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A COMPETITIVE MODEL OF COMMODITY DIFFERENTIATION*

by

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ABSTRACT

This paper develops a general, competitive model of commodity differentiation. The structure analyzed is sufficiently rich that the basic structures of many of the common models of commodity differentiation arise as special restrictions on the allowed preferences and production sets. Thus, the model provides a convenient, unifying framework within which alternative formulations of strategic product choice can be compared.

It it shown that, in contrast to strategic models, competitive equilibria exists only under model restrictions on the underlying economic structure. Further, it is shown that the equilibria of the model analyzed have a strong continuity property as endowments are adjusted. Finally, some results relevant to all models of commodity differentiation featuring price-taking consumers are presented. The results presented here are shown to point to some potentially important methodological restrictions.

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Section 1

Introduction

In recent years there has been a renewed interest in models in which one of the key strategic choice variables of firms is the characteristics of their products. The literature is highly diversified but can be roughly divided into three categories: variations on Hotelling's (14) model of location, the characteristics model of Lancaster (16) and the so-called general utility approach (10). All three approaches have led to useful, if sometimes opposing, insights. Unfortunately, this diversity of approach is a weakness of the literature. The various models are often so different that it is difficult to see what special features give rise to the results. Stated another way, since there is no common underlying framework for these models, we cannot view each model as specifying a set of restrictions on a common framework. It is clear that such a framework would facilitate these comparisons, thereby providing a deeper understanding of the special results. Another aspect of this problem is the lack of any general results concerning either the internal consistency or asymptotic properties of the equilibria of these models.

The aim of this paper is to provide a first step at rectifying these weaknesses by developing a general, internally consistent, competitive model of commodity differentiation. The model analyzed is of the general utility variety and has the advantage that the consumption sectors of many of the common models of commodity differentiation arise as restrictions on the preferences of agents in our model (see the remarks of Section 4).

The analyis presented here differs from other models of commodity differentiation in that attention is restricted to competitive (as opposed to strategic) equilibria. However, since most treatments of commodity differentiation do include price-taking consumption sectors, the results do provide some insights relevant to strategic equilibria in these models. The benefit to be reaped from the restriction to competitive analysis is the generality of the results that the approach affords. For example, in contrast with many versions of the location model (see Novshek (20)), competitive equilibria exist under fairly general circumstances. In addition, a rich competitive framework serves as a benchmark with which strategic equilibria of similar models can be compared. Finally, the model should serve as a useful backdrop for results on the behavior of strategic models of commodity differentiation in large economies. (Hart has provided one such use in (11).) In particular, it should provide a useful framework for the extension of the recent work on the asymptotic existence of strategic equilibria of Novshek and Sonnenschein (19).

The model presented here concentrates on several desirable properties of models of commodity differentiation. First, specialization is allowed. That is, the framework is rich enough to allow each producer to produce a distinct product, each consumer to favor a distinct commodity. (This is the essence of the Hotelling model of location.) Second, the model has very strong "continuity" properties. This is important since one of the issues of interest in this literature is a precise determination of the conditions under which monopolistically competitive equilibria (under various definitions) are approximately competitive in large economies. Finally, as a by-product of this development, the notion that commodities with nearly the same description in characteristics space are good substitutes is formalized. This intuition arises as a smoothness property of preferences and production functions and is a key factor in our approximations and limiting arguments. As a consequence of this substitutability, in equilibrium, prices will depend continuously on characteristics.

The model presented here is similar to those of Mas-Colell (18) and Hart (11). In Mas-Colell, a theorem on the existence of competitive equilibrium is presented, but the assumed non-convexity of consumption sets necessitates the inclusion of infinitely many agents. The smoothness assumptions on preferences and production sets allow us to avoid the non-convexity problem here and show that equilibria exist even with finitely many households. In Hart, the analysis is of the behavior of strategic equilibria in large economies; the question of the existence of equilibrium is not addressed.

The collection of commodities will be denoted by T (possibly infinite), where a t in T is viewed as a complete description of the economically relevant characteristics of the commodity in question. Consumers choose between alternative consumption bundles which are modeled as non-negative distributions (i.e., measures) on T. This structure has the flexibility to allow consumers to either specialize by choosing to buy only a few commodities or to spread consumption evenly over many commodities. Similarly, firms choose distributions (signed in this case; negative for inputs, positive for outputs) over T. Again, this formulation allows for firms to become either specialists or generalists. (Of course, prices adjust so that in equilibrium, these desires of firms and consumers coincide.)

The paper conducts a detailed competitive analysis of the model described above. The results presented are stated in the standard language of general equilibrium theory to facilitate comparisons with the common references on general competitive equilibrium (e.g., Debreu (8)).

In a model with finitely many consumers and firms, it is shown that, under conditions familiar in general equilibrium theory, competitive equilibria exist as long as preferences and production sets satisfy mild smoothness assumptions. As is standard, competitive equilibria are Pareto Optimal. Moreover, in the equilibrium shown to exist, prices depend continuously on commodity characteristics: Nearby commodities are priced nearly the same. This justifies the contention that under the allowed preferences, commodities with nearly the same characteristics are good substitutes. Relating to the point above concerning continuity results, it is shown that in those cases in which T is infinite, the equilibria of the full model shown to exist are very closely approximated by the equilibria of models with large finite dimensional commodity spaces.

As another example of the continuity properties of the model, it is shown that for the case of pure exchange, the map from endowment distributions (holding preferences fixed) to continuous price equilibria is an upper hemi-continuous correspondence. This result is particularly strong since the continuity in prices is in the sense of uniform convergence. Thus, if p^n are equilibrium prices for the endowment distributions e^n with $e^n \to e$, there is a subsequence n_k and a continuous function p such that $\sup_T |p^{n_k}(t) - p(t)| \to 0$ and p is an equilibrium price function for an economy having e as endowments.

As pointed out above, since most models of commodity differentiation feature competitive consumption sectors, the analysis presented is of relevance to models with strategic formulations. In particular, the approach provides some general methodological guidance in formulating preferences for these models. An example of this is included as Proposition 5 of Section 3. Section 4 contains further related comments.

Finally, the relationship between our approach and other models of commodity differentiatiion is discussed. Particular attention is paid to a discussion of the relationship between the consumption sectors of these models and the one analyzed here. It is shown that in many cases of interest the consumption sectors of these models can be arrived at by special restrictions on the allowed preferences.

Section 2 introduces the basic model to be analyzed and introduces notation and assumptions. The results on existence, optimality and continuity of equilibrium are presented in Section 3 along with some examples which suggest possible limitations on extensions of the main results. Section 4 is devoted to exploring the connection between this and other models. Examples arising from the literature on commodity differentiation are presented and the relationship between this and other work on competitive equilibrium with infinitely many commodities is discussed. Proofs of the main results are collected in Section 5.

Section 2

The Model: Notation and Assumptions

In this section, the formal description of the competitive model of commodity differentiation is presented for the case of finitely many households and firms. Since the basics of consumer choice are applicable for models beyond the competitive one considered here, this aspect of the model will be developed in more detail. To facilitate understanding of the basics, some examples from the literature on commodity differentiation are included. A more thorough exposition of these examples and their relation to the model presented here is included in Section 4.

As noted above, the collection of all possible commodities will be denoted by T. A t in T is to be interpreted as a complete description of all the economically relevant characteristics of the commodity in question. For example, $T = [0,1]$ with a $t \in T$ having the interpretation of location as in (14). If $T = [0,1] \times [0,1]$, the commodities considered could be all foods indexed by their content of Vitamin A and protein. Here, carrots would be represented by a t having a large first coordinate, steak a t with a large second component.

In what follows, T will be assumed to be a compact metric space with metric given by d although for most of the examples discussed, T will be a subset of R^n for suitably chosen n. Typical elements of T will be denoted by t and s. $\mathcal{B}(T)$ is the collection of Borel subsets of T with typical element B.

Consumers or households will be indexed by h, $h=1,\ldots,H$.

Firms will be indexed by j, $j=1,\ldots,J$.

Consumers are modeled as choosing non-negative distributions on T. That is, letting $\mathcal{M}$ be the collection of finite, signed measures on $(T, \mathcal{B}(T))$, and M those elements of $\mathcal{M}$ which are non-negative, a consumer will choose an m in M. Then, under the plan m, m(B) is the total amount of commodities consumed having characteristics in B. Two things should be noted. First, in looking at m(B), we are indeed adding apples and oranges (and calling the sum fruit). This is of no consequence however, since the value that m(B) takes on plays no essential role in the results except to guarantee that in equilibrium, the total amount of fruit consumed equals the amount available through endowments and production (the same is required of both apples and oranges individually as well). Second, as noted in the introduction, the pleasant feature of this structure is that it has sufficient flexibility to allow consumers to be either specialists or generalists. Again, if $T = [0,1]$, a typical consumer can specialize completely by choosing a distribution which puts all of its mass at one point (e.g., buy only carrots) or he can generalize by choosing a distribution with a density. (Thus, in certain cases, the results presented here apply to economies with consumption sets in L_∞ as well. See the remarks in Section 4 for details.)

Producers are modeled as usual in competitive analysis. Each firm can be described by the collection of technologically feasible alternatives available to it, Y_j, and by its ownership shares θ_{hj}. Of course, $0 \leq \theta_{hj} \leq 1$ and $\sum_h \theta_{hj} = 1$. The usual sign convention (negative for inputs, positive for outputs) is followed.

In this framework, prices can be interpreted in the usual way. That is, prices are a function which assign to each commodity t a non-negative real number $p(t)$. $p(t)$ is then the price of one unit of the good t. Hence, prices will be represented by $\mathcal{B}(T)$--measurable, bounded, non-negative, real-valued functions on T. Since the non-negative, continuous functions from T to R will play a special role here, these will be singled out as $\mathcal{C}(T)$.

Given a consumption plan m in M and a price function $p(t)$ (not necessarily continuous), value can be defined in the usual way:

$$p\cdot m = \int_T p(t)dm(t).$$

For example, if T is the interval $[0,T]$ and m has a density, ($m(B) = \int_B f(t)dt$ for all b), $p\cdot m = \int_0^T p(t)f(t)dt$. If m consists only of point masses of size a_i at points t_i in T, $p\cdot m = \sum_i p(t_i)a_i$.

Similar considerations apply to calculating the value of a production plan. Thus, this is a natural generalization of the usual formula for calculating value.

