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COMPARISON OF BARGAINING SOLUTIONS, UTILITARIANISM AND THE MINIMAX RULE BY THEIR EFFECTIVENESS

by

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Abstract

The two-person bargaining solutions of Nash, Kalai and Smorodinsky, and Rosenthal are generalized to apply to a larger class of games including most games with a finite set of possible agreements. This results in a simpler list of axioms for two of the theories. Several probability measures over bargaining games are stated and the three solutions are compared with each other on the basis of the expected payoff of a player and the expected minimum payoff between the two players. Other arbitration schemes such as the utilitarian rule, the maximin rule, and the choice of a Pareto-optimal point at random are compared with the bargaining models.

Sample calculations for a probability measure with the disagreement outcome a fixed value and the feasible set a random variable suggest that the Nash and the Kalai-Smorodinsky solutions are superior to the Rosenthal solution, but when the disagreement point is randomly chosen and the feasible set is fixed, the Rosenthal solution is superior. This is related to the axioms of the models and it is shown that there are practical as well as philosophical grounds for choosing these axioms.

§1. Introduction

Nash (1950) portrayed two-person bargaining as a pair (d,S), where $d = (d_1, d_2)$ is a point and S is a set of points in the players' utility plane. Set S contains the possible agreements the players can make and d is the outcome that occurs if no agreement is reached. A solution function f(d,S) associates a point $x^* = (x_1^*, x_2^*)$ to the pair (d,S). Point x^* is called the solution of the bargaining game.

Nash showed that under certain assumptions of f and S a unique solution function exists. Later, Kalai and Smorodinsky (1975) and Rosenthal (1976) modified Nash's axioms and derived two more solution functions.

These authors chose their axioms on philosophical grounds. For example, Nash accepted the axiom of independence of irrelevant alternatives whereas Rosenthal argued against it. Both authors resorted basically to intuition to justify their positions.

No decisive arguments have appeared for or against the axioms in the three systems, so one can say that all three solutions are plausible answers. The existence of several solutions is an embarrassment to bargaining theory as a whole, since players need a single point to agree on. Each of the authors took care to show that their axioms produced a unique solution but with several models and no guide to tell which one is most appropriate in the situation, the bargainers are still confronted with the problem of non-uniqueness.

Instead of asking which model has intuitively reasonable axioms, this paper will ask: for which types of bargaining problems will each of the models give the bargainers high utilities on the average? The question of the intuitive rationality of axioms is replaced by that of the average effectiveness of the solution, in a given class of situations.

Of course, the optimal model depends on the probability measure over bargaining games used, and there is no measure that is uniquely appropriate on _a priori_ grounds, but the issue of which measure is best can be compared with empirical evidence. Within a specific bargaining context one can investigate which types of games are most likely to arise. The unsolvable philosophical question of which axioms are most "rational" is replaced by the decidable empirical question of which probability measure over bargaining games is most realistic.

The three original theories assumed that the feasible set S is convex and compact, but in this paper the requirement of convexity is dropped. In §2, the theorems of Nash, Kalai and Smorodinsky, and Rosenthal are generalized to apply to compact but possibly nonconvex sets of outcomes.

The aim of this extension to nonconvex sets is to allow the calculations in §3. If S is nonconvex it may possibly contain a finite number of outcomes, so we can define a simple probability measure over the set S and calculate or estimate various statistics concerning the resulting solution payoffs. We will be able to answer questions such as: Which bargaining model gives a player the highest average utility in a long series of games? If utilities are comparable, how well on the average does each bargaining model treat the worse-off player in the game? How do the three bargaining models compare in these respects with other arbitration schemes such as choosing the agreement that maximizes total utility, that maximizes minimum utility, or choosing an agreement at random from the Pareto-optimal set?

§2. Generalizations of Existing Solutions

The solution functions of Nash, of Kalai and Smorodinsky, and of Rosenthal will be labelled f_N, f_{KS} and f_R.

The set $\mathcal{B}$ is defined as those bargaining games (d,S) with S compact, convex and with $x \geq d$ for every $x \in S$ and some $x \in S$ such that $x > d$. (This gives the players an incentive to bargain.) In the original papers each solution function had the set $\mathcal{B}$ as its domain. In this section the domains of the three solution functions are extended to three larger domains $\mathcal{B}_N$, $\mathcal{B}_{KS}$ and $\mathcal{B}_R$, which are supersets of the set $\mathcal{B}$.

The first step in generalizing Nash's solution is to define the function w_N:

$$w_N(x,d) = \begin{cases} (x_1 - d_1) \cdot (x_2 - d_2), & \text{if } x \geq d \\ 0, & \text{otherwise} \end{cases}$$

This can be regarded as a social objective function to be maximized, an indicator of the desirability of an outcome as a solution from Nash's viewpoint. (This claim will be substantiated by Theorem 1.) The curves of constant w_N form rectangular hyperbolae in the players' utility plane.

