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DUALITY THEORY FOR SOME STOCHASTIC CONTROL MODELS

by

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1. Introduction

This paper is concerned with a duality theory for discrete time stochastic control problems. The principal result is that, for rather general models, the dual variables are martingales.

The methods and main ideas of this paper originate from several papers in the literature. Although his results are stated in the context of a continuous time stochastic control problem with the underlying process being Brownian motion, Bismut's [1973] approach has several similarities to the one here. In particular, the variables are in spaces of stochastic processes, namely, well measurable functions on $\Omega \times [0, \infty)$, and convex optimization theory is used to establish the duality between the primal and dual problems. Although conditions are stated which guarantee the equality of the primal and dual optimal values, Bismut [1973] does not derive the kind of characterization of the dual variables sought in this paper.

With regard to discrete time models, there are two relevant lines of research in the literature. To briefly describe the first, consider an adapted stochastic process $Z = \{Z_n;\ n = 0, 1, \dots\}$ defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. Let $X = \{X_n;\ n = 1, 2, \dots\}$ be a predictable stochastic process on the same space. The <u>predictable transform</u> of Z by the process X is defined to be the stochastic integral of X with respect to Z evaluated at some time N, that is,

$$X \cdot Z_N = \sum_{n=1}^{N} X_n \Delta Z_n .$$

The problem is to maximize $E[X \cdot Z_N]$ over a class of predictable processes X for which this expectation is defined.

Problems of this sort were studied by Millar [1968], who assumed $N < \infty$, and Alloin [1969], who allowed $N = \infty$. Alloin also observed this problem can be viewed as a generalized optimal stopping problem because any stopping time T, taking positive values, gives rise to a predictable process X defined by setting $X_n = I_{\{T \geq n\}}$ and for which the predictable transform is the process $Z - Z_0$ stopped at time T.

Alloin [1969] assumed the admissible processes X in the optimization problem are those bounded by a specified scalar. More recently, Kennedy [1981] studied the more general problem where the admissible processes are those in the unit ball of an L^p space, where $p > 0$. He related the solution of the problem to an optimal stopping problem and showed that its form depends on whether $p \le 1$. In a second paper, Kennedy [1982] applied his results to the economic problem of optimally dividing (each period) a resource between consumption and investment, with the value next period of the invested portion being random. Using concave programming, he showed the Lagrange multipliers for his problem form a stochastic process that can be decomposed into the product of a martingale and a particular random discount factor (which corresponds to the rate of return for the investments). He also showed this dual process can be interpreted as a price system, for in an associated optimization problem where the decision maker can buy or sell unlimited quantitites of the resource at the prevailing price (i.e., current value of the dual process) the optimal consumption schedule turns out to be the same as before.

This last result by Kennedy, that the Lagrange multipliers are related to martingales, gets close to the focus of this paper. The other line of research on duality theory for discrete time stochastic models gets even closer and, indeed, was the original stimulus for this paper.

Rockafellar and Wets [1976] took a general version of a stochastic programming problem and derived a dual optimization problem in which the dual variables are stochastic processes satisfying the martingale type of conditional expectation relationship. To be more specific, and after transforming some of their stochastic programming terminology into probabilistic terms, their result is as follows.

A filtered probability space is specified, where the sample space Ω is a Borel subset of $\mathbb{R}^m$ and the filtration $\mathbb{F} = \{\mathcal{F}_0, \mathcal{F}_1, .., \mathcal{F}_{T-1}\}$, $T < \infty$. The problem is to choose a bounded, predictable process $X = \{X_t;\ t = 1, 2, .., T\}$ so as to minimize the expected value of $g(\omega, X_1, .., X_T)$, where $\omega \in \Omega$ and g is an inf-compact normal convex integrand. In their Theorem 2, Rockafellar and Wets [1976] show that with additional hypotheses there exists a natural dual optimization problem for which the dual variables are stochastic processes $Y = \{Y_t;\ t = 0, 1, .., T\}$ satisfying the martingale relationship

$$E[Y_{t+1} - Y_t | \mathcal{F}_t] = 0 \qquad t = 0, 1, .., T-1.$$

Although the general objective of the present paper is the same as that by Rockafellar and Wets [1976], there are several important differences. First, various assumptions are different. Their predictable processes are bounded, whereas those here are elements of an L^p space, $1 \le p < \infty$. Also, the objective function here is more general, and there is no assumption, such as made by Rockafellar and Wets [1976], implying the existence of a solution to the primal problem.

More importantly, the methods are significantly different. Rockafellar and Wets [1976] viewed each of the variables $X_1, X_2, .., X_T$ as an element of the space of bounded random variables and proved their result by induction on the time horizon T. The approach here is to view the whole stochastic process X as an element of an L^p space of stochastic processes and then derive the duality results in one step.