We will need a few more preliminary definitions and mathematical facts concerning $\mathcal{M}$.

We endow $\mathcal{M}$ with the weak* topology of the dual pairing $(\mathcal{M}, \mathcal{B}(T))$. This is just the usual notion of convergence in distribution familiar from probability theory extended to cover all of $\mathcal{M}$. All topological notions on (convergence, closedness, etc.) can be assumed to be in the sense of the weak* topology unless explicitly stated otherwise. It is well-known that under convergence in distribution, (variation norm) bounded subsets of $\mathcal{M}$ are conditionally compact and metrizeable.

Since the degenerate distributions play a special role in $\mathcal{M}$ under the weak* topology, we set aside a special notation for them: δ_t denotes the Dirac measure at t, $\delta_t(B) = 1$ if $t \varepsilon B$, 0 otherwise. In essence, the δ_t's are the only elements of $\mathcal{M}$ that matter in the weak* topology in the sense that if $\{t_i\}$ is any countable dense subset of T, $\mathcal{M}$ is the closure of the set of finite linear combinations of the δ_{t_i}. This fact serves as the basis of the approximations developed below. Given a collection of commodities $t_1,\ldots,t_n$ in T, $LS(t_1,\ldots,t_n)$ will denote the linear subspace of $\mathcal{M}$ generated by $\delta_{t_1}, \ldots, \delta_{t_n}$. Then, $M \cap LS(t_1,\ldots,t_n)$ represents a typical consumers consumption set when trading is restricted to the commodities $t_1,\ldots,t_n$.

Whenever it is necessary, the variation norm of an m in $\mathcal{M}$ will be denoted by $\|m\|$. Note that if $m \varepsilon M$, $\|m\| = m(T)$. A subset of $A \subset \mathcal{M}$ is bounded if $\{\|m\| \mid m \varepsilon A\}$ is bounded.

$\mathcal{M}$ has a natural notion of "greater than" associated with it: $m \geq m'$ if $m(B) \geq m'(B)$ for all B, $m > m'$ if $m \geq m'$ but $m \neq m'$. For example, if $T = [0,1]$ and m and m' have densities f and f', $m \geq m'$ if and only if $f \geq f'$.

For each $m\varepsilon\mathcal{M}$, the support of m is the smallest closed subset of T having full m-measure. Again, if T = [0,1] and m has a continuous density, f, supp m is the collection of points where f is non-zero.

The characteristics of an individual consumer, h, consist of his endowment e_h, in M, his preferences, $\succsim_h \subset M \times M$, and his firm shares θ_{hj}. It is interesting to note, although unimportant for our purposes here, that since bounded subsets of M are compact and metrizeable, weak* continuous preference orderings have utility representations (on bounded subsets of M).

Given prices, p, and a non-negative wealth, w, the budget set is defined in the usual way (recall that p is bounded above):

$$\beta(p;w) = \{m\varepsilon M | p\cdot m \leqslant w\} .$$

Note that if p is bounded below, β is bounded and if p is continuous, β is weak* closed. Thus, if p is continuous and bounded below, β is weak* compact.

We can make the usual definitions as follows. An <u>attainable allocation</u> is a H + J tuple of elements in M, $(m_1,\ldots,M_H, y_1,\ldots,y_J)$ with $m_h\varepsilon M$ for all h, $y_j\varepsilon Y_j$ for all j and $\Sigma m_h = \Sigma y_j + \Sigma e_h$. A <u>Pareto optimal</u> allocation is an allocation which is attainable and has the property that there is no other attainable allocation which makes everyone at least as well off and at least one person better off. The tuple $(p;m_1,\ldots,m_H;y_1,\ldots,y_J)$ is a competitive equilibrium if (m,y) is an attainable allocation, if for each h, m_h maximizes $\succsim_h$ on $\beta(p;w_h)$ where $w_h = p\cdot(e_h + \Sigma\theta_{hj}y_j)$ and for each j, y_j maximizes j's profits on Y_j.

The weak* topology on M has a very natural notion of closeness of commodities built into it. Given any m in M, a > 0 and any sequence t_k in T converging to t^*, $m + a\delta_{t_k} \to m + a\delta_{t^*}$ - weak*. Thus, under weak* continuous preference orderings, for large k, $m + a\delta_{t_k}$ is a good substitute for $m + a\delta_{t^*}$. Thus, we might expect that if all agents have weak* continuous preference orderings, equilibrium prices should be a continuous function of t.

In fact, the continuity of equilibrium prices is very important if we wish to retain the interpretation that the equilibria of the economy here represent an approximation of the equilibria of economies with a large but finite collection of commodities. That is, if we are to view (as we do here) the economy with infinite dimensional commodity space as a limiting ideal, it should be true that the equilibria in the limit are closely related to the equilibria of large finite dimensional approximations. Since we view this interpretation as essential, a principal part of the analysis will be directed at results guaranteeing the validity of this interpretation of the limiting economy. Thus, the proofs of the principal results will all be carried out through approximation. One thing that is necessary to guarantee that this can be done, is that the prices in the equilibria of the finite dimensional economies be sufficiently well-behaved to enable us to assign prices in the limit economy. A possible problem is the following: Suppose we are trying to define the limit price at a t which is not traded in the finite dimensional approximations, but that there are two sequences, s^n and t^n, of commodities both converging to t, both available for trading at the n-th stage in the approximation. If the equilibrium prices in the n-th stage, $p^n(s^n)$ and $p^n(t^n)$, have the same limiting value, there is no problem in assigning a price to commodity t in the limit. If, on the other hand, the limits of these two sequences differ there is, in general, no unambiguous way for choosing p(t). Thus, to guarantee the smooth operation of the limiting process, we must do something to guarantee that the prices of the approximations are well-behaved. As a first guess, one might think, by the substitutability argument outlined above, an assumption of weak* continuity of preferences might be sufficient. That is, since under weak* continuous preferences, s^n and t^n are good substitutes, they should have nearly the same prices in equilibrium.

Unfortunately, this is not quite correct. Intuitively, the problem with this argument is that it is possible for s^n and t^n to be good substitutes in the sense that preferences are weak* continuous yet $p^n(s^n)$ and $p^n(t^n)$ are far apart.

If we fix a t^* which is tradeable at all stages and fix its price at 1 at all stages, $p^n(s^n)$ and $p^n(t^n)$ are, respectively, the marginal rates of substitution between commodity t^* and s^n, and between t^* and t^n. Thus, $p^n(t^n)$ and $p^n(s^n)$ will be nearly the same if $MRS_{t^*t^n}$ and $MRS_{t^*s^n}$ are nearly the same. Stated another way, $p^n(t^n)$ and $p^n(s^n)$ will be nearly the same if s^n and t^n are good substitutes at the margin: $MRS_{t^ns^n} \approx 1$. This is exactly the assumption that we will need here.

It is easy to see that the condition $MRS_{st} \approx 1$ if s and t are sufficiently close can be interpreted as a condition on the smoothness of preferences. That is, this is the same as requiring that the marginal utility from consumption of commodity t depends continuously on t. (A similar assumption will be needed for producers.)

We should add that this is not an issue in Mas-Colell (18). Due to the invisibilities of the consumption decision in that model, the margin of choice is in the commodity chosen rather than in the quantity purchased of a given commodity. In the model presented here, agents are allowed to make choices over both of these margins. The result below shows that this simultaneous choice over both margins can be handled as long as the more important margin is the quantity one rather than the characteristics one. This will be true if the marginal utility depends continuously on the commodity description. In this case, the quantity margin is the more important in the sense that the dominant effect on utility of small changes in the consumption of commodities with similar characteristics is the change in the quantities rather than the differences in the characteristics. More explicitly, if we consider adding a small quantity, Δ, of commodity t to a given bundle, m, the resulting change in utility ΔU_t, is such that $\frac{\Delta U_t}{\Delta}$ converges (as $\Delta \to 0$) to a positive constant, the marginal utility at t. If we compare this with the effect of making this small change at commodity t' instead, the difference is the difference in the marginal utilities at t and t'. If t is close to t' and marginal utility depends continuously on the commodity, the characteristics effect is of the second order, $\frac{\Delta U_t}{\Delta} - \frac{\Delta U_{t'}}{\Delta} \to 0$ (as $t \to t'$ and $\Delta \to 0$). It is in this sense that the quantity margin is assumed to be the more important one in the results presented below.

Before turning to the assumptions, we should make one point clear. Assuming that preferences are weak* continuous does place a restriction on the units in which commodities are measured. The units of measurement cannot be chosen arbitrarily, rather, they must be chosen so that nearly the same quantities of nearby commodities are nearly perfect substitutes in consumption. This does not seem like an unduly restrictive requirement. Indeed, usually in economic models there is a "natural parameterization" of T which satisfies this restriction. Of course, this requirement can be weakened to: there is one parameterization of T such that preferences satisfy the continuity assumptions but this added generality seems superfluous. We should also point out that the fact that continuity assumptions can place economically meaningful restrictions on allowed preferences has been pointed out before. Brown and Lewis (4) provides a further example of this phenomenon.

We will make the following assumptions about the consumption sector of our economy which hereafter will be denoted by $\mathcal{E}$.

(H1)

(a) $\bigcap_h \text{supp } e_h \neq \emptyset$.

(b) For some $\underline{t} \varepsilon \bigcap_h$ supp e_h, preferences are strictly monotone at $\underline{t}$. For all $m \varepsilon M$, $\alpha > 0$, and for all h, $m + \alpha\delta_{\underline{t}} \succ_h m$.

The two parts of (H1) taken together imply that our households are "resource related"--each household has at its disposal some good which all consumers hold as desirable. This is necessary to guarantee that all consumers play active roles in our economy. This takes the place of non-satiation in the standard existence results and guarantees that, evaluated at the limit of the equilibria of the finite dimensional approximations, each household's endowment has positive value. This is essential if we are to conclude that the limiting prices/allocations are indeed an equilibrium, not just a quasi-equilibrium.

We will also use the following, slightly different assumption at some stages.

(H1)´

(a) For all h, h's preferences are strictly monotone--$m > m' \Rightarrow m \succ_h m'$

(b) For all h, $e_h > 0$.

(H2)

(a) For all h, $\succsim_h$ is complete, transitive and reflexive.

(b) For all h, $\succsim_h$ is weakly monotone--$m \geqslant m'$ implies $m \succsim_h m'$.