The domain of the generalized Nash solution function, the set $\mathcal{B}_N$, is defined as those bargaining games (d,S) such that

1) (compactness) S is compact

2) (incentive to bargain) for some $x \in S$, $x > d$

3) (uniqueness) a unique $x \in S$ maximizes $w_N(x,d)$ over S.

Comparing Nash's domain $\mathcal{B}$ with $\mathcal{B}_N$, Nash's requirement of convexity has been replaced by 3) which ensures that the solution using w_N will be unique. It may seem <u>ad hoc</u> to restrict the domain of f_N in this way but for practical purposes it is not a limitation on $\mathcal{B}_N$ since $\mathcal{B}_N$ contains "almost all" bargaining games satisfying 1) and 2).

The following is a version of Roth's simplification of Nash's axioms (Roth, 1975). Whereas Nash and Roth applied the axiom system to the set $\mathcal{B}$, here it is applied to $\mathcal{B}_N$. For $f: \mathcal{B}_N \Rightarrow \mathbb{R}^2$ and $(d,S) \in \mathcal{B}_N$,

<u>N1</u> (individual rationality). $f(d,S) > d$.

<u>N2</u> (symmetry-1). If (d,S) is invariant under an interchange of players, then $f(d,S)$ gives the players equal payoffs.

• <u>N3</u> (invariance under affine transformations). Let $A((x_1,x_2)) = (c_1x_1 + d_1, c_2x_2 + d_2)$ for some positive c_1, c_2. Then it is required that $f(A(d),A(S)) = A(f(d,S))$.

<u>N4</u> (independence of irrelevant alternatives). If (d,S) and (d,S') are bargaining games such that $S \subseteq S'$ and $f(d,S') \in S$, then $f(d,S) = f(d,S')$.

The <u>generalized Nash solution</u> $f_N(d,S)$ is now defined to be the outcome maximizing $w_N(x,d)$ for $x \in S$.

Theorem 1 characterizes the functions that satisfy these axioms. Parts 1) and 2) of Theorem 1 state that f_N satisfies the axioms and is unique in doing so. Parts 3) and 4) state that f_N is a generalization of Nash's solution, and that f_N cannot be further generalized to any reasonably rich set of compact games that is a superset of $\mathcal{B}_N$. By "reasonably rich" we mean a set of games that is closed under three operations: a switching of the players, an affine transformation of the utilities, and the formation of certain sum games using games already in the set.

<u>Theorem 1</u>. (Generalization of Nash solution)

1) f_N satisfies N1 through N4.

2) Any $f: \mathcal{B}_N \Rightarrow \mathbb{R}^2$ satisfying N1 through N4 is identical to f_N.

3) If $(d, S) \in \mathcal{B}$, then $f_N(d, S)$ is equal to the standard Nash solution.

4) Let $\mathcal{B}'$ be a set of games (d, S) such that

   i) S is compact and there is an $x \in S$ such that $x > d$,

   ii) $\mathcal{B}'$ is a proper superset of $\mathcal{B}_N$,

   iii) for any $(d, S) \in \mathcal{B}'$, $(T(d), T(S)) \in \mathcal{B}'$ where $T(d)$, $T(S)$ are the points generated from d and S by interchanging the players,

   iv) for any $(d, S) \in \mathcal{B}'$, if $(d, S') \in \mathcal{B}'$ then $(d, S \cup S') \in \mathcal{B}'$,

   v) for any $(d, S) \in \mathcal{B}'$ and affine transformation A, $(A(d), A(S)) \in \mathcal{B}'$.

   Then there is no $f\colon \mathcal{B}' \Rightarrow \mathbb{R}^2$ satisfying N1 through N4.

Theorem 1 is essentially saying that f_N is the right function to use in generalizing Nash's theory to non-convex games. It is proved in the appendix.

Turning now to the Kalai-Smorodinsky theory, the same steps are followed: we state an indicator function w_{KS}, define the domain, $\mathcal{B}_{KS}$, of the generalized solution function, f_{KS}, give a set of axioms defining the solution function, and then prove a theorem about the appropriateness and uniqueness of f_{KS}.

The indicator function is defined:

$$w_{KS}(x,d,S) = \text{Min}\left[\frac{x_1 - d_1}{c_1 - d_1}, \frac{x_2 - d_2}{b_2 - d_2}\right]$$

Here $b(S)$ is defined as the outcome (b_1,b_2) in S such that b_2 is a maximum for all outcomes in S, and b_1 is a maximum given a maximal b_2. The definition of $c(S)$ is analogous with the players' viewpoints interchanged. The desirability of an outcome is thus judged by the minimum of its payoffs, where the payoffs are normalized with respect to d and the extreme points, b and c, of S.

Since $x > d$ for some $x \in S$, it follows that $c_1 \neq d_1$ and $b_2 \neq d_2$ so there is no possibility that the denominators appearing in the definition of w_{KS} will be zero.

