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CORES AND VALUES OF MONOPOLISTIC MARKET GAMES:
ASYMPTOTIC RESULTS

by

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1. Introduction

The formulation and treatment of competitive markets as games, cooperative and noncooperative, is well established in the literature. Papers by Debreu (1952) and Shafer and Sonnenschein (1975), among many others, are representatives of the noncooperative approach. Papers by Debreu and Scarf (1963), Shapley (1964), Shapley and Shubik (1969) and Aumann (1964, 1975), among many others, are representatives of the cooperative treatment. In those papers the solution concepts of a noncooperative equilibrium (modeled after the Nash equilibrium), the core and the Shapley value were defined and analyzed for competitive market games.

Treatment of noncompetitive market games and, in particular, monopolistic and oligopolistic games is less common in the literature. In that respect, two main lines of analysis were used so far. Papers like Aumann (1973), Shitovitz (1973) and others formulate monopolistic situations as games where the set of competitive agents is a continuum and where a monopoly or an oligopoly is distinguished by its measure being positive. The papers in that tradition are interested mainly in the core of these games and its relation, allocation and utility wise, to the competitive equilibrium.

A different approach for noncooperative monopolistic games was taken by Kats (1973, 1974). In these papers a monopoly in a game of countably many players is distinguished not by its size, relative to the size of other agents, but rather by its ability to affect the actual set of alternatives available to the non-monopolistic agents.

Here we use the approach taken by Kats in dealing with situations where, on one hand, there is an agent (or agents) who, in some way, has

control over what other agents can do and, on the other hand, cooperation is possible. In particular we consider situations where agents are being endowed with a bundle of commodities, but where only one agent has an access to a production process, a process that can convert these initial bundles into bundles of consumable goods. Examples of cases that can be modeled this way are:

1. A governmental agency which has complete control over a trading market, like the Stock Exchange, where the trading of commodities can be viewed as a production process.

2. An industry where a patent is held by one agent and where no production can take place without the use of the patent.

3. A group of workers, like a labor union, which control an essential part of the production process in a plant, so that no production can take place unless management and that union cooperate.

We also consider situations where more than one agent has an access to production. More precisely we look at situations where going through production requires the consent of each of several distinguished monopolies, each controlling a different stage of the production process, as well as examining situations where several alternatives are available for production, each controlled by a different oligopoly. Examples are:

4. A plant where several labor unions control each of several critical stages of the production process.

5. A case of two companies each holding a patent to very similar production processes.

The above examples can be sorted out into two different categories. The first would include cases like the one about the government and the

stock exchange market, where the monopoly is not really needed for economic activity to take place. An argument can be made to the effect that trade can take place even without governmental intervention. One can consider the existence of the monopoly in this case as an imposition on the market. The second category would include cases like the one about the monopoly which developed a patent. The very existence of the monopoly improves the outcome of agents in the market using the patent.

In all of these examples one element repeats itself: There exist an agent (a monopoly) which can affect, up or down, the pay-off of all other agents in the market.

It is well known (Debreu and Scarf (1963) and Shapley (1964)) that in non-monopolistic market games with transferable utilities, when the economy is replicated repeatedly, the core and the Shapley value converge to the competitive equilibrium (in fact the convergence of the core holds true for economies without transferable utilities as well). Thus a natural question to ask is: Is there an analogous result for monopolistic situations? In this case, what is the monopoly's share of society's welfare or what is its "power"? Similarly, how is the welfare of the non-monopolistic players affected by the existence of the monopoly? In this paper we concentrate on asymptotic results with regard to the core and the Shapley value. We state them now for the case of an exchange economy with a presence of a monopoly (see example 1 above). Later we extend them to include economies with linearly homogenous production.

1. In the limit, the Shapley value of the monopoly is exactly one half the gains from trade of the non-monopolistic players, no matter what

the initial endowments or utility functions are. Moreover the Shapley value of each non-monopolistic player is just one half of his individual gains from trade.

2. The limit core consists of all imputations such that each non-monopolistic trader loses relative to its competitive payoff in a free market and the residual is received by the monopoly. This characterization does not hold in cases of finitely many replications.

3. The limit core described above has a center of symmetry and the Shapley value of the monopolistic game converges to that center of symmetry.

4. The Shapley values and the limit of the core are given for the cases of several monopolies or oligopolies. In the case of several monopolies, even though the limit of the core does not have a center of symmetry, the limit of the Shapley values is still contained in it. This inclusion result fails in the oligopoly case.

Another observation is that in the finite case it is possible that some of the nonmonopolistic players would benefit (in the core) by the very presence of the monopoly. However, no one of these nonmonopolists would gain if their number is not greater than three.

Those results are derived by using a model of a cooperative game with transferable utilities (with money). Coalition which does not contain the monopoly is assumed to be unable to take any economic activity. However, when the monopoly is a member of a coalition, the worth of that coalition is the maximum total utility that its members can achieve by trading their resources.

The paper is organized as follows: The basic model is laid out in sections 2 and 3. In Sections 4 and 5 we derive the basic asymptotic results of the Shapley value. In section 6 the cases of multimonomopolies and oligopolies are discussed. The limit core and the comparison between it and the limit Shapley values are discussed in section 7. In order to maintain continuity we gathered all the proofs together in section 8.

2. The Model

To simplify the presentation we discuss first a model of a trading economy without production. We consider a model of n traders, one monopoly, and ℓ goods, not including "money." Each trader i , $1 \leq i \leq n$, is assumed to have a utility function U^i of the form

$$U^i(x^i, \xi^i) = u^i(x^i) + \xi^i$$

where $u^i: E^{\ell} \rightarrow E^1$ is concave and differentiable and $\xi^i \in E^1$ represents the net change from the initial money position (ξ^i might be negative or positive).

Each i , $1 \leq i \leq n$, is endowed with an initial bundle $a^i = (a_1^i, \dots, a_\ell^i) \in E^{\ell}$ and starts with no money. Denote $\underline{a} = (a^1, \dots, a^n) \in E^{\ell \cdot n}$ and assume that

$$(1) \quad \sum_{i=1}^n a_j^i > 0, \quad j = 1, 2, \dots, \ell,$$

i.e., that each good is presented in the market. Before describing the role of the monopoly in this model it is clear that if traders are permitted to transfer goods and money at will then the above economic model can be formalized as a cooperative n -person game with side payments. Let $N = \{1, \dots, n\}$ be the set of all traders. The potential worth of a coalition S is given by

$$(2) \quad v(S) = \max \left\{ \sum_{i \in S} u^i(x^i) \mid \sum_{i \in S} x^i \leq \sum_{i \in S} a^i \text{ and } x^i \geq 0 \quad i = 1, 2, \dots, n \right\}.$$

Notice that by the continuity of the u^i and the compactness of the set of all reallocations $\underline{x} = (x^1, \dots, x^\ell) \in E_+^{n \cdot \ell}$ of \underline{a} , the maximum in (2) is achieved.