(c) For all h, $\succsim_h$ is convex--for all m, $\{m' | m' \succsim_h m\}$ is convex.

(H3)

For all h, $\succsim_h$ is weak* continuous-- $\succsim_h \subset M \times M$ is closed in the weak* $\times$ weak* topology.

This is the substitutability condition discussed above. For some of the results we will need.

(H4) supp $\Sigma e_h = T$.

This says that all commodities are available in the aggregate.

Finally, we have the conditions on marginal substitutability discussed above.

(HS1) For all h, and all sequences s^k, $t^k \varepsilon T$, $a^k, b^k > 0$ and $m^k \varepsilon M$ such that $s^k \to t^*$, $t^k \to t^*$, $m^k \to m^*$ and $\underline{\lim} \frac{a_k}{b_k} > 1$, there is a k (k may depend on h) such that:

$$m^k + a^k \delta_{t^k} \succsim_h m^k + b^k \delta_{s^k}$$

In words, (HS1) states that if t^k and s^k are close enough in commodity space, household h is willing to accept any trade of t^k for s^k in which the "terms" are strictly greater than 1. Roughly speaking, this says that, asymptotically, the marginal rate of substitution between s^k and t^k is 1. Thus if we hold consumption of all other commodities fixed and draw the indifference curves between s^k and t^k they look more and more like straight lines with slope -1 as k tends to infinity. That smoothness of preferences is sufficient to guarantee that (HS1) is satisfied is verified in Proposition 1 below.

To cover some of the cases of interest, we will need the following slightly stronger version of (HS1)

(HS2) For all h and all sequences B^k, $C^k \varepsilon \mathcal{B}(T)$ (closed),

$m^k \varepsilon M$, $\mu^k, \nu^k \varepsilon M$ such that $B^k, C^k \to \{t^*\}$ (in closed convergence), $t^* \varepsilon T$, $m^k \to m^*$, supp $\mu^k \subset B^k$, supp $\nu^k \subset C^k$ and $\underline{\lim} \frac{\|\mu^k\|}{\|\nu^k\|} > 1$, there is a k (which may depend on h) such that

$$m^k + \mu^k \succsim_h m^k + \nu^k.$$

That (HS1) is implied by (HS2) can be seen immediately by taking $\mu^k = a^k \delta_{t^k}$, $\nu^k = b^k \delta_{s^k}$.

Finally, to consider those cases where (H4) may not be satisfied we need:

(HS3) There is a $\underline{t}$ such that for all h and all sequences $t^k \varepsilon T$, $m^k \varepsilon M$, a^k, $b^k > 0$ such that $t^k \to t^*$, $m^k \to m^*$ and $\underline{\lim} \frac{a^k}{b^k} = \infty$, there is a k (k may depend on h) such that

$$m^k + a^k \delta_{\underline{t}} \succsim_h m^k + b^k \delta_{t^k}.$$

In terms of the marginal substitutability argument above, (HS3) guarantees that marginal rates of substitution are bounded uniformly. This is essentially a boundary condition to guarantee that prices can be assigned in a meaningful way to those commodities which are not available in the aggregate (when (H4) is not satisfied). Although this is not of crucial importance in the case of competitive analysis, if strategic equilibria are considered with only finitely many goods produced, it is useful to be able to find a set of simultaneous reservation prices for the goods which are not produced. To accomplish this, something like (HS3) is necessary.

To see that these assumptions are indeed conditions on the smoothness of preferences, we will need a definition. Suppose $\succsim$ has a utility representation, U. Define

$$MU(m;t) = \lim_{a \to 0} \frac{1}{a} [U(m + a\delta_t) - U(m)]$$

if this limit exists--where it is understood that $a \to 0^+$ if $m(t) = 0$. $MU(m;t)$ is the Gateaux derivative of U at m in the direction of δ_t. This is the obvious analogue of the normal directional derivative of multivariate calculus. (For a more detailed discussion of differentiation in infinite dimensional space, see Hille (13).) Further, we can define $R(m;t;a) = U(m + a\delta_t) - [U(m) + aMU(m;t)]$. We see that R is the remainder term after approximating $U(m + a\delta_t)$ by the first term in a Taylors' Series. If the limit in (1) exists:

$$\frac{1}{a} R(m;t;a) \to 0 \text{ as } a \to 0 .$$

Now, consider sequences as described in (HS1). If $MU(m;t)$ exists for all m and t,

$$U(m^k + a\delta_{t^k}) \approx U(m^k) + a^k MU(m^k;t^k) \text{ and}$$

$$U(m^k + b^k\delta_{s^k}) \approx U(m^k) + b^k MU(m^k;s^k) .$$

Thus, $U(m^k + a^k\delta_{t^k}) \geqslant U(m^k + b^k\delta_{s^k})$ if $\frac{a^k}{b^k} \geqslant \frac{MU(m^k;s^k)}{MU(m^k;t^k)}$ as long as MU is well-behaved. If $MU(m;t^*) > 0$ and MU is continuous, this will be satisfied so that (HS1) will hold. Thus, if MU is well-behaved, (HS1) will be satisfied. More formally, we make the following definition:

The preference ordering $\succsim$ is U-smooth if $\succsim$ has a utility representation, U, such that:

(a) For all m and t, $MU(m;t)$ exists.

(b) $MU(m;t)$ depends continuously on m and t.

(c) For all sequences m^k, t^k, a^k, with $m^k \to m$, $t^k \to t$ and $a^k \to 0$, $\frac{1}{a^k} \cdot R(m^k; t^k; a^k) \to 0$.

Intuitively, then, if $\succsim$ is U-smooth and $t^k \to t$, the indifference curves between commodities t^k and t (holding quantities of all other goods fixed) are like straight lines with slope -1 for large k. (Notice that condition (c) is implied by a uniform bound on second-derivatives in the finite dimensional case--see Section 5 in this regard.)

It is now straightforward to check that if preferences are U-smooth, (HS1) is satisfied. For future reference, we will set this aside as a proposition.

<u>Proposition 1</u>: If $\succsim_h$ is U-smooth and for all m and t, $MU(m;t) > 0$, $\succsim_h$ satisfies (HS1).

Similar statements can be made for (HS2) and (HS3).

Before going on to discuss assumptions concerning the production sector we should make two further comments about the smoothness condition.

First, since conditions like (HS1) play no role in results on the existence of competitive equilibrium in economies with finite dimensional commodity spaces and our result includes this as a special case (T finite and discrete) it is instructive to see what these assumptions reduce to in this case.

It is easy to see that if T is discrete, (HS1) and (HS2) are each implied by the weak monotonicity of preferences. Thus, we can see that although smooth preferences are a sufficient condition for (HS1) to be satisfied, this is not necessary. In particular, smooth preferences are not the reason for the Pareto optimality of equilibrium discussed below as is the case in

Hart (12). In fact, since there are only finitely many commodities in Hart's model, (HS1) is satisfied automatically. The reason our equilibria are Pareto optimal even if preferences are not smooth, whereas the same is not true in Hart, is that we consider only competitive equilibria here. The information carried by a full price system is sufficient to guarantee that all socailly beneficial commodities are in fact produced.

Second, as pointed out above, (HS3) is essentially a boundary condition to guarantee that finite prices can be assigned to all commodities when (H4) is not satisfied. This is not unique to this model. Something like (HS3) is needed even in the case with finitely many commodities when not all commodities are available. Boundary behavior of the type exhibited by, for example, Cobb-Douglas preferences must be eliminated for a meaningful equilibrium to exist.

Examples of preferences satisfying the various assumptions above which arise in models of commodity differentiation are presented in Section 4 below.

We now turn to the description of the production sector of the economy.

Recall that there are J firms indexed by j. Firms will maximize profits taking prices as given (we will think of J as large). These profits are distributed to the households according to the shares θ_{hj}. Most of the assumptions we will make concerning the nature of production are standard. No attempt was made to get the best possible assumptions--it is possible that weaker forms are available.

For notational convenience, let $Y = \Sigma Y_j$.

(F1) (a) Y is weak* closed.

(b) There is no free production--$Y \cap M = \{0\}$.

(F2) (a) For all j, Y_j is weak* closed.

(b) For all j, Y_j is convex.

(c) For all j, $-M \subset Y_j$ (free disposal).

(d) For all j, $0 \varepsilon Y_j$.

(F3) Boundedness--For all j and all bounded $A \subset M$, $Y_j \cap (M - A - \sum_{i \neq j} Y_i)$ is bounded.

This assumption states that, even using the outputs of the other firms as inputs, the feasible production plans of firm j are bounded. A similar assumption can be found in Bewley (3).

(F4) For all j there is a countable set $T_j \subset T$, $T_j = \{t_1^j, \ldots.\}$. such that $\bigcup_n LS(t_1^j, \ldots, t_n^j)$ is dense in Y_j.

That is, any production plan for firm j can be approximated (in Y_j) using only commodities in T_j as inputs and outputs. This will be used in the construction of our approximations to $\mathcal{E}$ and will allow us to restrict trading to a finite number of commodities while still guaranteeing that production is possible. (Notice that we have implicitly made a similar assumption for consumers by assuming that the consumption set for each agent is M.) Note that this is automatically satisfied in the case where T is finite.

We will also need a smoothness condition on firms' production sets similar to the restrictions on preferences. This along with (HS1) insure that the approximating equilibria "fit together" in a nice way.

<u>(FS1)</u> For all j and all sequences $y_j^k \varepsilon Y_j$, a^k, $b^k > 0$, $s^k, t^k \varepsilon T$ such that $s^k \to t^*$, $t^k \to t^*$, $y_j^k \to y_j^*$, $-y_j^k(s^k) \geqslant b^k$, $\underline{\lim}\, \frac{a^k}{b^k} > 1$, there is a k such that

$$y_j^k + b^k \delta_{s^k} - a^k \delta_{t^k} \varepsilon Y_j \ .$$

In words, (FS1) says that if s^k is being used as in input by firm j, and t^k is nearly the same as s^k, they can be exchanged at a rate which is nearly one to one without affecting outputs. Thus, if s^k and t^k are close in their characteristics descriptions they are good substitutes in the production process. Again, this is not an additional restriction if either T is finite or the number of potential inputs is finite. In these cases, (FS1) is implied by free disposability, (F2)(c).