The set of points x giving a constant value of $w_{KS}(x,d,S)$ is a horizontal half-line extending rightwards and a vertical half-line extending upwards, as shown in Figure 1.

As a step in stating the domain of the solution function, f_{KS}, three subsets of S will be defined. The set of points in S maximizing $w_{KS}(x,d,S)$ in S is labelled M(d,S). The set of points M(d,S) lie in two half-lines as depicted by Figure 1.

Figure 1 about here

The subset of M(d,S) that lies on the horizontal half-line will be labelled $M_1(d,S)$ and the subset of M(d,S) on the vertical half-line will be labelled $M_2(d,S)$.

The domain $\mathcal{B}_{KS}$ is then defined as those bargaining games (d,S), such that

1) S is compact,
2) for some $x \in S$, $x > d$,
3) $M(d,S) = M_1(d,S)$ or $M_2(d,S)$.

The third condition states that the points x in S maximizing $x_{KS}(x,d,S)$ must all lie on a vertical half-line or else all lie on a horizontal half-line. (It follows for example that the bargaining game of Figure 1 is not in $\mathcal{B}_{KS}$.) Clearly $\mathcal{B} \subset \mathcal{B}_{KS}$.

The following axioms correspond to those of Kalai and Smorodinsky except axiom KS3 which is a generalization of their axiom of Pareto-optimality. This allows it to be applied to the extended domain $\mathcal{B}_{KS}$. If KS3 is applied to Kalai and Smorodinsky's original domain $\mathcal{B}$, it is equivalent to their axiom.

<u>KS1</u> (symmetry-1). Identical to N2.

<u>KS2</u> (invariance under affine transformations). Identical to N3.

<u>KS3</u> (monotonicity). Define the following function for $x_1 \le c_1(S)$ $g_S(x_1) = x_2$ if x_2 is the least value such that (x_1,x_2) is not Pareto-dominated by any point in S.

It is required that if $c_1(S) = c_1(S')$ and $g_S \le g_{S'}$, then $f_2(d,S) \le f_2(d,S')$ (where $f_2(d,S)$ is the second component of $f(d,S)$).

If set S is compact and convex then the function defined in KS3, g_S will be part or all of the Pareto-optimal boundary of S. If S is finite, then g_S will be a step-function decreasing to the right.

The generalized Kalai-Smorodinsky solution will be defined as the highest point in $M_1(d,S)$ or $M_2(d,S)$.

$f_{KS}(d,S) = x \in M(d,S)$ such that if $x \in M_1(d,S)$, then x_1 is maximum or if $x \in M_2(d,S)$ then x_2 is maximum.

Theorem 2 states that f_{KS} satisfies the axioms, is the unique function to do so over $\mathcal{B}_{KS}$, is a generalization of the Kalai-Smorodinsky solution and cannot be generalized further. It is proved in the appendix.

<u>Theorem 2</u>. (Generalization of the Kalai-Smorodinsky solution) Theorem 1 holds <u>mutatis</u> <u>mutandis</u> for f_{KS}, $\mathcal{B}_{KS}$ and KS1 through KS3.

For Rosenthal's solution, the following indicator function is used:

$$w_R(x,S) = \text{Min}\left[\frac{x_1 - b_1}{c_1 - b_1}, \frac{x_2 - c_2}{b_2 - c_2}\right] \quad \text{if } b \ne c$$

$$= \text{Min}\left[x_1, x_2\right] \quad \text{if } b = c$$

The development proceeds in a way similar to the generalization of the Kalai-Smorodinsky solution. The sets M(S), $M_1(S)$ and $M_2(S)$ are defined based on $w_R(x,S)$, and the domain $\mathcal{B}_R$ is similarly defined. The following axioms are required of any solution function f(d,S).

R1 (symmetry-2). If S is invariant under an interchange of players then f(d,S) gives the players equal payoffs, $f_1(d,S) = f_2(d,S)$.

R2 (invariance under affine transformations). Identical to N3.

R3 (monotonicity-2). Let P be that part of the weakly Pareto-optimal set of S lying between b(S) and c(S). Let P' be the corresponding set for S' and suppose that $b(S) = b(S')$, $c(S) = c(S')$. If the convex hull of P contains that of P', it is required that $f(d,S) \geq f(d,S')$.

The generalized Rosenthal solution is defined as the highest point on $M_1(S)$ or $M_2(S)$:

$f_R(S) = x \in M(S)$ such that if $x \in M_1(S)$ then x_1 is a maximum, or if $x \in M_2(S)$ then x_2 is a maximum, for $(x_1, x_2) \in S$.

Theorem 3. (Generalization of Rosenthal's solution).

Theorem 1 holds *mutatis mutandis* for f_R, $\mathcal{B}_R$ and R1 through R3.