The monopoly, denoted by o , has the power to block any trade of goods within a coalition $S \subset N$ as long as it is not a member of that coalition. In that case any member of S can consume his initial endowment only. Let $N_0 = N \cup \{0\}$ and let v_0 be the game on N_0 describing the monopolistic case, i.e., for $S \subseteq N_0$

$$(3) \quad v_0(S) = \begin{cases} v(S) & 0 \in S \\ \sum_{i \in S} u^i(a^i) & 0 \notin S \end{cases} .$$

The following properties of the nonmonopolistic case are important to the sequel. Let $b = (b^1, \dots, b^n)$ be an optimal allocation for the market when trade is permitted. i.e.

$$(4) \quad \sum_{i=1}^n u^i(b^i) = v(N), \quad \sum_{i=1}^n b^i = \sum_{i=1}^n a^i \quad \text{and} \quad b^i \geq 0 \quad i = 1, 2, \dots, n.$$

Let u_j^i denote the partial derivative of u^i with respect to the j^{th} good. Then $b_j^i > 0$ implies that $u_j^i(b^i) \geq \hat{u}_j^i(\hat{b}^i)$ for every i (otherwise i could transfer some amount of j to i to increase the total utility). Notice that by (1) for each j , $1 \leq j \leq \ell$, there is an i , $1 \leq i \leq n$, such that $b_j^i > 0$. Hence no confusion would result if the competitive prices $\pi = (\pi_1, \dots, \pi_\ell)$ are defined by

$$(5) \quad \pi_j = u_j^i(b^i) \quad \text{for all } i \text{ such that } b_j^i > 0, \quad 1 \leq j \leq \ell.$$

Define the competitive imputation by

$$(6) \quad w^i = u^i(b^i) + \pi \cdot (a^i - b^i) \quad , \quad i=1, \dots, n.$$

It is easy to verify that the prices π and the imputation $w=(w^1, \dots, w^n)$ are independent of the choice of the optimal allocation b . To understand (6) notice that $\pi \cdot (x^i - a^i)$ is the amount of money that trader i must pay under π in order to buy the bundle x^i , over and above the amount that he gets by selling his initial bundle a^i . This amount should be subtracted from $u^i(x^i)$ in order to yield the net "income" of i , and it is this income that i wishes to maximize. The vector $b=(b^1, \dots, b^n)$ is a vector of such maximizers and thus (π, b) is an equilibrium point for the economy.

3. Replication

Consider now n types of traders with k traders of each type. Traders of the same type have identical utility functions and identical initial bundles. We shall continue to use the notation of the preceding sections but with the understanding that the index "i" hereafter refers to types, not individuals. We thus have k identical economies regarded as a single economy, having kn traders of n different types. The competitive price vector of the enlarged market is again π , while the competitive imputation is just the kn -dimensional vector (w, w, \dots, w) (k times). The characteristic function v^k of the enlarged market is defined on $N^k = \{1, \dots, nk\}$ by

$$(7) \quad v^k(S) = \frac{1}{k} \max \left\{ \sum_{i=1}^n s^i u^i(x^i) \mid \sum_{i \in S} s^i x^i \leq \sum_{i \in S} s^i a^i \text{ and } x^i \geq 0 \quad i=1, 2, \dots, n \right\}$$

where s^i is the number of traders of type i in S . Thus $v^k(S)$ is the per replica worth of the coalition S when trades are permitted.

With the presence of a monopoly who controls all trades, the corresponding game v_0^k on $N_0^k = N^k \cup \{0\}$ is defined by

$$(8) \quad v_0^k(S) = \begin{cases} v^k(S) & o \in S \\ \sum_{i \in S} \frac{s^i}{k} u^i(a^i) & o \notin S \end{cases} .$$

4. The Shapley Value

There are few equivalent definitions for the value of a game. We use here the one described in Shapley (1953). Intuitively, the value of a game to a given player is the average of his marginal contribution over all possible coalitions. In other words it is his expected marginal worth in a coalition chosen at random. Thus we define

$$(9) \quad \phi^i = \sum_{S: i} \{v(S) - v(S \setminus \{i\})\}$$

where the probabilities to be associated with the expectation symbol $\sum_{S: i}$ are such that each coalition size from 1 to n has probability 1/n and all coalitions of the same size are equally likely. Hence

$$\phi^i = \sum_{|S|=1}^n \frac{(|S|-1)!(n-|S|)!}{n!} (v(S) - v(S \setminus \{i\})) .$$

5. The Shapley Value of the Monopolistic Game, Asymptotic Results

In this section we derive the Shapley value of the monopoly as well as of each type of traders when the number of replications tends to infinity.

Denote by $\phi(p, k)$ and $\phi_0(p, k)$ the Shapley value of trader p in the games v^k and v_0^k respectively. Let

$$\phi^i(k) = \sum_{p \in N^k} \phi(p, k) \quad i = 1, \dots, n$$

$$(10) \quad \phi_0^i(k) = \sum_{p \in N^k} \phi_0(p, k) \quad i = 1, \dots, n$$

$$\phi_0^0(k) = \phi_0(o, k)$$

where N_i^k is the set of traders of the i^{th} type in N^k . Thus $\phi^i(k)$ and $\phi_0^i(k)$ are the values of type i in the games v^k and v_0^k respectively and $\phi_0^0(k)$ is the value of the monopoly in v_0^k . Shapley (1964) proved the following seminal theorem:

Theorem 1 (Shapley)

The value of v^k converges to the competitive pay-off vector, namely

$$\lim_{k \rightarrow \infty} \phi^i(k) = w^i \quad \text{for each } i, 1 \leq i \leq n,$$

We prove that for monopolistic games the following holds:

Theorem 2 For each $i, 1 \leq i \leq n$,

$$(I) \quad \lim_{k \rightarrow \infty} \phi_0^i(k) = \frac{1}{2} [w^i + u^i(a^i)] \quad \text{for each } i, 1 \leq i \leq n.$$

$$(II) \quad \lim_{k \rightarrow \infty} \phi_0^0(k) = \frac{1}{2} \left[\sum_{i=1}^n w^i - \sum_{i=1}^n u^i(a^i) \right].$$

Namely, in the limit the value of the monopoly is one half the total net "income" yielded by the monopoly's permission to trade. On the other hand, each type loses one half of its net "income" in comparison to its value in the case where all trades are permitted (namely, compared to the non-monopolistic game). The proof of the theorem, along with all other proofs, is stated in Section 8.