The approach taken here has two important consequences. First, the dual variables are shown to be martingales, the key point being that the dual stochastic processes are shown here to be adapted to the original filtration, a result Rockafellar and Wets [1976] did not state in their Theorem 2. Secondly, by not being an induction proof, the approach taken here has potential for being applied to continuous time stochastic control problems.

After some preliminaries, which include the key result that the orthogonal complement of the subspace of predictable processes consists of the martingale difference processes, the basic duality results are presented. These results are rather general, so two examples are discussed in the succeeding section. The paper concludes with some remarks about the economic interpretation of the duality results as well as how things might go with continuous time models.

2. Preliminary Results

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space, where the filtration $\mathbb{F} = \{\mathcal{F}_t;\ t = 0, 1, .., T\}$, $T < \infty$, $\mathcal{F}_0$ consists of Ω and all the null sets of P, and $\mathcal{F}_T = \mathcal{F}$.

Let $(S, \underline{\underline{O}}(\mathbb{F}), m)$ be the measure space with $S = \Omega \times \{0, 1, .., T\}$ and $\underline{\underline{O}}(\mathbb{F})$ the optional σ-field, that is, the σ-field generated by the adapted stochastic processes. Moreover, m is the bounded measure defined for $A \in \underline{\underline{O}}(\mathbb{F})$ by

$$m(A) = E\Big[\sum_{t=0}^{T} 1_A(\omega, t)\Big]$$

For $1 \le p < \infty$, let $L^p = L^p(S, \underline{\underline{O}}(\mathbb{F}), m)$ denote the L^p-space corresponding to $(S, \underline{\underline{O}}(\mathbb{F}), m)$. Thus L^p consists of all adapted stochastic processes $X = \{X_t;\ t = 0, 1, .., T\}$ such that $|X_t(\omega)|^p$ is m-integrable over S.

As usual, let q denote the conjugate exponent of p so that L^q is the dual space of L^p. Thus each bounded linear functional f(X) on L^p can be represented in the form $f(X) = E\Big[\sum_{t=0}^{T} X_t Y_t\Big]$ for some $Y \in L^q$. In particular, if $p = 1$, then L^∞, the space of bounded, adapted processes, is the dual of L^1.

Let $\underline{\underline{D}}$ denote the set of all predictable stochastic processes in L^p. In other words, $\underline{\underline{D}}$ consists of all the stochastic processes X in L^p satisfying $X_t \in \mathcal{F}_{t-1}$ for $t = 1, 2, .., T$.

Let $\underline{\underline{D}}^\perp$ denote the orthogonal complement of $\underline{\underline{D}}$, that is, all the elements in L^q orthogonal to every element of $\underline{\underline{D}}$. To be more specific,

$$\underline{\underline{D}}^\perp = \{Y \in L^q:\ E\Big[\sum_{t=0}^{T} X_t Y_t\Big] = 0 \ \text{ for all } X \in \underline{\underline{D}}\}.$$

If X is a stochastic process, then let ΔX denote the corresponding difference process, that is, $\Delta X_0 = 0$ and $\Delta X_t = X_t - X_{t-1}$ for $t = 1, 2, .., T$.

The key result of this paper is the following.

(1) Proposition. The stochastic process $Y \in \underline{\underline{D}}^{\perp}$ if and only if there exists a martingale $M \in L^q$ such that $Y = \Delta M$.

Remark Since each bounded linear functional $f(X)$ on L^p can also be written in the form $f(X) = E[X_0 Y_0 + \sum_{t=1}^{T} X_t \Delta Y_t]$ for some $Y \in L^q$, Proposition (1) can be restated to say that $\underline{\underline{D}}^{\perp}$ consists of all the martingales in L^q that are null at zero.

Proof. The sufficiency is easy to demonstrate. For an arbitrary martingale $M \in L^q$, set $Y = \Delta M$. Under the convention $\Delta M_0 = 0$, the linear functional $f(X)$ corresponding to Y can be written as $f(X) = E[\sum_{t=1}^{T} X_t \Delta M_t]$. Now for arbitrary $X \in \underline{\underline{D}}$ the stochastic process $\sum_{s=1}^{t} X_s \Delta M_s$, being the stochastic integral of a predictable process with respect to a martingale, is itself a martingale. It is also null at zero, so $E[\sum_{t=1}^{T} X_t \Delta M_t] = f(X) = 0$ for all $X \in \underline{\underline{D}}$, in which case $Y = \Delta M \in D^{\perp}$.

Conversely, suppose $Y \in \underline{\underline{D}}^{\perp}$. Clearly $Y_0 = 0$, for $Y_0 \in \mathcal{F}_0$ implies Y_0 is constant, and any non-zero constant Y_0 would lead to $E[\sum_{t=0}^{T} X_t Y_t] \neq 0$ for some $X \in \underline{\underline{D}}$.