One interesting feature of the generalized Kalai-Smorodinsky and Rosenthal solutions is that the individual axioms given above are fewer in number and weaker than those in the original papers. Specifically, in our generalized Kalai-Smorodinsky solution, the axiom of symmetry is weakened, and in the generalized Kalai-Smorodinsky and Rosenthal solutions, Pareto-optimality is not required as an axiom but follows from the other axioms. Of course each system is not weaker as a whole since the solution function must obey the axioms over a wider domain of bargaining games, but in terms of individual axioms generalizing these theories has made them simpler.

One aspect of the generalized theories that is more complex than the originals is the definitions of the three domains $\mathcal{B}_N$, $\mathcal{B}_{KS}$ and $\mathcal{B}_R$.

The sets $\mathcal{B}_N$ and $\mathcal{B}_{KS}$ include "almost all" finite bargaining games -- those games not included form a set of measure zero. In the definition of f_N for example, the only games that are unsolvable are those in which more than one outcome maximizes w_N.

For the generalized Rosenthal solution, f_R, it is not true that almost all finite games are solvable. If a game has exactly two Pareto-optimal outcomes, both will maximize w_R and there will be no way to choose between them. How likely is this to happen? If the players' payoffs for the m outcomes better than d are drawn independently from continuous distributions, then the probability p_m of exactly two Pareto-optimal outcomes satisfies the recursive equation (O'Neill, 1981)

$$p_1 = 0$$

$$p_m = [(m - 1)\ p_{m-1} + 1/(m - 1)]/m.$$

Some values of p_m are $p_2 = 1/2$, $p_5 = .417$, $p_{15} = .218$ and $p_{50} = .090$. Thus games with small numbers of outcomes may be unsolvable by f_R, but the likelihood of this declines with the number of possible alternatives.

§3. Some Probability Calculations

Several probability measures over games will be defined now as a way of evaluating the behaviour of the three solution methods. Some other solution rules will be included in the comparison. We define S_r as the strictly individually rational outcomes in S, that is $S_r = \{x|\ x \in S \text{ and } x > d\}$. We will investigate the following rules.

f_{DICT}: the outcome chosen maximizes x_1 in S_r

f_{MXMN}: the outcome chosen maximizes $Min(x_1, x_2)$ over S_r

f_{MXAV}: the outcome chosen maximizes $(x_1 + x_2)/2$ over S_r

F_{PO}: the outcome is chosen by sampling from the Pareto-optimal set, such that each outcome in the Pareto-optimal set of S has equal probability.

Using f_{DICT}, player 1 is a dictator who can select the best possible outcome. Player 2 has no bargaining power, but only the option of withdrawing from the interaction if player 1 grants the former no better than the disagreement payoff d_2. These are the best and worst positions any bargainers could hope for. All other bargaining models will be between these solutions so they serve as two extremes for comparison.

The function f_{MXMN} is the maximin rule of choice discussed by Rawls (1971) and others. Function f_{MXAV} is the utilitarian rule of choice.

The final bargaining method, F_{PO} suggests a situation in which the players can agree that the outcome should be Pareto-optimal but have no further guidance in selecting an outcome so they decide to choose a Pareto-optimal point using a random device. A common approach in welfare economics is to accept a certain allocation scheme as satisfactory if it can be proved that it yields a Pareto-optimal distribution, so our calculation of the expected utility using F_{PO} can be regarded as testing the validity of this approach. We want to know if Pareto-optimality typically means high benefits, at least for the probability measures used here.

Notice that two of the methods, f_{MXAV} and f_{MXMN}, rely on interpersonal comparisons of utilities, but that the others do not.

These four rules, plus the three generalized solution functions f_N, f_{KS} and f_R will be applied to games and the following two statistics will be calculated: (Here the solution by each rule is denoted by X^*.)

$E(X_i^*)$, the expected value to player i when the rule is in use.

$E[\text{Min}\ (X_1^*, X_2^*)]$, the expectation of the minimum of the two payoffs.

The first statistic measures the <u>effectiveness</u> of the rule, in terms of the average benefit to a player who will use the rule repeatedly. The second measures the <u>stability</u> of the rule, since the lower the benefits to the less contented player, the less incentive will that player have to stay in the agreement or even to make the agreement in the first place.

Calculation of the average $E(X_i^*)$ does not assume interpersonal comparability of utilities but calculation of the minimum of the two payoffs does.

<u>Model 1</u>. (Fixed disagreement outcome, exponential and uniform distribution of feasible set of outcomes.)

Statistics for the seven bargaining rules listed above were determined for the following type of probability measure over the bargaining games

1) The set of possible agreements has fixed and finite size m.
2) The payoff to player i of agreement j is the random variable X_{ij}, $i = 1, 2$; $j = 1,\ldots m$. All X_{ij} are independent and have a common probability distribution $F(x)$. The values of the X_{ij} are known to the bargainers.
3) The disagreement point is $(0,0)$.