In order to capture markets with production as well, the above model can be generalized as follows: Let $G: E^{\ell} \times E^m \rightarrow E^m$ be a production function. The initial endowment of agent i is $a^i \in E^{\ell}$. The utility function will be $U^i: E^{\ell+1} \times E^m \rightarrow E^1$ of the form $U^i(x^i, \xi^i, y^i) = u^i(x^i, y^i) + \xi^i$ where consumer i can consume part or all of the total initial endowments, as well as commodities

produced through the production process. As before, we assume that u^i is concave and differentiable. With production, the per-replication potential worth of a coalition S , $S \subseteq N$, with access to the production process, is

$$\bar{v}^k(S) = \frac{1}{k} \max \left\{ \sum_{i \in S} s^i u^i(x^i, y^i) \mid \sum_{i \in S} s^i x^i \leq \sum_{i \in S} s^i a^i, \sum_{i \in S} s^i y^i \leq G \left(\sum_{i \in S} s^i a^i - \sum_{i \in S} s^i x^i \right) \right.$$

and $x^i, y^i \geq 0$, $i = 1, 2, \dots, n$.

The characteristic function of the game is defined now exactly the same as in (8), except that $\bar{v}^k(S)$ will replace $v^k(S)$.

In case $k=1$ we denote $\bar{v} = \bar{v}^1$. Notice also that a pure exchange economy is a special case where $G(x)=0$.

We can state now the analogue of Theorem 2 for production economies:

Theorem 2*

$$\lim_{k \rightarrow \infty} \phi_0^0(k) = \lim_{k \rightarrow \infty} \int_0^{\frac{1}{k} \bar{v}^k} (N^{\lceil tk \rceil}) dt - \frac{1}{2} \sum_{i=1}^n u^i(a^i, 0),$$

where $\lceil a \rceil$ is the smallest integer $\geq a$.

Thus, if the production function G is homogeneous of degree 1 then

$$\lim_{k \rightarrow \infty} \phi_0^0(k) = \frac{1}{2} \left[\bar{v}(N) - \sum_{i=1}^n u^i(a^i) \right],$$

namely in the limit the value of the monopoly is exactly one half the net "income" yielded by the use of its homogeneous technology.

6. The Multi Monopolies and Oligopolies Cases

We move now to consider cases of more than one entity having an access to the production process. We will distinguish between two different situations. In the first there is a set $M = \{M_0, \dots, M_{m-1}\}$ of m entities each of which controls a different part of the production process, so that no

production can take place by the coalition S unless $M \subseteq S$. Examples can be a case of several labor unions in the same firm where the members of each are in charge of a different segment of the production process. Only a coalition of management with all unions will enable production to take place. Another example can be a case where each part of the production involves a use of a different patent and each patent is held by a different entity. In this multi-monopolies case the characteristic function v_M^k for the k -fold replica market is defined on $N^k \cup M$ by

$$v_M^k(S) = \begin{cases} v^k(S) & \text{if } M \subseteq S \\ \sum_{i \in S} \frac{s_i}{k} u^i(a^i) & \text{otherwise.} \end{cases}$$

In the definition of v_M^k we use the game v^k rather than the more general one \bar{v}^k just to simplify the presentation.

The second case is one where an access to the production process can be gained through each of several entities. In this case of several oligopolies we denote $L = \{L_0, \dots, L_{m-1}\}$ and define the characteristic function v_L^k by

$$v_L^k(S) = \begin{cases} v^k(S) & \text{if } S \cap L \neq \emptyset \\ \sum_{i \in S} \frac{s_i}{k} u^i(a^i) & \text{otherwise.} \end{cases}$$

Theorem 3. Let $\phi_M^i(k)$ and $\phi_L^i(k)$ be the Shapley value of the games v_M^k and v_L^k respectively. Then

$$(I) \quad \lim_{k \rightarrow \infty} \phi_M^i(k) = \frac{m}{m+1} \left(\sum_{i=1}^n w^i - \sum_{i=1}^n u^i(a^i) \right)$$

and

$$\lim_{k \rightarrow \infty} \phi_L^i(k) = \frac{1}{m+1} (w^i - u^i(a^i)) \quad , \quad i = 1, \dots, n \quad .$$

$$(II) \quad \lim_{k \rightarrow \infty} \phi_L^L(k) = \frac{1}{m+1} \left(\sum_{i=1}^n w^i - \sum_{i=1}^n u^i(a^i) \right)$$

and

$$\lim_{k \rightarrow \infty} \phi_L^i(k) = \frac{1}{m+1} (w^i - u^i(a^i)) \quad , \quad i = 1, \dots, n$$

where $\phi_M^M(k)$ is the Shapley value of the set of monopolies M in v_M^k and $\phi_M^i(k)$ is the Shapley value of the i^{th} type in v_M^k . The terms $\phi_L^L(k)$ and $\phi_L^i(k)$ are defined similarly.

In the first case, the more monopolies there are, the greater $\left(\frac{1}{m+1}\right)$ is their share in the net "income" while in the second case as the number of oligopolies increases their value drops.

Notice that since all the monopolies (oligopolies) are symmetric players then by the last theorem the value $\phi_{M_t}^{M_t}(k)$ and $\phi_{L_t}^{L_t}(k)$ of each monopoly M_t and each oligopoly L_t in the games v_M^k and v_L^k respectively is asymptotically given by

$$\lim_{k \rightarrow \infty} \phi_{M_t}^{M_t}(k) = \frac{1}{m+1} \left(\sum_{i=1}^n w^i - \sum_{i=1}^n u^i(a^i) \right) \quad , \quad t = 0, \dots, m-1,$$

$$\lim_{k \rightarrow \infty} \phi_{L_t}^{L_t}(k) = \frac{1}{m(m+1)} \left(\sum_{i=1}^n w^i - \sum_{i=1}^n u^i(a^i) \right) \quad t = 0, \dots, m-1.$$

7. The Core

This section deals with the limit core of the various games described above, namely, the limit of the core of the k -fold replication games with one or more monopolies or oligopolies. We also explore here the relations of the limit core of these games to their asymptotic Shapley value. In order to simplify the exposition we assume that $u^i(a^i) = 0$, $i = 1, 2, \dots, n$.