As in the case of preferences, (FS1) can be given a smoothness interpretation. For example, we have:

Proposition 2: If there is an $F_j : M \to \mathbf{R}$ such that $Y_j = \{y \varepsilon M | F_j(y) \leq 0\}$ and F_j is U-smooth with $MF_j(y;t) > 0$ for all y and t, Y_j satisfies (FS1).

The proof is similar to that of Proposition 1.

Again, when (H4) is not satisfied, we will need to make an assumption about production analogous to (HS3). To this end, we have:

(FS2) There is a $\underline{t}$ such that for all j and all sequences $y_j^k \varepsilon Y_j$, a^k, $b^k > 0$, $t^k \varepsilon T$ such that $t^k \to t^*$, $y_j^k \to y_j^*$, $-y_j^k(t^k) \geq b^k$ and $\underline{\lim} \frac{a^k}{b^k} = \infty$, there is a k such that

$$y_j^k + b^k \delta_{t^k} - a^k \delta_{\underline{t}} \varepsilon Y_j$$

Restated, this says that there is some common factor, $\underline{t}$, which can always be substituted for any other input at a bounded rate. That is, the marginal rate of technical substitution between t^k and $\underline{t}$ is never infinite. However, it may be very large.

We turn now to a presentation of the results.

Section 3 Results

In this section, results on the existence and continuity of equilibria for the model discussed in Section 2 are presented. In addition, some examples which suggest the limitations of possible extensions of the results are discussed. Complete proofs of the main propositions are included as Section 5.

First, we discuss sufficient conditions for the existence of competitive equilibrium. We give two sets of sufficient conditions for those economies in which (H4) is satisfied, one for those cases in which it is not.

We will use $\mathcal{E}$ to denote the economy (preferences, endowments, production sets and firm ownership) outlined in Section 2.

Theorem 1

(a) If the consumption sector of $\mathcal{E}$ satisfies (H1), (H2), (H3), (H4) and (HS1) and the production sector satisfies (F1), (F2), (F3), (F4) and (FS1), $\mathcal{E}$ has a competitive equilibrium supported by continuous prices (i.e., $p^* \varepsilon \ \mathcal{C}(T)$).

(b) If the consumption sector satisfies (H1)$'$, (H2), (H3), (H4) and (HS1) and the production sector satisfies (F1), (F2), (F3), (F4) and (FS1), $\mathcal{E}$ has a competitive equilibrium supported by continuous prices.

(c) If the consumption sector satisfies (H1)$'$, (H2), (H3) and (HS1), the production sector satisfies (F1), (F2), (F3), (F4) and (FS1) and there is some $\underline{t}$ such that (H1), (HS3) and (FS2) are satisfied at this $\underline{t}$, $\mathcal{E}$ has a competitive equilibrium supported by continuous prices.

Thus, if preferences and production sets are sufficiently smooth, an equilibrium with continuous prices exists. More formally, the following corollary is immediate.

<u>Corollary</u> If $\mathcal{E}$ satisfies (H1), (H2), (H3), (H4), (F1), (F2), (F3), (F4) and the assumptions of Propositions 1 and 2 are satisfied, $\mathcal{E}$ has a competitive equilibrium supported by continuous prices.

We should point out here that the additional assumptions we have made to cover the case of infinite T are minimal. In fact, because of the way we have stated the smoothness conditions, the only assumptions which are more restrictive than the usual conditions for existence with finite T are (H4)--which can probably be relaxed--and (F3).

The continuity of equilibrium prices is very satisfying in the sense that it implies that under the conditions of the theorem, nearby commodities are good substitutes. That is, although it is always true that $\frac{p^*(t_1)}{p^*(t_2)}$ units of commodity t_2 is, at the margin, a perfect substitute for one unit of good t_1, the fact that prices are continuous shows that nearby commodities are good substitutes at a more basic level. This justifies our earlier contention that the weak* continuity and smoothness, (HS1), of preferences can be correctly interpreted as conditions on the substitutability of commodities with similar characteristics.

In fact, the proof of Theorem 1 shows more than just that equilibria exist. Since the proof proceeds by approximation, the equilibrium shown to exist can be very well approximated by the equilibria of economies with a large, but finite, number of commodities. This is an important point as far as interpretation is concerned. That is, with the view that $\mathcal{E}$ is a limiting ideal, those equilibria of $\mathcal{E}$ which are approximable are the only ones which matter in the sense that these are the only ones which give us any information about equilibria in economies with large numbers of differentiated commodities. In this interpretation, any other equilibria of $\mathcal{E}$ are solely artifacts of the specification with infinitely many commodities and are, in this sense, less important. We will go into this in more detail in the discussion below.

Notice that by applying Theorem 1 (a) or (b) to Theorem 1 (c) we can guarantee that an equilibrium with continuous prices exists when trade is restricted to those commodities in supp (Σe_h) without assuming (HS3) and (FS2). Thus, the addition of (HS3) and (FS2) allow us to extend this result to all of T even if it is not possible to produce all commodities.

Concerning the optimality of equilibrium, we have the usual result:

Proposition 3 If $(p^*;m^*,y^*)$ is a competitive equilibrium for the economy $\mathcal{E}$, p^* is bounded, and $\mathcal{E}$ satisfies (H1) or (H1)′, (H2)(a) and (H2)(b), (m^*, y^*) is Pareto Optimal.

The usual argument for establishing the inverse to Proposition 3, that all optima are equilibria, will not work in our case. That is, the agrument usually given relies as its basis on a form of the separating hyperplane theorem. In our case, the relevant result is the Hahn-Banach Theorem in one of its versions. Unfortunately, the Hahn-Banach Theorem does not hold for our model since M, the consumption sets of individual agents, has an empty interior. That this could cause difficulties was first noted in Debreu (7). This point is discussed in more detail in Section 4.

Before turning to a discussion of the continuity properties of $\mathcal{E}$ we will answer a question raised by Theorem 1: If $(p^*;m^*,y^*)$ is a competitive equilibrium of $\mathcal{E}$, is it necessarily true that p^* is continuous? At this level of generality, the answer is no. If, for example, we consider the case of pure exchange, then, starting from a competitive equilibrium we can arbitrarily raise prices on sets which have $\sum_h e_h$ — measure zero and retain an equilibrium. Of course, the same does not hold for lowering prices. Despite this, the question can be answered in the affirmative for those equlibria which arise as the units of economies with finitely many commodities. To see this we will need a few concepts.

Let $T^n = \{t_1, \ldots, t_n\}$ be an increasing sequence of subsets of T such that $T^* = \bigcup_n T^n$ is dense in T. Consider a sequence of economies

$$\mathcal{E}^n = ((Y_j^n)_{j=1}^{J_n}, (\succsim_h^n, X_h^n, e_h^n, \theta_{hj}^n)_{h=1}^{H_n}).$$

(The usual notation is followed here, X_h^n is h's consumption set.)

We will say that $\mathcal{E}^n$ is a <u>T^n based approximating sequence</u> for if:

(1) For all n, $H_n = H$, $J_n = J$, $\theta_{hj}^n = \theta_{hj}$.

(2) For all n and all h, $X_h^n = M^n = M \cap LS(t_1, \ldots, t_n)$.

(3) For all n and all h, $\succsim_h^n = \succsim_h \cap M^n \times M^n$.

(4) For all h, $e_h^n \varepsilon M^n$, $e_h^n > 0$ and $e_h^n \to e_h$.

(5) For all n, $Y_j^n = Y_j \cap LS(t_1, \ldots, t_n)$.

(6) For all j, $\bigcup_n Y_j^n$ is dense in Y_j.

(7) For all n and all $t \varepsilon T^n$, $\sum_h e_h^n(t) > 0$.

Roughly, then, the economy $\mathcal{E}^n$ is just $\mathcal{E}$ with trading restricted to T^n.

With this definition, the proof of Theorem 1 can be easily outlined. The proof that proceeds by constructing a sequence T^n and a T^n based approximating sequence for $\mathcal{E}$, $\mathcal{E}^n$, such that the competitive equilibria of $\mathcal{E}^n$ have as a limit point a continuous price competitive equilibrium for $\mathcal{E}$.

Letting CE(•) denote the equilibrium correspondence, we will say that a competitive equlibrium $(p^*; m^*, y^*) \varepsilon CE(\mathcal{E})$ is <u>approximable</u> if there is a sequence T^n as above, and a T^n based approximating sequence, $\mathcal{E}^n$, such that (p^*, m^*, y^*) is a limit point of the $CE(\mathcal{E}^{n_k})$ such that:

(1) $m_h^{n_k} \to m_h^*$ for all h.

(2) $y_j^{n_k} \to y_j^*$ for all j.

(3) For all $t \varepsilon T$, there is a sequence $t^k \varepsilon T^{n_k}$ such

that $p^*(t) = \lim_{k\to\infty} p^{n_k}(t^k)$.

Thus, an equilibrium is approximable if it can, in any reasonable sense, be thought of as arising as a limit of equilibria of economies in which trading is restricted to only finitely many commodities. As argued above, these are the equilbria of $\mathcal{E}$ which are of principal interest since they contain the most information about economies with large but finite numbers of commodities. Of course, Theorem 1 guarantees the existence of approximable equilibria. In fact, something slightly stronger than (3) can be said about the equilibria of Theorem 1: for any t, $p^*(t) = \lim_k p^{n_k}(t^k)$ for any sequence of $t^k \varepsilon T^{n_k}$ such that $t^k \to t$. We have:

<u>Proposition 4</u>: Under the assumptions of Theorem 1(a), if the competitive equilibrium of $\mathcal{E}$, $(p^*;m^*,y^*)$, is approximable, p^* is continuous. (Similar results hold under the assumptions of Theorem 1(b) and (c)).

Restated, this proposition justifies the statement that, in economies with a large number of related commodities, good substitutes must have similar prices in equilibrium. This is true independent of the fact that vastly differing quantities of the commodities might be available. (For example, t could be an atom with no other nearby atoms.) Intuitively, each potential buyer of commodity t considers nearby commodities as alternatives, if t is priced even slightly higher then these alternatives, no t can be sold. This is in sharp contrast to the usual intuition about economies with infinitely many commodities arising from models based on control theory. This is commented on in more detail in Section 4.