These bargaining games have a fixed disagreement point, but the set of possible agreements varies from game to game.

Statistics were calculated for the unit exponential distribution, $F(x) = 1 - e^{-x}$, and for the uniform distribution on [0,1], $F(x) = x$, $0 \leq x \leq 1$. Two values of the size of S were chosen, $m = 5$ and $m = 15$. Results are shown in Tables 1 and 2.

Tables 1 and 2 and Figure 2 about here

Some of the values could be calculated analytically (formulae are given in the appendix) but others had to be estimated by Monte Carlo methods. Each of the simulated values is marked by a dagger in the tables, each is based on 20,000 trial games, which were enough so that each is at least three standard deviations away from the nearest value in the column.

In the case of f_R sometimes the game happened to have exactly two Pareto-optimal points and was thus unsolvable. In these instances one of the two was chosen as the solution randomly assigning each probability 1/2.

The tables suggest that the ranking of the rules by their effectiveness, $E(X_i^*)$, is independent of the number of outcomes m and almost independent of the choice of underlying distribution. The same can be said for the ranking according to minimum payoff. The Nash solution is more effective, but the Kalai-Smorodinsky solution is more stable, i.e., has a higher expected value of $\min(X_1^*, X_2^*)$. Both are preferable to Rosenthal's solution for effectiveness and stability, as shown in Figure 2.

The Rosenthal solution is not always inferior as the next example shows.

<u>Model 2.</u> (Fixed feasible set, variable disagreement outcome)

Here the feasible set is the unit disc $x_1^2 + x_2^2 \leq 1$, and the disagreement point is chosen from this set at random, from a uniform distribution over the disc.

Some of the statistics were calculated analytically as detailed in the appendix, and others had to be simulated. The results are shown in Table 3.

Table 3 and Figure 3 about here

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§4. Discussion

In the fixed disagreement point example, model 1, the Nash solution does extremely well. As shown in Figure 2, it is a good compromise between maximizing payoff and maximizing the minimum of the two payoffs.

We would like to know if this is a general property of the Nash solution, true for a wider class of situations than the probability models specified here. Perhaps it is, for the following reason. Each of the three methods in effect lays down a family of curves in the utility plane and chooses the outcome on the highest curve. The utilitarian solution, f_{MXAV}, uses straight lines of slope -1. The maximin solution lays down right-angled corners, each curve comprising the points directly above and to the right of a point on the diagonal $x_1 = x_2$. The Nash solution uses rectangular hyperbolae with the utility plane axes as the axes of each hyperbola.

If the Nash solution gives outcomes between f_{MXAV} and f_{MXMN}, it may be because its curves combine features of the latters' curves. For "unfair" points (x_1, x_2) far from the diagonal, Nash's hyperbolae become almost parallel to the axes, like f_{MXMN}. But if fairer points are available, Nash tries more or less to maximize average utility, i.e., near the diagonal $x_1 = x_2$ the hyperbolae approximate straight lines of slope -1. This argument about the Nash model does not rely on any specific assumption about the probability distribution of the points in the feasible set so it may be a characteristic type of behavior, independent of the probability model assumed.

Another possible general property of the models is shown by the poor behavior of the Rosenthal solution in model 1, where the feasible set was a random variable. We might regard the three bargaining models as different ways of finding a high utility outcome, the three being subject to certain restrictions like invariance under affine transformations of utility. The

Nash solution chooses an outcome according to its position relative to the disagreement point. The positions of the other possible outcomes are irrelevant since Nash requires the axiom of independence of irrelevant alternatives. Rosenthal, on the other hand, adopts the monotonicity axiom, and makes the solution independent of the position of the disagreement point for a given feasible set. In his model a solution is chosen because of its position relative to the extreme values of the feasible set, which in our model 1 are random variables. An analogy would be that Nash and Rosenthal are both shooting at a moving object, but Rosenthal is standing on a moving platform.

It is understandable that Rosenthal's method does better in model 2, where the disagreement point is the value of a random variable. Nash's solution does poorly here. The Kalai-Smorodinsky solution depends partly on d and partly on S, and gives results between the other two.

The axioms of most bargaining models can be divided into two types, which could be termed <u>basic</u> and <u>novel</u> axioms. The basic axioms are those such as Pareto-optimality, symmetry and invariance under utility transformations. These are common to almost all models and essentially define the system as a bargaining model. The novel axioms on the other hand are variable from model to model and are more controversial since they reflect each author's special philosophy about what must be added to the basic axioms to get a unique solution. The aim of this paper is to show that effectiveness as well as philosophical considerations play a role in choosing the novel axioms. The calculations show that the differences in effectiveness are sometimes quite large and can be related to the content of the axioms.