Let \tilde{v} be a game with side-payments in a characteristic function form defined on a set \tilde{N} of players. The core $C\tilde{v}$ of the game \tilde{v} is the set of all imputations $\alpha = (\alpha^1, \dots, \alpha^n)$ that satisfy

$$(i) \text{ For each } S \subseteq \tilde{N} \quad \tilde{v}(S) \leq \sum_{i \in S} \alpha^i$$

and

$$(ii) \quad v(\tilde{N}) = \sum_{i \in \tilde{N}} \alpha^i .$$

If the game \tilde{v} is a market game (i.e. is defined as in (2)) then \tilde{v} is totally balanced and hence has a nonempty core. Thus for each k the game v^k defined in (7) has a nonempty core $C(k)$. Using the result of Debreu and Scarf (1963), $C(k)$ "shrinks" to the competitive imputation $w=(w^1, \dots, w^n)$. It will be shown below that in the case of one monopoly the core $C_0(k)$ of the single monopoly games converge to the set of all imputations such that each type i gets an amount smaller or equal to w^i and the monopoly collects the residual or the loss of all the types together. Thus an imputation is in the limit core $\lim_{k \rightarrow \infty} C_0(k)$ if and only if each type of traders is losing (in the weak sense) relative to the competitive imputation $w = \lim_{k \rightarrow \infty} C(k)$. Similarly, an imputation is in the limit core $\lim_{k \rightarrow \infty} C_M(k)$ of the multimonoopolies games if and only if each type of traders is losing (in the weak sense) relative to w and the monoopolies share the excess income in any way possible. Finally, the only imputation in the limit core $\lim_{k \rightarrow \infty} C_L(k)$ of the oligopolistic games is the vector $(0, w)$. Namely the oligopolies, as a result of an internal competition between them, end up with zero profits and each type i of nonmonoopolistic players gets the competitive payoff w^i .

To state these results precisely notice first that the concavity of the utility functions u^i imply that any imputation in $C(k)$, $C_o(k)$, $C_M(k)$ and $C_L(k)$ treats equally any two traders of the same type. Therefore a vector in $C_o(k)$ can be represented by a vector in E_+^{1+n} of the form

$$(\beta^o, \alpha) = (\beta^o, \alpha^1, \dots, \alpha^n)$$

where β^o is the payoff to the monopoly and $\alpha^1, \dots, \alpha^n$ the total payoffs to each of the n types respectively¹. Similarly an imputation in $C_M(k)$ or in $C_L(k)$ is represented by a vector

$$(\beta, \alpha) = (\beta^o, \dots, \beta^{m-1}, \alpha^1, \dots, \alpha^n)$$

in E_+^{m+n} .

We can state now:

Theorem 4. The following characterizations of the limit cores hold:

$$(I) \quad \lim_{k \rightarrow \infty} C_o(k) = \{(\beta^o, \alpha) \in E_+^{1+n} \mid \alpha^i \leq w^i \text{ and } \beta^o + \sum_{i=1}^n \alpha^i = \sum_{i=1}^n w^i\}$$

$$(II) \quad \lim_{k \rightarrow \infty} C_M(k) = \{(\beta, \alpha) \in E_+^{m+n} \mid \alpha^i \leq w^i \text{ and } \sum_{i=0}^{m-1} \beta^i + \sum_{i=1}^n \alpha^i = \sum_{i=1}^n w^i\}$$

$$(III) \quad \lim_{k \rightarrow \infty} C_L(k) = \{(0, w)\}.$$

The equalities in Theorem 4, in general do not hold for a finite k .

To that end denote

¹Notice that α^i is the total payoff to type i and not the payoff to each trader in type i . Recall that the game v_0^k is normalized and measures the per replica worth of each coalition.

$$A^k = \{ \alpha \in E^n \mid \exists y \in C v^k \text{ for which } \alpha^i \leq y^i, \quad i=1,2,\dots,n \}.$$

For each k , A^k consists of payoff vectors that relative to some imputation y in the core of v^k represent losses (in the weak sense) to each type of players.

It is easy to verify that

$$(11) \quad C_0(k) \supseteq \{ (\beta^0, \alpha) \in E^{1+n} \mid \alpha \in A^k \text{ and } \beta^0 + \sum_{i=1}^n \alpha^i = \sum_{i=1}^n w^i = v(N) \}.$$

However the inclusion in (11) can not, in general, be replaced by an equality sign. To see this consider the nonmonopolistic game (N, v) given by $N = \{1, 2, 3\}$ and $v(1, 2, 3) = 3.4$, $v(1, 2) = v(1, 3) = 2$, $v(2, 3) = 2.9$ and $v(S) = 0$ for all other $S \subseteq N$. It is easy to check that this game is totally balanced and therefore (see Shapley and Shubik, 1969) is a market game. It is also easy to verify that the vector $(0.2, 0.4, 1.4, 1.4)$ is in the core C_0 of the corresponding monopolistic game. However, there is no vector $(0.4 + \varepsilon_1, 1.4 + \varepsilon_2, 1.4 + \varepsilon_3)$ in the core of the nonmonopolistic game, where $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0.2$, $\varepsilon_1 + \varepsilon_2 \geq 0.2$, $\varepsilon_1 + \varepsilon_3 \geq 0.2$ and $\varepsilon_2 + \varepsilon_3 \geq 0.1$. The inability to write (11) as an equality suggests that some type might be better off with the monopoly rather than without it. This indeed is the case. Moreover, it can be shown that some type might get in the presence of the monopoly more than the maximum payoff that he can get in the core of the non-monopolistic game.

Example. Consider the following 4-person game v :

$$v(1,2,3,4) = 4, v(1,2,3) = 3, v(1,2,4) = 3.5,$$

$$v(1,3,4) = v(2,3,4) = 2.5, v(1,2) = v(1,3) = v(2,3) = 2 \text{ and}$$

$$v(S) = 0 \text{ otherwise.}$$

It is again easy to verify that the game v is totally balanced and that its core consists of just one imputation

$$\bar{\alpha} = (1.5, 1.5, 0.5, 0.5).$$

If v_0 is the monopolistic game corresponding to v then clearly the imputation

$$(\beta^0, \alpha) = (1, 1, 1, 0, 1)$$

is in the core C_0 of v_0 . Hence trader 4 who gets 1 in this imputation is improving upon the unique vector $\bar{\alpha}$ in the core C of v . Theorem 4 above asserts that such a phenomena is impossible in the limit. In that respect we assert that a game of not more than three types also has this property. Namely,

Proposition 5. Let $N = \{1,2,3\}$ and (N,v) be totally balanced. Denote

$\bar{\alpha}^i = \max_{\alpha \in C} \alpha^i$, $i = 1,2,3$. Then for each $(\beta^0, \alpha) \in C_0$, $\bar{\alpha}^i < \alpha^i$, where C and C_0 are the cores of v and v_0 respectively.