Setting $M_0 = 0$ and $M_t = Y_t + M_{t-1}$ for $t = 1, 2, .., T$, it remains to show that M is a martingale. Since M is clearly adapted, it suffices to show

(2) $$E[\Delta M_t | \mathcal{F}_{t-1}] = E[Y_t | \mathcal{F}_{t-1}] = 0, \qquad t = 1, 2, .., T.$$

To do this, let $t \geq 1$ and $B \in \mathcal{F}_{t-1}$ be arbitrary, and set $X_t = 1_B$ and $X_s = 0$ for all $s \neq t$. Since $X \in \underline{\underline{D}}$, it follows that

$$E[\sum_{s=0}^{T} X_s Y_s] = E[1_B Y_t] = 0.$$

This verifies (2) because $B \in \mathcal{F}_{t-1}$ is arbitrary, so this proof is completed.

3. Basic Duality Results

Many discrete time stochastic control problems can be cast in the following (primal) form:

$$\text{(P)} \qquad \text{minimize} \qquad f(X)$$
$$\text{subject to} \qquad X \in \underline{\underline{C}} \cap \underline{\underline{D}}$$

where $\underline{\underline{C}}$ is a convex subset of L^p, $f\colon \underline{\underline{C}} \to \mathbb{R}$ is a convex functional, and $\underline{\underline{D}}$, as above, is the subspace of predictable processes. This being the case, it is natural to apply classical optimization theory in order to establish the duality theory for the primal problem (P).

One line of approach is as follows. Let $\underline{\underline{C}}^*$ denote the conjugate set

$$\underline{\underline{C}}^* = \{Y \in L^q: \sup_{X \in \underline{\underline{C}}} [E[\sum_{t=0}^{T} X_t Y_t] - f(X)] < \infty\},$$

and let f^* denote the conjugate functional on $\underline{\underline{C}}^*$

$$f^*(Y) = \sup_{X \in \underline{\underline{C}}} \{E[\sum_{t=0}^{T} X_t Y_t] - f(X)\}.$$

These lead to the dual problem

$$\text{(D)} \qquad \text{maximize} \qquad -f^*(Y)$$
$$\text{subject to} \qquad Y \in \underline{\underline{C}}^* \cap \underline{\underline{D}}^{\perp}.$$

Observe, by Proposition (1), that the variables in dual problem (D) are martingale difference processes. Moreover, for $X \in \underline{\underline{C}}$ and $Y \in \underline{\underline{C}}^*$ the definition of f^* gives

$$-f^*(Y) \leq -E[\sum_{t=0}^{T} X_t Y_t] + f(X),$$

so $-f^*(Y) \leq f(X)$ for all $Y \in \underline{\underline{C}}^* \cap \underline{\underline{D}}^{\perp}$ and all $X \in \underline{\underline{C}} \cap \underline{\underline{D}}$. In other words, the optimal value of the dual problem (D), abbreviated sup D, is less than or equal to the optimal value of the primal problem (P), abbreviated inf P. An application of the Fenchel Duality Theorem (see, for example, Luenberger [1969, p. 201]) then gives a sufficient condition for equality to actually hold:

(3) Theorem. (Fenchel Duality) Suppose $\underline{\underline{C}} \cap \underline{\underline{D}}$ contains points in the relative interior of $\underline{\underline{C}}$ and $\underline{\underline{D}}$, the epigraph of f over $\underline{\underline{C}}$ has a nonempty interior, and inf P is finite. Then

$$\inf P = \sup D,$$

and there exists a solution Y_0 to the dual problem (D).

Alternative conditions can be given for inf(P) = sup(D). See, for example, Rockafellar [1974, pp. 56-57]. Moreover, conditions can be given that are sufficient for ensuring there exists a solution to primal problem (P). Rather than pursuing these matters any further, however, the discussion will turn to the application of these duality results to some particular control problems.

4. Two Examples

This first example is similar to the stochastic programming model studied by Rockafellar and Wets [1976].

For the primal problem (P) one has $\underline{\underline{C}} = L^p$ and

$$f(X) = E[\sum_{t=0}^{T} g(t, \omega, X_t(\omega))],$$

where g: $\{0, 1, .., T\} \times \Omega \times \mathbb{R} \to \mathbb{R}$ is a function such that $g(t, \omega, \cdot)$ is convex for each fixed (t, ω). Thus f is a convex integral functional, provided it is well-defined in the sense that $g(t, \omega, X_t(\omega))$ is a measurable function of (t, ω) for every $X \in L^p$.

