition 1 this result will also be satisfied if the sequence of optimal allocations is monotonically nondecreasing. By lemma 2 this is always true for economies of pure public goods or pure public services since in these cases assumption vi) is automatically satisfied.

Corollary to theorem 1 If assumptions i) to v) are satisfied for a sequence of expanding economies of pure public goods (or pure public services), then for any type of consumers, the Clarke taxes converge to zero.

Conclusion

We have generalized the DRM and defined the set of conditions under which this mechanism is not only incentive compatible but also 'limiting efficient'. This is a possibility result and its main interest may be in the fact that the class of economies for which this result holds is a 'very large subset' of the class in which the DRM is IC.

Although Roberts (1976) has a general impossibility result for limiting public goods economies, our results are not in contradiction with his, not because we consider only a particular class of economies, but for two reasons: first, he accepts that a mechanism in which taxes converge to zero may be limiting incentive compatible (proposition 3); second, and more importantly, he studies resource allocation rules that are utility-respecting (proposition 1), arguing that in the limit the mechanism cannot be IC, which is exactly the same argument used in proving the limiting IC of the price

mechanism.

In public goods environments, however, it seems more appropriate to require IC in the sequence and study the efficiency properties as an asymptotic result. First, because it is known that there does exist a mechanism with such properties - the DRM. In fact, Robert's approach is based on the knowledge that the price mechanism is utility-respecting but not IC in finite economies. Second, because when one compares economies that differ only in size, the existence of a free rider problem in the smaller economy generally will not disappear and will probably worsen in the large economy.

Green, Kohlberg and Laffont (1976) have results similar to ours. They study a simple pivotal mechanism with a costless public project that is either accepted or rejected. They study the statistical properties of the mechanism, showing that the expected total surplus grows only at the rate of the square root of the population size. A modification of the DRM is introduced, distributing the total surplus equally across the entire population, and thereby destroying the IC property for finite economies. They prove that, given such a rate of growth of the total surplus and the assumption of equal beliefs concerning the distribution of tastes in the population, the DRM mechanism is nonwasteful and limiting IC. Our result is somewhat stronger and simpler.

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