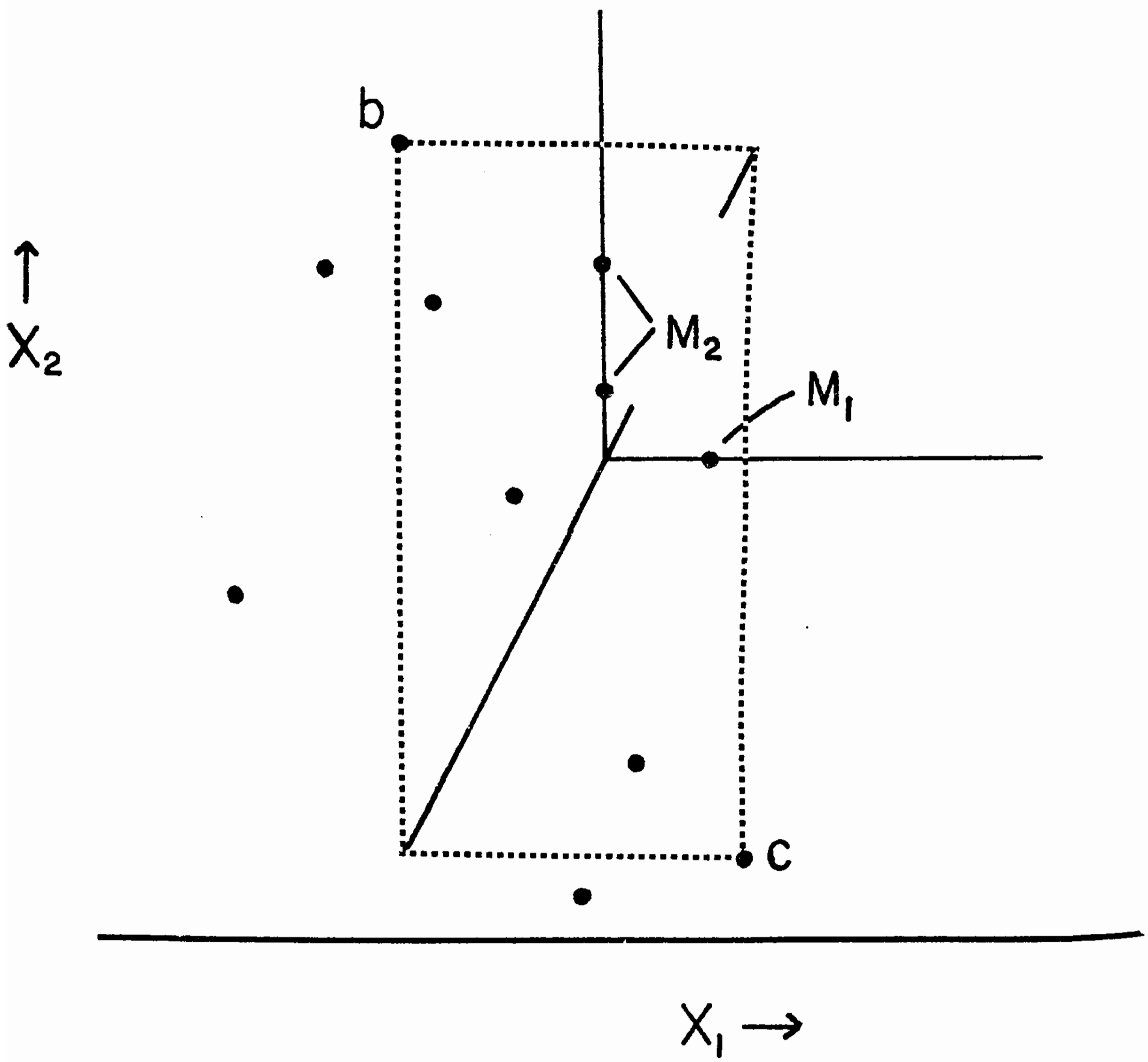

Figure 1

Exponential Distribution		
Solution Function	Number of outcomes	
	m = 5	m = 15
DICT. (Player 1)	2.28	3.32
MAXAV	1.90	2.51
NASH	1.82†	2.36†
K-S	1.76†	2.22†
MAXMIN	1.64	2.16
RANDOM P.O.	1.60	2.05
ROSENTHAL	1.56†	1.91†
DICT. (Player 2)	1.00	1.00

Uniform Distribution		
Solution Function	Number of outcomes	
	m = 5	m = 15
DICT. (Player 1)	.833	.938
MAXAV	.739	.842
NASH	.736†	.842†
K-S	.730†	.834†
MAXMIN	.723	.833
ROSENTHAL	.679†	.790†
RANDOM P.O.	.670	.744
DICT. (Player 2)	.500	.500

Table 1. Randomly chosen feasible set S -- expectations, $E(X_i^*)$. † indicates a value determined by simulation.