In fact, something similar to Proposition 4 holds in institutional frameworks much richer than the competitive framework considered thus far. As long as consumers act as price takers a similar result will hold in equilibrium independent of how firms compete. Consider an economy with a consumption sector as in $\mathcal{E}$ of Section 2. Assume that consumers take both prices and profit shares from firms as given when they make their decisions. Then, we have:

Proposition 5: If the consumption sector satisfies (H1), (H2), (H3) and (HS1) and $(m_1,\ldots,m_H)$ is such that for some $(p^*;w_1,\ldots,w_H)$, with $w_h > 0$ for all h, and m_h maximizes $\succsim_h$ on $\beta(p^*;w_h)$ for all h, then:

(a) If t^* is at atom of $\sum_h m_h$ and $t_k \to t^*$, $p^*(t^*) \leq \underline{\lim}\, p^*(t_k)$.

(b) If t^* and t_k are all atoms of $\sum_h m_h$ and $t_k \to t^*$, $p^*(t^*) = \lim p^*(t_k)$.

To better understand this result and its implications we will give one interepretation of the proposition: consider a game among firms in which one of the strategic choice variables is the characteristics of the commodity they choose to produce. Firms aim to maximize profits calculated from sales to a consumption sector in which individuals take firms' prices as given and maximize utility subject to a budget constraint which includes, as a part of its specification, the distributed profits of the firm. (We set $p(t) = \infty$ if t is not chosen by any firm.)

In this situation, the Proposition states that if a firm wishes to market a product similar to those chosen by other firms, it must either charge a price no higher than its nearby competitors' or produce a very small (non-atomic) quantity. Thus, the only way in which the firm has any "upward" power over its price is to isolate itself by producing a commodity which has no close substitutes.

This is a very satisfying result in that it has a great deal of intuitive appeal. It lends strength to the claim the the approach presented here is in fact a reasonable way to model the consumption sector of an economy in which commodity differentiation plays a key role.

Note that an analogue of Proposition 5 will carry over to limiting arguments involving sequences of consumption sectors as long as (HS1) is satisfied uniformly over all considered preferences. (Given any t and $\varepsilon > 0$, there is a $\delta > 0$ such that if $d(s,t) < \delta$, $|MRS_{ts} -1| < \varepsilon$ for all allowed preferences.) Clearly, this will be a useful property for limiting results.

We should also point out that Proposition 5 has the following restatement:

If m maximizes $\succsim_h$ on $\beta(p;w_h)$ with $w_h > 0$ and $p(t^*) > \overline{\lim}\ p(t^k)$ for a sequence t^k with $t^k \to t^*$, then $m(t^*) = 0$. Thus, as in the location model, if firm j is producing commodity t^* and charging $p(t^*)$, by charging $\alpha p(t^*), \alpha < 1$, and producing a commodity with a description sufficiently close to t^*, a competing firm can usurp all of j's sales. (This is true as long as j does not respond--the standard Nash assumption.) Thus, this is a simple consequence of the smoothness of preferences. (This is not true of other representations of preference with an infinitely many commodities--see the remarks in Section 4.)

We turn now to a study of the continuity properties of the equilibrium correspondence. To simplify this, we will restrict attention to the pure exchange case and consider only changes in the endowment distributions.

Let CPCE($\mathcal{E}$) be competitive equilibria of $\mathcal{E}$ for which the prices are continuous and let M^* be the collection of H-tuples of endowment distributions satisfying (H1)′(b) and (H4). Then, we have:

Theorem 2: If for each household h, $\succsim_h$ satisfies (H1)′(a), (H2), (H3) and (HS2) the map CPCE(•) is non-empty, compact-valued and upper hemi-continuous on bounded subsets of M^*.

The continuity of the equilibrium price correspondence is in the topology of uniform convergence--$p^n \to p$ if and only if $\sup_{t\varepsilon T} |p^n(t) - p(t)| \to 0$. This is a particularly strong result and implies that one can approximate p(t) by $p^n(t^n)$ for n sufficiently large and t^n sufficiently close to t.

To gain more insight into the workings of the model and identify possible limitations on extensions of the results, we will examine three examples.

First, we present an example which shows that the smoothness assumptions are indeed necessary to guarantee the continuity of prices in equilibrium. This is a one consumer example with no production.

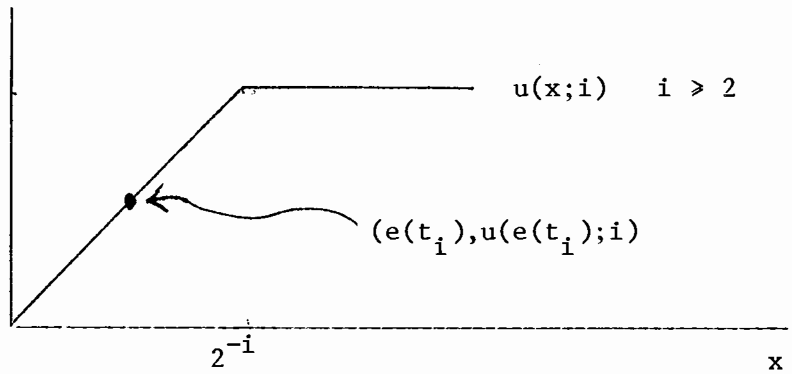
Example 1: Let $T = \{0,1,1/2,1/3,...\} = \{t_0,t_1,t_2,...\}$ (i.e., $t_n = 1/n$ for $n \geq 1$). Our agent will have a utility function given by

$$U(m) = \sum_{i=0}^{\infty} u(m(t_i);i)$$

where $$u(x;i) = \begin{cases} x & 0 \leq x \leq 2^{-i} \\ 2^{-i} & 2^{-i} < x \end{cases} \quad \text{for } i \geq 2,$$

$$u(x;1) = x \quad \text{for all } x, \text{ and}$$

$$u(x;0) = 0 \quad \text{for all } x.$$



Thus, the agent is eventually satiated in all commodities except commodity 1. The level at which he becomes satiated depends on the commodity and shrinks to 0 at commodity 0 about which he is totally indifferent.

We let the agent's endowment be given by e:

$$e(t_i) = 2^{-i-1} \text{ for } i \geqslant 1,\ e(t_0) = 1.$$

We see that $MU(e;t) = 1$ if $t \neq t_0$, $MU(e;t_0) = 0$. It is straightforward to check that all of the conditions of Theorem 1 (a) are satisfied other than (HS1) and that no equilibrium with continuous prices exists for this one agent economy. There is, however, an equilibrium with discontinuous prices given by $(p^*;e)$ where $p^*(t) = 1$ if $t \neq 0, p^*(0) = 0$.

At first glance, one might think that it is either the satiation or the kink in $u(\cdot;i)$ which is causing the problem. This is not true, however. Similar examples can be constructed having neither of these features.

Thus (HS1) is necessary to guarantee that an equilibrium with continuous prices exists. This example suggests that it might be possible to prove that an equilibrium without continuous prices exists without assuming (HS1). The problem lies in guaranteeing that the prices of the equilibria of the approximating sequence of economies converge, in some sense, to a set of prices which are sensible for the limit economy. We can rephrase this as: Given the equilibria of the approximating economies, how should prices be set for the limit economy?

Under the assumptions of Theorem 1, this is not a problem. There is a subsequence, n_k, such that given any t in T and any two sequences $\{s^k\}$ and $\{t^k\}$ in T^{n_k}, converging to t, $\lim p^{n_k}(t^k) = \lim p^{n_k}(s^k)$. Thus, we can just set $p^*(t)$ equal to this common limit. If $p^{n_k}(s^k)$ and $p^{n_k}(t^k)$ have different limits, however, there is no unambiguous way of setting the price at t. This would not matter if exactly how the price is set is unimportant. Unfortunately, this can be crucial as Example 2 shows.

Example 2: Again, this is a one-agent economy with no production. We will set $T = [0,2]$ and $e = \delta_1$, i.e., the agent's endowment is a point mass at $t = 1$. Let u be any strictly concave, differentiable, strictly increasing function with $u(0) = 0$ and $u'(0) < \infty$. For any bounded function $x(t)$ on T, define $u(x) = \int_0^2 u(x(t))dt$. Now, if $m \varepsilon M$, define $y(m) = \{x | x \geq 0$ and for all s, $\int_0^s x(t)dt \leq m[0,s]\}$. Finally, define V on M by $V(m) = \sup\limits_{x \varepsilon y(m)} U(x)$.

It is shown elsewhere, (15), that V defined in this way satisfies all the restrictions on preferences of Theorem 1(c) other than (HS1).

It is straightforward to check that the only equilibrium of this one agent economy which is approximable is given by $(p^*;e)$ where

$$p^*(t) = \begin{cases} u'(0) & 0 \leq t < 1 \\ u'(1) & 1 \leq t \leq 2 \end{cases}$$

The problem occurs at t=1. One way to approximate this economy is to restrict trading to $T^n = \{\frac{k}{n};\ k=0,\ldots,2n\}$ at the n-th iteration and let $e^n = \delta_{\frac{n+1}{n}}$. We see that in equilibrium

$$p^n(t) = \begin{cases} u'(0) & t = \frac{k}{n} \quad k \leq n \\ u'(\frac{n+1}{n}) & t = \frac{k}{n} \quad k \geq n+1 \end{cases}.$$

We can see that there are two possible ways to set p^* at 1. We could choose $p^*(1) = u'(1)$ giving us the equilibrium above, or we could set $p^*(1) = \lim p^n(1) = u'(0)$. Unfortunately, this second path leads us astray:

$$(p;e) \text{ with } p = \begin{cases} u'(0) & 0 \leq t \leq 1 \\ u'(1) & 1 < t \leq 2 \end{cases}$$

is not an equilibrium. The agent's demand at these prices is δ_0, he buys only commodity 0.

Although we can fix the problem in this case, this phenomenon is quite general and there does not seem to be any general method for resolving it.

As a final example, consider the following line of reasoning: By assuming that marginal utilities are continuous, we have been able to conclude that equilibrium prices are continuous. Could we, by assuming that marginal utilities are smooth, if T is such that this is sensible, guarantee the existence of an equilibrium with smooth prices? That this is not possible in general is shown by Example 3. Again, no production is allowed.

Example 3: We will let T be the unit interval [0,1]. There are two agents with preferences defined by the utility functions:

$$U_1(m) = \int_0^1 t \, dm(t)$$

$$U_2(m) = \int_0^1 (1-t) \, dm(t) .$$

Thus, agent 1 likes goods with indices near 1 better while agent 2 prefers those with indices near 0. If we let e_1 be Lebesgue measure on [0,1/2] and e_2 be Lebesgue measure on [1/2,1] (i.e., $e_1(B) = \int_{B \cap [0,1/2]} 1\, dt$), it is straightforward to check that $m_1 = e_2$, $m_2 = e_1$ and $p(t) = \max(t, 1-t)$ is (up to sets of measure 0) the only equilibrium.