The multi-monopolies case is similar. It can be checked easily that

$$(12) \quad C_M(k) \supseteq \{(\beta, \alpha) \in E_+^{m+n} \mid \alpha \in A^k \text{ and } \sum_{i=0}^{m-1} \beta^i + \sum_{i=1}^n \alpha^i = \sum_{i=1}^n w^i\}$$

and the inclusion in (12) cannot in general be replaced by the equality sign.

Finally it is clear that each oligopoly in $C_V^k_L$ should get zero and thus

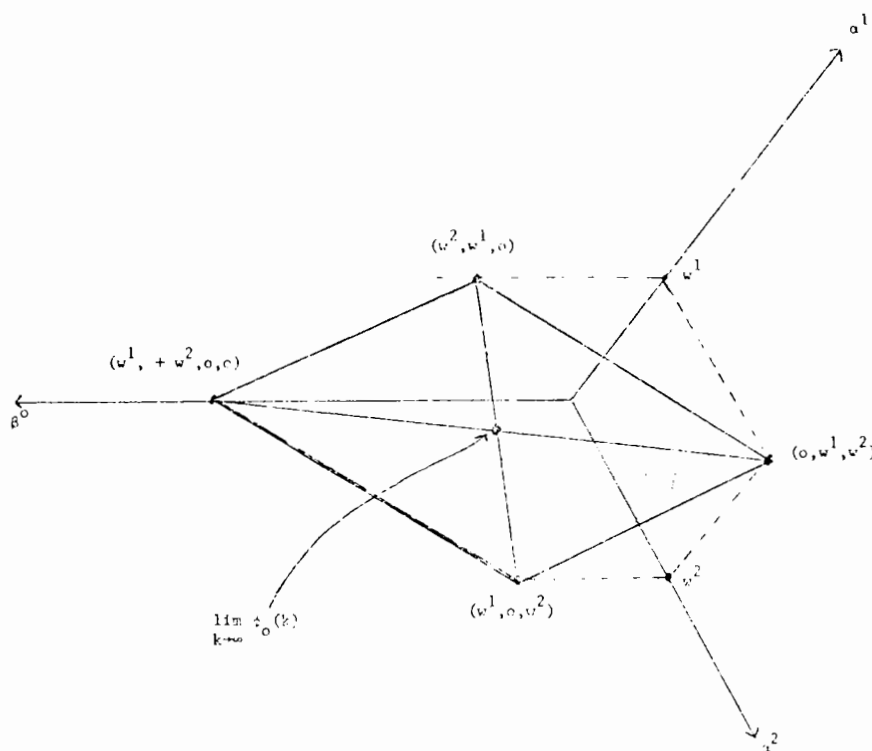
$$(13) \quad C_L^k(k) = \{(0, \alpha) \in E_+^{m+n} \mid \alpha \in C_V^k\}.$$

Using Theorems 2, 3 and 4 above we can state precisely the asymptotic relations between the core and the Shapley value of each of the games v_o^k , v_M^k and v_L^k .

Theorem 6. Let $\phi_o(k)$, $\phi_M(k)$, $\phi_L(k)$ be the Shapley value of the games v_o^k , v_M^k and v_L^k respectively then

- (I) $\lim_{k \rightarrow \infty} \phi_o(k)$ is the center of symmetry of the limit core $\lim_{k \rightarrow \infty} C_o(k)$.
- (II) If $m > 1$, $\lim_{k \rightarrow \infty} C_M(k)$ does not have a center of symmetry but $\lim_{k \rightarrow \infty} \phi_M(k)$ is always an element in $\lim_{k \rightarrow \infty} C_M(k)$.
- (III) If $m > 1$, $\lim_{k \rightarrow \infty} \phi_L(k)$ is not an element of the limit core $\lim_{k \rightarrow \infty} C_L(k)$.

A geometrical example for $N = \{1,2\}$ is presented in the following diagram.



7. Proofs of the Results

In this section we present the proofs of the results stated above.

W.l.o.g. assume that $u^i(a^i) = 0$ for all $i=1,2,\dots,n$.

Proof of Theorem 2. We follow the notations and the basic arguments of Shapley (1964) who proved that $\lim_{k \rightarrow \infty} \phi(k) = w$.

Let F be a function defined on n -tuples $s = (s^1, \dots, s^n) \in \mathbb{E}_+^n$ by

$$(14) \quad F(s) = \max_{i=1}^n \{ \sum_{i=1}^n s^i u^i(x^i) \mid x^i \geq 0 \text{ and } \sum_{i=1}^n s^i x^i \leq \sum_{i=1}^n s^i a^i \}.$$

If s is a vector of non-negative integers then $w(s)$ denotes the competitive payoff vector of the market consisting of s^i traders of the i^{th} type for $i=1, \dots, n$, i.e.

$$(15) \quad w^i(s) = u^i(b^i) + \sum_j (a_j^i - b_j^i) \cdot \frac{\partial u^i(j)}{\partial x_j^i} (b^i(j))$$

where b is any maximizer in (14) and $i(j)$ is such that $b_j^{i(j)} > 0$ for each j .

Lemma 7 (Shapley). F is homogeneous of degree one and concave. Furthermore, F has continuous first order partial derivatives for all $s > 0$, given by

$$\frac{\partial F}{\partial s^i}(s) = w^i(s).$$

Since $\frac{\partial F}{\partial s^i}$ is homogeneous of degree zero $\frac{\partial F}{\partial s^i}(s) = \frac{\partial F}{\partial s^i}(\bar{s})$ where $\bar{s} = s / \sum_{i=1}^n s^i$

and hence $w^i(s) = w^i(\bar{s})$. Thus from now on we will refer to $w(\cdot)$ as a function on the simplex $S^{n-1} = \{(x^1, \dots, x^n) \in E^n \mid \sum_{i=1}^n x^i = 1\}$.

Let $\bar{n} = (1/n, \dots, 1/n)$ then $w^i(\bar{n}) = w^i$ where w^i is the competitive payoff to type i in a market where all the n types have the same number of traders. Let

$$D^i(s) = F(s^1, \dots, s^n) - F(s^1, \dots, s^{i-1}, s^i - 1, s^{i+1}, \dots, s^n).$$

By the concavity of F , $D^i(s) \geq w^i(\bar{s})$. Hence, by the continuity of $\frac{\partial F}{\partial s^i}$, for each $\varepsilon > 0$ there is $\delta = \delta(\varepsilon)$ such that

$$(16) \quad \|\bar{s} - \bar{n}\| < \delta \text{ implies } D^i(s) \geq w^i - \varepsilon \text{ for each } i=1, \dots, n$$

(here $\|x\|$ denotes $\max |x^i|$).