Exponential Distribution		
Solution Function	Number of outcomes	
	m = 5	m = 15
MAXMIN	1.14	1.66
K-S	1.10†	1.60†
NASH	1.09†	1.56†
ROSENTHAL	1.08†	1.43†
MAXAV	.952	1.28
RANDOM P.O.	.875	1.10
DICTATOR	.833	.938

Uniform Distribution		
Solution Function	Number of outcomes	
	m = 5	m = 15
MAXMIN	.631	.778
K-S	.626†	.774†
NASH	.623†	.769†
MAXAV	.609	.763
ROSENTHAL	.563†	.698†
RANDOM P.O.	.500	.573
DICTATOR	.476	.496

Table 2. Randomly chosen feasible set S expected minimum payoff, $E(\mathrm{Min}\,(X_1^*,X_2^*))$. † indicates a value determined by simulation.

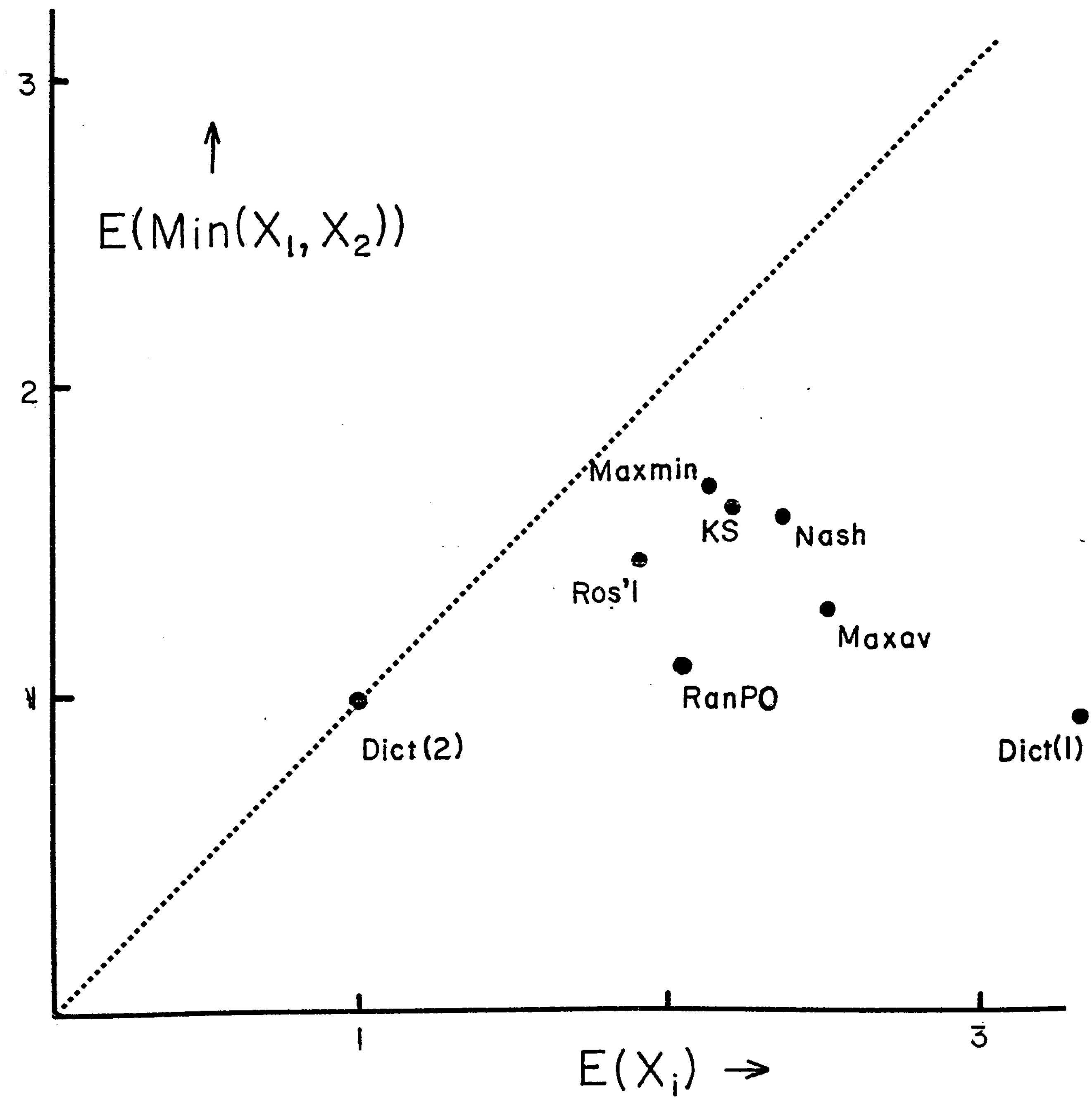
3
2
1
E(Min(X1, X2))
Maxmin
KS
Nash
Ros'l
Maxav
RanPO
Dict(2)
Dict(1)
1
3
E(Xi)

Figure 2

Solution Function	$E(X_i^*)$
DICT (Player 1)	.924
MAXAV MAXMIN	.707
ROSENTHAL	.694†
K-S	.689†
NASH	.679
RANDOM P.O.	.645
DICT (Player 2)	.137

Solution Function	$E(\mathrm{Min}(X_1^*, X_2^*))$
MAXAV MAXMIN	.707
ROSENTHAL	.598†
K-S	.562†
NASH	.518
RANDOM P.O.	.333
DICT	.145

Table 3. Randomly chosen disagreement point, d, expected payoff and expected minimum payoff. † indicates a value determined by simulation.