Here, $MU_1 = t$ and $MU_2 = 1-t$ independent of m so that the marginal utilities are as smooth as is possible, yet no equilibrium with smooth prices exists. This is primarily due to complementary slackness conditions. That is, utility maximization requires that $\frac{MU(m;t)}{MU(m;t')} \leq \frac{p(t)}{p(t')}$ as long as commodity t' is being purchased (i.e., $t' \varepsilon$ supp m). However, equality will hold only if commodity t is being purchased as well. Since there are infinitely many commodities, strict inequality is possible. Intuitively then, if in equilibrium all agents purchase commodity t^*,

$$p(t) = p(t^*) \cdot \max_h \left[\frac{MU_h(m_h;t)}{MU_h(m_h;t^*)}\right].$$

Thus, p need not be smooth no matter how well-behaved the MU's of the various agents are.

From these considerations, it is clear that the example is quite robust. The only really important feature is that MU_1 and MU_2 intersect transversally at $t = 1/2$. It is possible that some condition on the disperseness of preferences might smooth out this kink, but this is unsure.

Section IV

Complements and Remarks

In this section we present a few remarks concerning the relationship between the model analyzed here and other literature and point out some generalizations of the results of Section 3.

As a first step, we will fulfill an earlier promise by giving two examples of consumption sectors satisfying the assumptions of Theorem 1 which have appeared in the literature on commodity differentiation.

(1) The first example we will consider is one version of Hotelling's location model (14). Many versions of this model have appeared and a complete survey of this literature would be beyond the scope of this paper. We will note that the version we will discuss is a differentiated commodity rather than a strictly locational interpretation.

Not all of the locational models have consumption sectors as described here. For simplicity, it is often assumed that the consumption decision is either unspecified or exogenous (6). What is described here is, I believe, a reasonable description of a complete model of consumer choice in these models. There is some overlap between our remarks and those of Novshek (20).

We will consider a model in which locations are along the line segment $[0,1]$. The circle and higher dimensional "location" problems could be described similarly.

We will also suppose that there are L "other" goods. Thus, $T = [0,1] \cup \{t_1,\ldots,t_L\}$. Then, M can be represented as the product of the non-negative distributions on $[0,1]$ with $\mathbf{R}^L_+$. We will write this as $M = Z \times \mathbf{R}^L_+$.

In most locational treatments, for each $t\varepsilon[0,1]$ there is a consumer whose most preferred location or commodity in the interval is t. (Although Theorem

1 as presented covers only those cases with finitely many customers, it can be extended to models with infinitely many agents--see remark (6) below.) For each $t\varepsilon[0,1]$ let $f_t(s)$ be a continuous, non-negative, real-valued function which is maximized at $s=t$. If $z\varepsilon Z$, define $U_t(z) = \int_0^1 f_t(s)dz(s)$ and if $m = (z,x)\varepsilon M$, let $U_t(z) = V(x;u_t(z))$ where V is any standard utility function on $\mathbf{R}_+^{L+1}$. Thus, we have assumed that all agents have the same preferences over the "other" goods. With these preferences, agent t prefers good t to all others in [0,1] since this is were f_t is maximized. Also, since $u_t(z)$ is linear in z, agents will generally buy only one of the commodities in [0,1], that one which maximizes $\frac{f_t(s)}{p(s)}$.

In the standard location model, $f_t(s) = a - c|s-t|$ where c is interpreted as the unit transportation cost. More generally, a form like $f_t(s) = a - cg(|s-t|)$ allows for transportation costs which are non-linear in distance.

It is straightforward to check that as long as f_t is continuous, the preferences defined above satisfy (H3), (HS1) and (HS2). Under reasonable conditions on V, (HS3) will be satisfied as well. Thus, this form of the location model gives us one example of preferences which satisfy the assumptions of our model.

(2) The second example is the orginal characteristics model of Lancaster (16). Here, there will be L characteristics which describe commodities. The range of possible amounts of the i-th characteristic is given by I_i, a closed bounded interval. Thus, $T = I_1 \times \ldots \times I_L$. In the original formulation of this model, agents care only about the total amount of each characteristic available under a consumption plan. Clearly, the total amount of characteristic i available under the plan m is given by $\int_T t_i dm(\underset{\sim}{t}) \equiv u_i(m)$. Thus, preferences are of the form $U(m) = V(u_1(m),\ldots,u_L(m))$ where $V:\mathbf{R}_+^L \to \mathbf{R}$ is a typical utility function on L commodities. Note that, as in (1), $u_i(m)$ is

linear in m. Again, it is straightforward to check that these preferences satisfy (H3), (HS1) and (HS2).

More generally, if $f_1,\ldots,f_r$ are non-negative, real-valued, continuous functions on T, $u(m) = V(\int_T f_1(t)dm(t),\ldots,\int_T f_r(t)dm(t))$ can be seen to satisfy (H3), (HS1) and (HS2). This holds for any r and T: Again, u will satisfy (HS3) as well as long as V is suitably restricted. We can see that the preferences in both (1) and (2) are of this form.

This points out one similarity between the location and characteristics models of commodity differentiation. The preferences in both are, in an important way, linear in their treatment of the differentiated sector.

The difference is that in the locational example, different consumers have different linear forms whereas in the characteristics model, there is more agreement about the relative values of the different commodities.

In this sense, we can see that the more recent work on the characteristics model, (17), more closely resembles the location model as outlined above.

The preferences analyzed here are not restricted to be of this form but fall under the "general utility" heading. One insight that we obtain here is exactly how the consumption sector of the location and characteristics models arise as special restrictions on preferences from the general utility approach. This also highlights a common fallacy, namely, the contention that the specification of T as a line segment necessarily implies that a firm has only two competitors. This is often seen to be a weakness of the locational models because of its restrictive nature. We can see that this contention is only true for very special restrictions on preferences. That is, the number of competitors a firm faces is determined by preferences and is only affected by the underlying specification of T through this relationship. (Of course, it is true that these restrictions on preferences are usually made in treatments of the location model.)

We turn now to a few remarks concerning the relationship between the model analyzed here and other competitive models featuring infinitely many commodities.

(3) First, we consider the analysis of competitive equilibrium with commodity space L_∞ presented in Bewley (3). Note that if T is a compact subset of R^L and we denote Lebesgue measure on T by λ, $L_\infty(T,\lambda)$ provides a special case of Bewley's model. This framework also is a special case of the model considered here with the additional restriction that the only distributions which consumers purchase are those which have bounded densities with respect to λ.

If we consider the case of pure exchange, under suitable restrictions on preferences and endowments, both formulations give rise to results on the existence of competitive equilibrium. If preferences are Mackey continuous, Bewley shows that there is a competitive equilibrium with prices lying in L_1. Thereom 1 of Section 3 shows that if preferences can be extended to all of M such that (H3) and (HS1) are satisfied, there is a competitive equilibrium with continuous prices. (Note that the equilibrium distributions will have densities if endowments do follows from the equilibrium conditions.)

Since our result makes a stronger statement about prices it should not be surpirsing that our assumptions on preferences are more restrictive. In fact, it has been shown elsewhere (15), that if preferences can be extended to satisfy (H3) and (HS1), the restriction to L_∞^+ is Mackey continuous. This explains the stronger result in prices. To see how much more restrictive our assumptions on preferences are we will give an example.

Example 4: Let T be the unit interval, [0,1] and for $x(t)\varepsilon L_\infty$, define

$$U(x) = \int_0^1 u(x(t))dt \text{ where u is increasing and strictly concave.}$$

Bewley (2) has shown that U is Mackey continuous. It is straightforward to check that U cannot be extended to all of M such that (H3) is satisfied.

This should not be surprising as we know that if we consider a one-agent economy with preferences given by U and an endowment given by the density which is 1 on [0,1/2] and 2 on [1/2,1], equilibrium prices are given by:

$$p^*(t) = \begin{cases} u'(1) & t \leq 1/2 \\ u'(2) & t > 1/2 \end{cases} \quad \text{if u is differentiable.}$$

Thus, this economy cannot satisfy the assumptions of Theorem 1. We can, however, alter these preferences only slightly such that the assumptions of Theorem 1 are satisfied. For $x \varepsilon L_\infty^+$, define $F_x(s) = \int_0^s x(t)dt$. If we let

$$U_\varepsilon(x) = \int_0^1 u(\tfrac{1}{2\varepsilon}[F_x(t+\varepsilon) - F_x(t-\varepsilon)])\, dt \quad \left(F_x = \begin{cases} 0 \text{ for} & t < 0 \\ F_x(1) & t > 1 \end{cases}\right),$$

for $\varepsilon > 0$, U_ε can be extended to M in such a way as to satisfy the assumptions of Theorem 1. That is, if we let

$$U_\varepsilon(m) = \int_0^1 u(\tfrac{1}{2\varepsilon}[m[0,t+\varepsilon] - m[0,t-\varepsilon]])\, dt,$$

these preferences satisfy both (H3) and (HS1). Thus, although the additively separable case is not covered by Theorem 1, economies with preferences arbitrarily nearby (ε small) are covered.

The interpretation of U_ε is clear, the increment to utility at t depends on the local average of consumption around t rather than the "instantaneous" consumption at t (x(t)). It is only the limiting case where the increment depends on the instantaeous which causes problems for Theorem 1. For $\varepsilon > 0$, the competitive prices are given by

$$p^*_\varepsilon(t) = MU_\varepsilon(e;t) = \frac{1}{2\varepsilon}\int_{\max(0,t-\varepsilon)}^{\min(1,t+\varepsilon)} u'(\frac{1}{2\varepsilon}[F_e(s+\varepsilon) - F_e(s-\varepsilon)])ds$$

where e is the agents' endowment distribution. This clearly depends continuously on t. In fact, if we choose e to be as in Example 4, we see that $p^*_\varepsilon \to p^*$ as $\varepsilon \to 0$. Note that this example gives us another example of preferences which satisfy all the assumptions of Theorem 1.