A coalition $S \subseteq \underset{o}{N^k}$ has a one to one correspondence with a profile (s^o, s) of $n+1$ nonnegative numbers where s^i is the number of traders of type i and $s^o = 0$ or 1 depending on the monopoly o being in S or not, respectively. A coalition S is called " δ -diagonal" (or " δ -balanced") if the profile s of the n types of traders satisfies $\|\bar{s} - \bar{n}\| < \delta$. Then, given an ε and a δ as above, there is an integer $r_0 = r_0(\varepsilon)$ large enough such that for each integer $r \geq r_0$ the probability is greater than $1-\varepsilon$ that an r -element set is δ -diagonal if it is formed by choosing the type of element at random "without replacement" from a finite collection in which there are k elements of each type $1, \dots, n$ and there is only one element of type o , namely the monopoly (for more details see Shapley (1964)). Hence, if $r \geq r_0$ for a random r -member coalition in the k -fold market with a monopoly we have

$$\text{Prob}\{\|\bar{s} - \bar{n}\| < \delta\} > 1-\varepsilon.$$

Since $\text{Prob}\{s^0=1\} = \frac{r}{nk+1}$ for a random r -member coalition where

$$r > r_0(\varepsilon)+1, \delta=\delta(\varepsilon) \text{ and } k > \frac{r-1}{n},$$

$$(17) \quad \text{Prob}\{\|\bar{s} - \bar{n}\| < \delta \text{ and } s^0=1\} > \frac{r}{nk+1} (1-\varepsilon).$$

Since all the k traders of type i are symmetric we have by the definition of v_o^k that for each $i, 1 \leq i \leq n,$

$$(18) \quad \phi_o^i(k) = \sum_{p \in N_i^k} \phi_o(p, k) = \frac{k}{nk+1} \sum_{r=1}^{nk+1} \left[E \left\{ \frac{1}{S} D^i(s) \cdot s^0 \mid |S|=r, p \in S \right\} \right]$$

where $\phi_o(p, k)$ is the Shapley value of trader p of type i in the game v_o^k .

Since $u^i(a^i) = 0$ then $D^i(s) \geq 0$. Therefore by (17) and (18) for

any $p \in N_i^k$

$$\phi_o^i(k) \geq \frac{1}{nk+1} \sum_{r=r_0+1}^{nk+1} \left[E \{ D^i(s) \cdot s^0 \mid |S|=r, p \in S \} \right].$$

This together with (17) imply

$$\phi_o^i(k) \geq (1-\varepsilon) \cdot \frac{1}{nk+1} \sum_{r=r_0+1}^{nk+1} \frac{r}{nk+1} \left[E \{ D^i(s) \mid |S|=r, p \in S, s^0=1 \text{ and } \|\bar{s}-\bar{n}\| < \delta \} \right].$$

Hence by (16)

$$\phi_o^i(k) \geq \frac{1-\varepsilon}{(nk+1)^2} \sum_{r=r_0+1}^{nk+1} r(w^{i-\varepsilon}) = \frac{(1-\varepsilon)(w^{i-\varepsilon})}{(nk+1)^2} \cdot \frac{(nk+r_0+2)(nk+1-r_0)}{2}$$

or

$$\begin{aligned}\phi_o^i(k) &\geq \frac{(1-\varepsilon)(w^{i-\varepsilon})}{2(nk+1)^2} \cdot (nk+1+r_o)(nk+1-r_o) = \\ &= \frac{(1-\varepsilon)(w^{i-\varepsilon})}{2(nk+1)^2} [(nk+1)^2 - r_o^2] = \frac{(1-\varepsilon)(w^{i-\varepsilon})}{2} - \frac{r_o^2(1-\varepsilon)(w^{i-\varepsilon})}{2(nk+1)^2}.\end{aligned}$$

Choose $k_o = k_o(\varepsilon) = \frac{r_o(\varepsilon)}{n\sqrt{\varepsilon}} - \frac{1}{n}$. Then if $k > k_o$

$$(19) \quad \phi_o^i(k) \geq \frac{(1-\varepsilon)(w^{i-\varepsilon})}{2} - \frac{\varepsilon(1-\varepsilon)(w^{i-\varepsilon})}{2} = \frac{(1-\varepsilon)(w^{i-\varepsilon})}{2} = \frac{w^i}{2} + o^i(\varepsilon).$$

By estimating $\phi_o^o(k)$ we shall prove that the inequality in (19) can be reversed. Notice first that by the linear homogeneity and the concavity of F (Lemma 7) for each r -member coalition $S \subseteq N^k$ with profile s we have

$$\begin{aligned}F(r \cdot \bar{n}) - F(s) &= r[F(\bar{n}) - F(\bar{s})] \leq r(\bar{n} - \bar{s}) \nabla F(\bar{s}) \\ &\leq r \cdot \sum_{i=1}^n \left(\frac{1}{n} - \bar{s}_i\right) (w^{i+\varepsilon}) < r \cdot \delta \sum_{i=1}^n (w^{i+\varepsilon}),\end{aligned}$$

where ε and δ are chosen as in (16). Thus

$$(20) \quad F(s) \geq F(r\bar{n}) - r \cdot \delta \sum_{i=1}^n (w^{i+\varepsilon}).$$

Now by the definition of $\phi_o^o(k)$

$$\begin{aligned}\phi_o^o(k) &= \frac{1}{nk+1} \sum_{r=1}^{nk+1} \left[\mathbb{E} \left\{ \frac{1}{S} F(s) \mid |S| = r, o \in S \right\} \right] \\ &> \frac{1-\varepsilon}{k(nk+1)} \sum_{r=r_o+1}^{nk+1} \mathbb{E} \{ F(s) \mid |S| = r, o \in S \text{ and } \|\bar{s} - \bar{n}\| < \delta \}.\end{aligned}$$

By (20)

$$\phi_o^o(k) > \frac{1-\varepsilon}{k(nk+1)} \sum_{r=r_o+1}^{nk+1} [F(r\bar{n}) - r \cdot \delta \sum_{i=1}^n (w^i + \varepsilon)].$$