The functional f will be well-defined, according to Rockafellar [1968], if g is what is called a normal convex integrand, a general condition that is implied by a variety of specific situations. Under this condition, Rockafellar [1968] went on to derive an explicit formula for the conjugate functional f^*. Let g^* be the integrand conjugate to g, that is,

$$g^*(t, \omega, y) = \sup_{x \in \mathbb{R}} \{xy - g(t, \omega, x)\}.$$

If $g(t, \omega, X)$ is majorized by an integrable function of (t, ω) for at least one choice of $X \in L^p$ and if $g^*(t, \omega, Y)$ is majorized by an integrable function of (t, ω) for at least one choice of $Y \in L^q$, then Rockafellar [1968] showed that $\underline{\underline{C}}^* = L^q$ and

$$f^*(Y) = E[\sum_{t=0}^{T} g^*(t, \omega, Y_t(\omega))].$$

Thus the dual problem (D) is to maximize $-f^*(Y)$ over the subspace of martingale difference processes in L^q. The first hypothesis in Theorem (3) is automatically satisfied, and the other two are easy to check for particular cases, so if they both hold then inf(P) = sup(D) and there exists a solution to (D).

The second example comes from Pliska's [1982] discrete time stochastic decision model. Let the sample space Ω be the 2^T T-dimensional vectors whose components are either 1 or -1. Let the probability measure P be arbitrary, subject only to the requirement that $P(\omega) > 0$ for all $\omega \in \Omega$. Define a stochastic process Z = $\{Z_t;\ t = 1, 2, .., T\}$ by setting $Z_t(\omega) = \omega_t$, the t^{th} component of ω. Let the filtration $\mathbb{F}$ be the one generated by Z. The problem is to minimize $E[u(\sum_{t=1}^{T} X_t Z_t)]$ subject to $X \in \underline{\underline{D}}$, where u is a specified convex function.

This problem fits in the framework of primal problem (P), for $\underline{\underline{C}} = L^p$ and $f(X) = E[u(\sum_{t=1}^{T} X_t Z_t)]$ is convex on $\underline{\underline{C}}$. With suitable assumptions about u the hypotheses in Theorem (3) will be easy to verify, in which case inf(P) = max(D). This will be the case, for example, if u is strictly convex and decreasing, has a continuous second derivative, and satisfies either $u'(x) \to 0$ as $x \to \infty$ or $u'(x) \to -\infty$ as $x \to -\infty$. In fact, as shown in Pliska [1982], under these particular conditions there will exist a solution to the primal problem (P).

5. Concluding Remarks

Besides being of general theoretical interest, the primary importance of the duality theory has to do with applications of stochastic control models where the dual variables can be interpreted as prices. Knowing by the duality theory presented above that the price processes are actually martingales may have significant economic implications. Of course, the interpretation of Lagrange multipliers and dual variables as prices is well-known for economic models. This interpretation is discussed in the context of linear programming in Dantzig [1963], for example. Some references involving more general settings are Arrow, Hurwicz and Uzawa [1958], Baumol [1977], and Gale [1960]. The paper by Rockafellar and Wets [1976] enumerates several additional papers dealing with price systems associated with constraints appearing in multistage stochastic programs.

One economic application where duality theory may be of interest is the area of consumption-investment problems. Such problems, as well as variations such as optimal capital accumulation under uncertainty and resource allocation under uncertainty, have been extensively studied in the economics literature; see, for example, Arrow and Hurwicz [1977], Brock and Mirman [1973], Mirman [1971], and Lucas and Prescott [1971]. The basic idea is that each period the decision maker must divide his wealth between current consumption, yielding immediate utility, and investment, which becomes worth a random amount next period. This leads to a trade-off between current and future consumption. Welch [1979], Zilcha [1976], and, as mentioned at the beginning of this paper, Kennedy [1982] studied particular versions of this problem and showed the Lagrange multipliers could be interpreted as prices.

Rockafellar and Wets [1976] derived their results in the context of the variables being bounded stochastic processes. The specific approach taken here would not work in that case, because the dual of L^∞ is not L^1 and so one could not proceed as in Proposition (1). However, it is possible that one could apply one aspect of their approach in order to overcome this difficulty without resorting to their induction proof methodology. This would be to add more structure to the convex functional f in primal problem (P) and then apply the results of Rockafellar [1971] to show that $\underline{\underline{C}}^* \cap D^\perp$ is actually in L^1, even though L^1 is not the dual of L^∞.

The principal advantage of the approach here over that by Rockafellar and Wets [1976] is that it does not rely on a proof by induction on the time variable. Thus this method has the potential of being suitable for continuous time stochastic control problems. Indeed, for $p < \infty$ one could proceed as above and compute $\underline{\underline{D}}^\perp$ but with linear functionals of the form $E\left[\int_0^T X_t Y_t dt\right]$ one would get the uninteresting result

$\underline{\underline{D}}^{\perp} = \emptyset$. On the other hand, with linear functionals of the form $E\left[\int_0^T X_t dY_t\right]$ one sees that $\underline{\underline{D}}^{\perp}$ contains all the martingales in L^q null at zero. These interesting issues deserve further study.

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