This example suggests the following methodological point: If the properties of demand outlined in Proposition 5 and the comments following it are desireable properties for all models of commodity differentiation (it is hard to argue otherwise), the computational convenience of additively separable preferences must be adandoned. With additively separable preferences T is merely an index set--the closeness of s and t implies nothing about their substitutability. In fact, in the example given, for <u>any</u> s and s', the commodites s and s' are equally good substitutes for t (cf. (10))!

(4) Concerning the last remark we should point out the relationship between the result on the existence of equilibrium presented here and the general existence theorems presented in (1) and (22). These results are basically generalizations of the arguments of Bewley (3).

The first thing to note is that this argument will not work in our case since M has an empty interior (even in the norm topology) in most cases of interest. The examples of remarks (1)-(3) are in this category. Second, even in those cases included in our formulations where these more general arguments are appropriate they give rise to equilibria supported by price systems in the norm dual of $\mathcal{M}$. Since no useful characterization of this space exists, this approach is not fruitful in our case.

As a final remark on this point we recall that the possibility of problems caused by the non-empty interior of consumption sets in models with infinitely many commodities was first pointed out in Debreu (7). The fact that general existence arguments will not work in our case is just one aspect of this problem.

Another aspect of this problem arises as Example 1 of Section 3. That is, in a one-person economy, the problem of existence reduces to one of the existence of a price system which separates the endowment from the preferred set. If the consumption set has a non-empty interior and preferences are continuous the existence of such a separator is guaranteed by the Hahn-Banach Theorem. Moreover, the separator lies in the dual (under the topology such that preferences are continuous and the consumption set has a non-empty interior) of the underlying space. When the consumption set has an empty interior, this no longer holds. Examples 1 and 2 provide examples of this phenomenon. To solve the existence problem in situations such as this, additional restrictions, such as those in Theorem 1, must be placed on the allowed preferences.

We will conclude this section with two comments on the extension of the results included here.

(5) Theorem 1 has been generalized to include models like that presented in Example 2 (see (15)). It is shown that if T is an interval in the real line the smoothness restrictions on preferences can be dropped as long as preferences satisfy a stronger monotonicity property. that is, if we define $mTDm'$ by

$$mTDm' \text{ if and only if } m[0,t] \geq m[0,t] \text{ for all } t$$

and preferences satisfy

$$mTDm' \Rightarrow m \succsim_h m',$$

an equilibrium exists. TD is a form of stochastic dominance in the sense that mTDm' iff those commodities purchased under m are more concentrated around 0. It is easy to see that the preferences defined in Example 2 satisfy this restriction. Example 2 also shows that under these conditions equilibrium prices need not be continuous. However, they will be non-increasing and continuous on the right.

(6) Finally, Theorem 1 has been generalized to cover the case of infinitely many consumers for the case of exchange (see (15)). An equilibrium with continuous prices exists as long as the restrictions on preferences of Theorem 1 hold uniformly over all consumers. This is a type of compactness of characteristics assumption similar to those needed to ensure existence even in the case with finitely many commodities. Thus, this restriction is not surprising.

This result is relatively unique. Results on the existence of competitive equilibrium with both infinitely many consumers and infinitely many commodities are rare. Exceptions to this which are particularly relevant are Mas-Colell (18) and Bewley (2).

Section V

Proofs

In this section, we present the proofs of the results of the earlier sections. We restrict the presentation to outlines for those results which are straightforward.

Propositions 1 and 2

The proof of these propositions is by standard Taylors' Series techniques and will not be included here. Since condition (c) of the definition of U-smooth appears unusual we will spend a moment to give sufficient conditions for it to be satisfied. We restrict attention to a discussion of preferences. A similar statement concerning the hypotheses of Proposition 2 can be made.

Suppose h's preferences can be represented by the real-valued function U.

$$\text{Let } H(m;t;a) = U(m + a\delta_t) \text{ for } m \in M,\ a \geq 0,\ t \in T.$$

If H, as a function of a, is twice differentiable, we have:

$$\frac{R(m;t;a)}{a} = \frac{1}{2}\, aH_{33}(m;t;\theta a) \text{ where } H_{33} = \frac{\partial^2 H}{\partial a^2} \text{ and } 0 \leq \theta \leq 1$$

as a standard form for the remainder. Thus, as long as H_{33} is bounded we see that $\frac{R(m;t;a)}{a} \to 0$ as $a \to 0$.

Before turning to the proof of Theorem 1, we need a few definitions and preliminary results based on the work of Mas-Colell (18).

Let T^n be a sequence of a compact subsets of T, p^n a sequence of non-negative, continuous, real-valued functions on T^n. If $p \in \mathcal{C}(T)$ is non-negative, we write $(T^n,p^n) \to (T,p)$ if $T^n \to T$ in closed convergence and for all sequences n_k, t^{n_k}, with $t^{n_k} \varepsilon T^{n_k}$ and $t^{n_k} \to t$, $p^{n_k}(t^{n_k}) \to p(t)$. We have:

Lemma 1 (Mas-Colell): Suppose $(T^n,p^n) \to (T,p)$ and for a bounded sequence m^n, with supp $m^n \subset T^n$, $m^n \to m$ then $p^n \cdot m^n \to p \cdot m$. (Of course $p^n \cdot m^n = \int_{T^n} p^n(t)dm^n(t)$.)

Further, we will define (T^n,p^n) to be equicontinuous if for all $\varepsilon > 0$, there is a $\delta > 0$ such that for all n and all t, $t' \varepsilon T^n$ with $d(t,t') < \delta$, $|p^n(t) - p^n(t')| < \varepsilon$. This is the analogue of the normal definition of equicontinuity defined on functions with restricted domains. We have:

Lemma 2 (Mas-Colell): Let T^n, p^n be as above with $T^n \subset T^{n+1}$ and $T^n \to T$ in closed convergence. If (T^n,p^n) is equicontinuous and the p^n are uniformly bounded, there is a subsequence n_k and a $p \varepsilon \mathcal{C}(T)$ such that $(T^{n_k},p^{n_k}) \to (T,p)$.

We turn now to the proof of Theorem 1. Since the three parts of the Theorem have similar proofs, we only prove part (a) in detail. Descriptions of the modifications necessary to prove parts (b) and (c) are also given.

Theorem 1. Proof:

The strategy of the proof is to construct a sequence of finite subsets of T, T^n, and a T^n based approximating sequence of economies $\mathcal{E}^n$ such that $CE(\mathcal{E}^n)$ have as a limit point a competitive equilibrium for $\mathcal{E}$. If $(p^n;m^n,y^n) \varepsilon CE(\mathcal{E}^n)$, that the (m^n,y^n) are bounded follows from (F3) and the equilibrium conditions. We can therefore assume that they converge. The key then is to show that the (T^n,p^n) are bounded and equicontinuous. We now begin the formal constructions.

By (F4) and (H1), there is a countable dense subset of T, $T^* = \{t_1,....\}$ with the following two properties:

(1) For all j, $\bigcup_n LS(t_1,....,t_n)$ is dense in Y_j.

(2) For all h, $t_1 \varepsilon$ supp e_h and h's preferences are strictly monotone at t_1,

$$m + \alpha\delta_{t_1} \succ_h m \quad \text{for all } h,\ \alpha > 0.$$

For this T^*,

let $T^n = \{t_1,\ldots,t_n\}, \mathcal{M}^n = LS(t_1,\ldots,t_n)$, $M^n = M \cap \mathcal{M}^n$, $Y_j^n = Y_j \cap \mathcal{M}^n$, $\succsim_h^n = \succsim_h \cap M^n \times M^n$, and $\theta_{hj}^n = \theta_{hj}$. To complete the approximation, we need only construct approximate endowments. There are several possible ways to do this. The only requirements we make are that:

(a) For all n,h, $e_h^n(t_1) > 0$.

(b) For all n, all $t \varepsilon T^n$, $\sum_h e_h^n(t) > 0$.

(c) For all h, $e_h^n \to e_h$.

One way to do this is to take disjoint measureable sets B_i^n,i=1,...,n with $\bigcup_i B_i^n = T$, $t^i \varepsilon B_i^n$ for all i,n, sup diam $(B_i^n) \to 0$, and B_i^n contains the ε_n-neighborhood of t^i for some $\varepsilon_n > 0$ ($\varepsilon_n \to 0$ necessarily). We then define $e_h^n = \sum_{i=1}^n e_h(B_i^n)\delta_{t_i}$.

Let e_h^n, h=1,...,n=1,2,... be any collection of sequences satisfying (a), (b), and (c) above. Note that by (H4) (b) can be satisfied.

Now, consider the sequence of economies $\mathcal{E}^n$ described by the consumption sets $-M^n$, preferences $- \succsim_h^n$, production sets $-Y_j^n$, firm shares $-\theta_{hj}^n$, and endowments $-e_h^n$. This gives rise to n commodity economy. It is straightforward to check that $\mathcal{E}^n$ satisfies all of the assumptions of Debreu (9) and thus a quasi-equilibrium $(p^n;m^n,y^n)$ with $p^n \neq 0$ exists.

Since $-M^n \subset Y_j^n$, $p^n \geq 0$. Therefore $p^n \cdot e_h^n > 0$ for some h. Thus, for this household $p^n \cdot [e_h^n + \sum_j \theta_{hj} y_j^n] > 0 = \min p^n \cdot M^h$ ($p^n \cdot y_j^n \geq 0$ for all j since $0 \varepsilon Y_j^n$). Hence, since for this household preferences are strictly monotone for commodity t_1, we can conclude that $p^n(t_1) > 0$. Since $e^n_h(t_1) > 0$ for all n and h, we can now conclude that $(p^n;m^n,y^n)$ is in fact an equilibrium for $\mathcal{E}^n$.

By (F3), the m^n_h and y^n_j are bounded. Thus, by taking subsequences, we can assume that the m^n_h and y^n_j converge to m^*_h and y^*_j respectively. Clearly $m^*_h \varepsilon M$, that $y^*_j \varepsilon Y_j$ follows by (F2).

Since demand and supply decisions are homogeneous of degree 0 in prices, we can, without loss of generality, assume that $\sup_{t \varepsilon T^n} p^n(t) = 1$. Thus, the p^n are bounded. We must show that the (T^n,p^n) are equicontinuous. We proceed by contradiction.