Since

$$F(r \cdot \bar{n}) = \frac{r}{n} \cdot F(1, \dots, 1) = \frac{r}{n} v(N) = \frac{r}{n} \sum_{i=1}^n w^i,$$

we obtain

$$\begin{aligned}\phi_o^o(k) &> \frac{1-\varepsilon}{k(nk+1)} \sum_{r=r_o+1}^{nk+1} \left[\frac{r}{n} \sum_{i=1}^n w^i - \delta \cdot r \sum_{i=1}^n (w^i + \varepsilon) \right] \\ &= \frac{1-\varepsilon}{k(nk+1)} \cdot \left[\frac{1}{n} \sum_{i=1}^n w^i - \delta \sum_{i=1}^n (w^i + \varepsilon) \right] \cdot \sum_{r=r_o+1}^{nk+1} r \\ &= \frac{1-\varepsilon}{k(nk+1)} \left[\frac{\sum_{i=1}^n w^i}{n} - \delta \cdot \sum_{i=1}^n (w^i + \varepsilon) \right] \frac{(nk+2+r_o)(nk+1-r_o)}{2} \\ &> \frac{(1-\varepsilon)(nk+1-r_o)}{kn} \cdot \frac{\sum w^i}{2} - \frac{(1-\varepsilon)(nk+2+r_o)\delta}{k} \cdot \frac{\sum_{i=1}^n (w^i + \varepsilon)}{2}.\end{aligned}$$

Thus if $\delta = \delta(\varepsilon)$ is chosen as in (16) and also $\delta < \varepsilon$ then it is easy to

verify that for $k > k_0(\varepsilon)$ where $k_0(\varepsilon) = \frac{r_0(\varepsilon)-1}{n\varepsilon}$

$$(21) \quad \phi_o^i(k) \geq (1-\varepsilon) \cdot \frac{\sum_{i=1}^n w^i}{2} + o(\varepsilon^2) \cdot \sum_{i=1}^n (w^{i+\varepsilon}) = \frac{\sum_{i=1}^n w^i}{2} + o^0(\varepsilon).$$

Now since $\sum_{i=0}^n \phi_o^i(k) = v^k(N^k) = \sum_{i=1}^n w^i$, we have by (19) and (21)

$$\phi_o^i(k) = \frac{w^i}{2} + \sum_{h \neq i} o^h(\varepsilon)$$

for k sufficiently large. Thus

$$\lim_{k \rightarrow \infty} \phi_o^i(k) = \frac{w^i}{2}$$

and the proof of Theorem 2 is complete.

Proof of Theorem 3. The proof of this theorem is similar to the previous one.

Therefore we shall only sketch it briefly. First, consider the game v^k .

Denote a profile of a coalition $S \subseteq N^k \cup M$ by $(s^{M_0}, s^{M_1}, \dots, s^{M_{m-1}}, s^1, \dots, s^n)$

where for h , $0 \leq h \leq m-1$,

$$s^{M_h} = \begin{cases} 1 & \text{if } M_h \in S \\ 0 & \text{otherwise} \end{cases}.$$

It is easy to verify that for a random r -member coalition in the k -fold market with m monopolies

$$\text{Prob}\{s^{M_0} = s^{M_1} = \dots = s^{M_{m-1}} = 1\} = \prod_{h=0}^{m-1} \frac{r-h}{nk+m-h}.$$

Using the same arguments used in the proof of Theorem 2 we can show that for $i = 1, \dots, n$

$$\begin{aligned} \phi_M^i(k) &> \frac{(1-\varepsilon)(w^{i-\varepsilon})}{nk+m} \prod_{h=0}^{m-1} \frac{1}{nk+m-h} \cdot \sum_{r=r_0+m}^{nk+m} \prod_{h=0}^{m-1} (r-h) \\ &> (1-\varepsilon)(w^{i-\varepsilon}) \cdot \frac{1}{\prod_{h=0}^{m-1} (nk+m-h)} \int_{r_0+m-1}^{nk+m} \prod_{h=0}^{m-1} (r-h) dr. \end{aligned}$$

The denominator $\prod_{h=0}^{m-1} (nk+m-h)$ is asymptotically $(nk)^{m+1}$ and the integrand is a polynomial of degree m where r^m appears with a coefficient 1. Obviously, all the other monomials are negligible since their integral is asymptotically of the order $(nk)^q$ for $q < m$ which is small relative to $(nk)^{m+1}$. Hence since

$$\int_{r_0+m-1}^{nk+m} r^m = \frac{(nk+m)^{m+1}}{m+1} - \frac{(r_0+m-1)^{m+1}}{m+1}$$

we obtain asymptotically

$$(22) \quad \phi_M^i(k) > \frac{(1-\varepsilon)(w^{i-\varepsilon})}{m+1} = \frac{1}{m+1} w^i + o^i(\varepsilon).$$

Similarly it can be shown that

$$\phi_M^{M_h}(k) > \frac{1}{m+1} \sum_{i=1}^n w^i + o^{M_h}(\varepsilon)$$

which together with (22) and the fact that $\sum_{i=1}^n \phi_M^i(k) + \sum_{h=0}^{m-1} \phi_M^{M_h}(k) = \sum_{i=1}^n w^i$ imply the result.

Consider now the game v_L^k . For a random r -member coalition in the k -fold market with m -oligopolies

$$\text{Prob}\{\text{for at least one } h, s \stackrel{L_h}{=} 1\} = 1 - \prod_{h=0}^{m-1} \frac{(nk+1-(r-h))}{mk+m-h} \equiv a(r).$$

Asymptotically

$$a(r) \approx 1 - \frac{(nk-r)^m}{(nk)^m} = \frac{r}{(nk)^m} ((nk)^{m-1} + (nk)^{m-2}(nk-r) + \dots + (nk-r)^{m-1})$$

and as in v_M^k , all other terms except $\frac{r}{(nk)^m} \cdot (nk-r)^{m-1}$ are negligible. Thus asymptotically for $i=1, \dots, n$

$$(23) \quad \phi_L^i(k) \geq \frac{(1-\varepsilon)(w^i-\varepsilon)}{(nk)^{m+1}} \int_{r_0+m-1}^{nk+m} r(nk-r)^{m-1} dr.$$

Integrating the last integrand by parts we obtain

$$\begin{aligned} \int_{r_0+m-1}^{nk+m} r(nk-r)^{m-1} dr &= -\frac{(nk+m)(-m)^m}{m} + \frac{(r_0+m-1)(nk-r_0-m+1)^m}{m(m+1)} \\ &\quad -\frac{(-m)^{m+1}}{m(m+1)} + \frac{(nk-r_0-m+1)^{m+1}}{m(m+1)}. \end{aligned}$$

All the summands but the last one are negligible relative to $(nk)^{m+1}$ hence by (23) we obtain that asymptotically

$$(24) \quad \phi_L^i(k) \geq \frac{(1-\epsilon)(w^i-\epsilon)}{m(m+1)} = \frac{1}{m(m+1)} w^i + o^i(\epsilon).$$

On the other hand in a similar way it can be shown that for $h = 0, \dots, m-1$

$$\phi_L^{L_h}(k) \geq \frac{1}{m(m+1)} \sum_{i=1}^n w^i + o^{L_h}(\epsilon)$$

which together with (24) and the fact that $\sum_{i=1}^n \phi_L^i(k) + \sum_{h=0}^{m-1} \phi_L^{L_h}(k) = \sum_{i=1}^n w^i$ imply the result.

Proof of Theorem 4

(I) Clearly $\lim_{k \rightarrow \infty} A^k = \{\alpha \in E^n \mid \alpha \leq w\}$. Thus by (11)

$$\lim_{k \rightarrow \infty} C_0(k) \supseteq \left\{ (\beta^0, \alpha) \in E^{1+n} \mid \alpha \leq w \text{ and } \beta^0 + \sum_{i=1}^n \alpha^i = \sum_{i=1}^n w^i \right\}.$$

It is left to show that the reverse inclusion holds. Indeed, let $(\beta^0, \alpha) \in \lim_{k \rightarrow \infty} C_0(k)$. Then there exists a sequence $\{(\beta^0(k), \alpha(k))\}_{k=1}^{\infty}$ in $\{C_0(k)\}_{k=1}^{\infty}$ respectively such that

$$\lim_{k \rightarrow \infty} (\beta^0(k), \alpha(k)) = (\beta^0, \alpha).$$

Let S^k be a coalition of traders in N^k with a profile $s_k = (k, \dots, k, k-1, k, \dots, k)$ i.e. $s_k^{i'} = k$ for $i' \neq i$ and $s_k^i = k-1$. Then²

²See Footnote 1.

$$(25) \quad \beta^0(k) + \sum_{i'=1}^n \frac{s_k^{i'}}{k} \alpha^{i'}(k) > \frac{1}{k} F(k, \dots, k, k-1, k, \dots, k)$$

and

$$(26) \quad \beta^0(k) + \sum_{i'=1}^n \alpha^{i'}(k) = \frac{1}{k} F(k, \dots, k).$$

Subtracting (25) from (26) and using the concavity of F we have

$$\frac{1}{k} \alpha^i(k) < \frac{1}{k} [F(k, \dots, k) - F(k, \dots, k, k-1, k, \dots, k)] < \frac{1}{k} \frac{\partial F}{\partial x_i}(k, \dots, k, k-1, \dots, k).$$

Since $\frac{\partial F}{\partial x_i}$ is homogeneous of degree zero

$$\alpha^i(k) < \frac{\partial F}{\partial x_i} \left(\frac{k}{nk-1}, \dots, \frac{k}{nk-1}, \frac{k-1}{nk-1}, \frac{k}{nk-1}, \dots, \frac{k}{nk-1} \right).$$

Taking the limit of both sides of the inequality as k tends to infinity we obtain by the continuity of $\frac{\partial F}{\partial x_i}$

$$\alpha^i < \frac{\partial F}{\partial x_i}(\bar{n}) = w^i,$$

and the proof of the first part of the theorem is complete.

(II) The proof is completely analogous to the proof of the first part of the theorem.

(III) Follows from the theorem of Debreu-Scarff and from

$$C_L(k) = \{(0, \alpha) \in E_+^{m+n} \mid \alpha \in Cv^k\},$$

which holds for each non-negative integer k .

Proof of Proposition 5. We prove a somewhat stronger version of Proposition 5.

Let (N,v) be a non-decreasing three person game such that

- (a) $Cv \neq \phi$
- (b) $v(i) = 0$ for $i=1,2,3$.

Then

$$(27) \quad \bar{\alpha}^i = v(N) - v(N-i).$$

Indeed it is easy to verify that $\bar{\alpha}^i \leq v(N) - v(N-i)$. It is therefore sufficient to find one imputation α in C such that $\alpha^i = \bar{\alpha}^i$. W.l.o.g. assume $i=1$. Then again it is easy to verify that $(\alpha^1, \alpha^2, \alpha^3)$ defined by

$$\alpha^1 = v(N) - v(2,3)$$

$$\alpha^2 = v(2,3) - \alpha^3$$

and where α^3 is a number satisfying

$$\text{Max}\{0, v(1,3) + v(2,3) - v(N)\} \leq \alpha^3 \leq \text{Min}\{v(2,3), v(N) - v(1,2)\},$$

is in the core C of v .

Now, let (β^0, α) be in C_0 . Then

$$\beta^0 + \alpha^2 + \alpha^3 \geq v(2,3)$$

and

$$\beta^0 + \alpha^1 + \alpha^2 + \alpha^3 = v(N)$$

Hence $\alpha^1 \leq v(N) - v(2,3) = \bar{\alpha}^1$ and the proof is complete.

Proof of Theorem 6. Parts (2) and (3) follow directly from Theorems 3 and 4.

We shall prove the first part of the theorem. Let A be a subset of an

Euclidean space. A point $\gamma \in A$ is a center of symmetry of A iff for each x in the space, $(\gamma+x) \in A$ implies that $(\gamma-x) \in A$. We claim that

$$(\beta^0, \alpha) = \left(\frac{\sum_{i=1}^n w^i}{2}, \frac{w^1}{2}, \dots, \frac{w^n}{2} \right) = \lim_{k \rightarrow \infty} \phi_0(k)$$

is the center of symmetry of $\lim_{k \rightarrow \infty} C_0(k)$. Indeed, let $(b^0, e) \in E^{1+n}$ and assume that $((\beta^0, \alpha) + (b^0, e)) \in \lim_{k \rightarrow \infty} C_0(k)$. This happens if and only if

$$0 \leq \frac{1}{2} w^i + e^i \leq w^i \quad \text{for all } i = 1, 2, \dots, n$$

(28) and

$$b^0 + \sum_{i=1}^n e^i = 0 \quad .$$

This implies that for each i , $-\frac{1}{2} w^i \leq e^i \leq \frac{1}{2} w^i$ or equivalently -

$$\frac{1}{2} w^i \leq -e^i \leq \frac{1}{2} w^i. \quad \text{Thus}$$

$$0 \leq \frac{1}{2} w^i - e^i \leq w^i$$

and

$$-b^0 - \sum_{i=1}^n e^i = 0.$$

Hence $((\beta^0, \alpha) - (b^0, e)) \in \lim_{k \rightarrow \infty} C_0(k)$ and the proof is complete.

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