If the (T^n,p^n) are not equicontinuous, there are sequences $T^{n_k}, s^{n_k}, t^{n_k}$, such that $t^{n_k} \to t^*$, $s^{n_k} \to t^*$ and for some $\varepsilon > 0$, $p^{n_k}(s^{n_k}) > p^{n_k}(t^{n_k}) + \varepsilon$. Thus, $\lim \frac{p^{n_k}(s^{n_k})}{p^{n_k}(t^{n_k})} \geqslant 1+\varepsilon'$ for some $\varepsilon' > 0$.

For notational convenience, we will drop the subscript k. Now, since $\sum_h e^n_h(s^n) > 0$, either there is some h such that $m^n_h(s^n) > 0$ for infinitely many n or there is some j with $y^n_j(s^n) < 0$ for infinitely many n. Suppose the former is true and let $1 < \gamma < 1+\varepsilon'$. Then, define

$$\bar{m}^n_h = m^n_h - m^n_h(s^n)\delta_{s^n} + \gamma m^n_h(s^n)\delta_{t^n}$$

By construction, for infinitely many n, $\bar{m}^n_h \cdot p^n < m^n_h \cdot p^n$. Thus, for infinitely many n, $\bar{m}^n_h$ is in h's budget set. By (HS1), for some such n, $\bar{m}^n_h \succsim_h m^n_h$. Thus, by adding $\alpha\delta_{t_1}$ for some positive α, we have constructed a bundle both affordable to and strictly preferred by h. This contradicts the fact that $(p^n;m^n,y^n)$ is a competitive equilibrium for $\mathcal{E}^n$. A similar argument uses (FS1) to show that y^n_j cannot maximize j's profits for some n if $y^n_j(s^n) < 0$ for infinitely many n. This contradiction establishes the fact that the (T^n, p^n) are indeed equicontinuous.

By extracting subsequences, then, there is a $p^* \varepsilon\ \mathfrak{C}(T)$ with $(T^{n_k}, p^{n_k}) \to (T, p^*)$, $p^n \cdot e_h^n \to p^* \cdot e_h$ and $p^n \cdot m_h^n \to p^* \cdot m_h^*$ for all h.

Since $\bigcup_n \mathcal{M}^n$ is dense in Y_j for all j, it is straightforward to check that y_j^* maximizes j's profit.

Thus, we need only show that m_h^* maximizes $\succsim_h$ on h's budget set. For some t, $p^*(t) = 1$. Hence, by (H4), for some h, $p^* \cdot e_h > 0$. For this h $p^* \cdot (e_h + \sum_j \theta_{hj} y_j^*) > 0$. The argument that m_h^* maximizes $\succsim_h$ on h's budget set follows the usual line. From here, it follows that $p^*(t_1) > 0$, hence $p^* \cdot e_h > 0$ for all h.

Thus, m_h^* maximizes $\succsim_h$ on h's budget set for all h. Hence $(p^*; m^*, y^*)$ is indeed a competitive equilibrium for $\mathcal{E}$.

(b) The alterations from the proof given in (a) are relatively minor. The only steps to be altered are the argument which shows that the quasi-equilibrium of $\mathcal{E}^n$ is in fact an equilibrium and the proof that m_h^* maximizes $\succsim_h$ on h's budget set for all h. In both cases, it is enough to show that all agents' endowments have positive value.

For $\mathcal{E}^n$, it is shown as in (a) that $p^n \cdot e_h^n > 0$ for some h. Again, it is argued from this that m_h^n maximizes $\succsim_h^n$ on h's n-th budget set. One then concludes that $p^n \gg 0$ and hence $p^n \cdot e_h^n > 0$ for all h.

(c) The significant difference between the proof of this section and the others is that since not all commodities are available in the aggregate, an extra step is needed to show that $p^* \cdot e_h > 0$ for some h. That is, it is possible p^* is identically 0 on supp $\sum_h e_h$. That this does not occur follows from the fact that for some $\underline{t_h}$ in supp e_h, (HS3) is satisfied. That is,

$p^*(t^*) > 0$ for some t^*, so that $\lim \frac{p^n(t^n)}{p^n(\underline{t}_h)} = \infty$ for some sequence t^n converging to t^*. (Of course, we construct T^n so that $\underline{t}_h \varepsilon T^n$ for all h and n.) One now uses (HS3) and the fact that $(p^n;m^n,y^n)$ is a competitive equilibrium for $\mathcal{E}^n$ to form a contradiction. Thus we can conclude that $p^*(\underline{t}_h) > 0$ for some h. The argument now proceeds along familiar lines. **Q.E.D.**

Proposition 3: We discuss the case in which (H1) is satisfied, if (H1)′ is satisfied instead, similar reasoning applies.

Since p^* is bounded, by (H1)(b) and the fact that $(p^*;m^*,y^*)$ is a competitive equilibrium:

(i) $m \succsim_h m^*_h \Rightarrow p^n \cdot m \geqslant p^* \cdot m^*_h$

(ii) $m \succ_h m^*_h \Rightarrow p \cdot m > p^* \cdot m^*_h$

(iii) $p^* \cdot m^*_h = p^* \cdot (e_h + \sum_j \theta_{hj} y^*_j)$

Thus, if $(\bar{m},\bar{y})$ is a Pareto superior feasible allocation,

$$p^* \cdot \sum_h e_h = p^* \cdot (\sum_h \bar{m}_h - \sum_j y_j) > p^* \cdot (\sum_h m^*_h - \sum_j y_j) = p^* \cdot \sum e_h$$

a contradiction. **Q.E.D.**

Proposition 4: This follows as in the proof of Theorem 1(a). That is, if p^* is not continuous at t, there are sequences t^n, s^n converging to t with $p^n(s^n) > p^n(t^n) + \varepsilon$ for all n and some positive ε. Now, (HS1) is applied as in Theorem 1(a). **Q.E.D.**

Proposition 5: Again, the arguments are similar to those of Theorem 1.

(a) Suppose that for some sequence $t^k \to t^*$, $p^*(t^k) \to p^*(t^*) - \varepsilon$ for some $\varepsilon > 0$. By assumption, there is some h with $m_h(t^*) > 0$. Consider the sequence $\bar{m}^k = m_h - m_h(t^*)\delta_{t^*} + rm_h(t^*)\delta_{t^k}$ for some such r such that

$1 < r < \frac{p^*(t^*)}{p^*(t^*) - \varepsilon}$. Eventually, $p^*\cdot\bar{m}^k < p^*\cdot m_h \leqslant w_h$ and by (HS1), for some k,

$\bar{m}^k \succsim_h m_h$. Again, adding some positive amount of a commodity for which h's preferences are strictly monotonic (guaranteed to exist by (H1)) we have constructed an element of $\beta(p^*;w_h)$ which h strictly prefers to m_k, a contradiction.

(b) By (a), $p^*(t) \leqslant \underline{\lim}\, p^*(t^k)$. We need only show that $p^*(t^*) \geqslant \overline{\lim}\, p^*(t^k)$. If not, take a sequence t^k with $\lim p^*(t^k) > p^*(t^*) + \varepsilon$ for some positive ε. Since the t^k are all atoms of $\sum_h m_h$, there is some h with $m_h(t^k) > 0$ for infinitely many k. One now proceeds to construct a contradiction for this h as in (a). **Q.E.D.**

Theorem 2:

That CPCE($\cdot$) is non-empty valued on M^* follows from Theorem 1. Let $(e_1^n,\ldots,e_H^n)$ be a sequence of endowment distributions in M^* converging to $(e_1,\ldots,e_H)\varepsilon M^*$. Denote the associated economies by $\mathcal{E}^n$ and $\mathcal{E}$ and take $(p^n;m^n)\varepsilon$ CPCE($\mathcal{E}^n$), i.e., $p^n \varepsilon\ \mathcal{C}(T)$. Without loss of generality we assume $m_h^n \to m_h^*$ for all h. Of course, $\sum_h m_h^* = \sum_h e_h$.

We can assume that $p^n(t) \leqslant 1$ for all t with equality for some t. We will show that the p^n are equicontinuous. We proceed by contradiction. If they are not, there is some t^* such that for some $\varepsilon > 0$ and all $\delta > 0$ there are infinitely many n such that:

$$\text{There are } t^n, p^n \text{ with } d(t^n,t^*) < \delta \text{ and } |p^n(t^n) - p^n(t^*)| > \varepsilon.$$

By taking $\delta_k = \frac{1}{k}$ successfully, we find an increasing sequence n_k and t^{n_k} with $t^{n_k} \to t^*$ and $|p^{n_k}(t^{n_k}) - p^{n_k}(t^*)| > \varepsilon$ for all k.

We can assume that $p^{n_k}(t^*) \to \Pi^*$ without loss of generality. We will suppose that $p^{n_k}(t^{n_k}) > p^{n_k}(t^*) + \varepsilon$ for all k. By extracting subsequences either this is true or $p^{n_k}(t^{n_k}) < p^{n_k}(t^*) - \varepsilon$. The proof in the two cases being similar, we cover only the first.

Since p^{n_k} is continuous, there are neighborhoods, C_k, of the t^{n_k} with $p^{n_k}(t) > p^{n_k}(t^*) + \frac{\varepsilon}{2}$ for all $t \varepsilon C_k$ (we choose C_k closed and with non-empty interior). Clearly, we can take $C_k \to t^*$ since $t^{n_k} \to t^*$. Since $\mathcal{E}^{n_k}$ satisfies (H4) for all k, there is some h such that for infinitely many k, supp $m_h^{n_k} \cap C_k \neq \emptyset$.

Define ν^k by $\nu^k(B) = m_h^{n_k}(B \cap C_k)$. Then

supp $\nu^k \subset C_k$ and choosing r so that $1 < r < 1 + \frac{\varepsilon}{2p^{n_k}(t^*)}$ and letting $\bar{m}_h^k = m_h^{n_k} - \nu^k + r\nu^k(C_k)\delta_{t^*}$ we can apply (HS2) to see that for some k, $\bar{m}_h^k \succsim_h m_h^{n_k}$ for some k ($B^k = \{t^*\}$ for all k). By adding a positive quantity of any commodity, we have constructed a consumption bundle for h in h's n_k-th budget set and strictly preferred to $m_h^{n_k}$, a contradiction. Thus, the p^n are equicontinuous. Hence a subsequence converges to p^* for some $p^* \varepsilon\, \mathcal{C}(T)$. The argument now proceeds along the lines of the proof of Theorem 1(a) using the fact that the endowment distributions satisfy (H4) for all n.

Q.E.D.

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