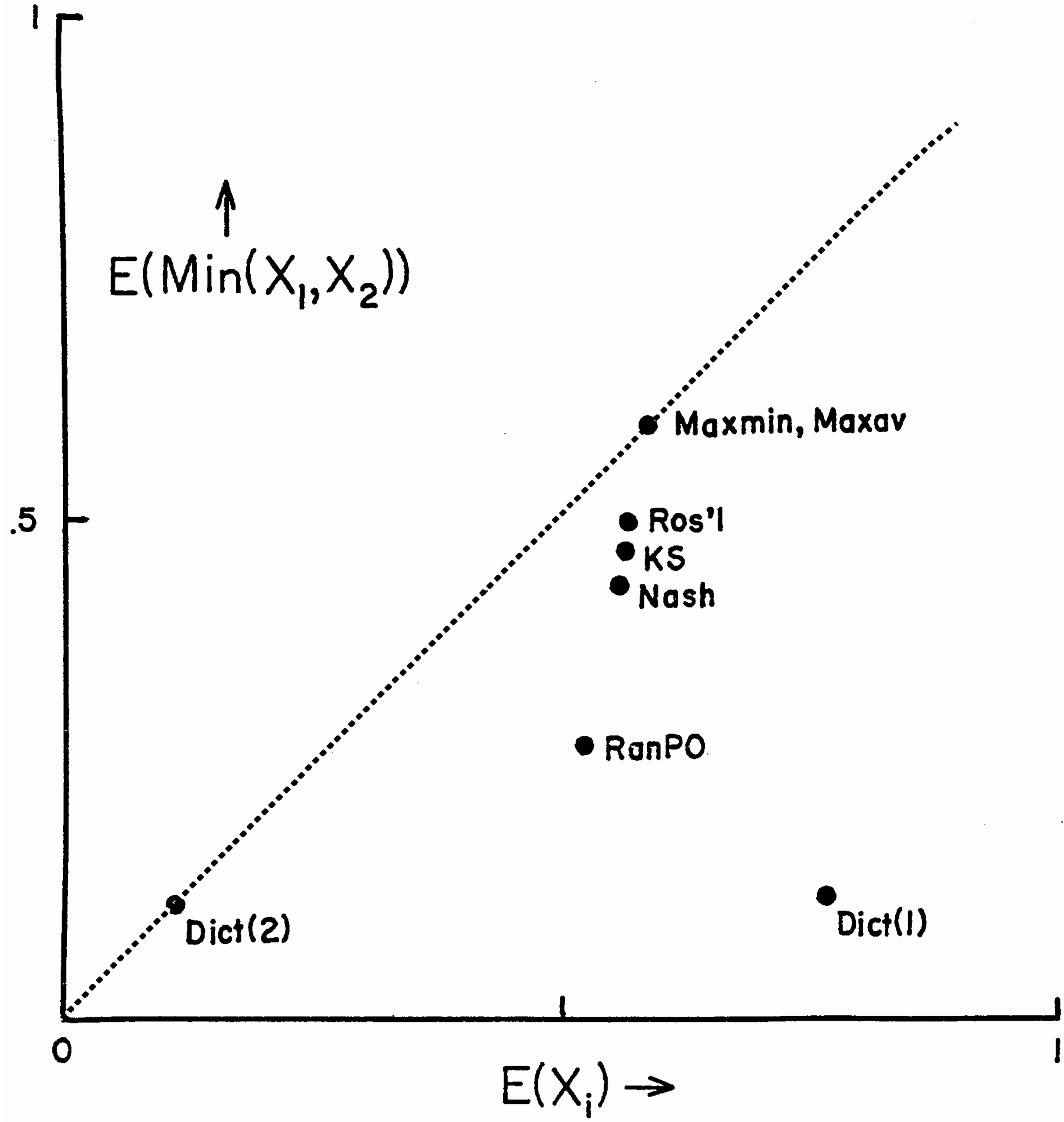
1
.5
0
1
E(Min(X1,X2))
E(Xi)
Maxmin, Maxav
Ros'l
KS
Nash
RanPO
Dict(2)
Dict(1)

Figure 3

APPENDIX

Proof of Theorem 1.

1) It may be verified directly that f_N satisfies each axiom.

2) For some $(d,S) \in \mathcal{B}_N$, let x^* be the outcome in S maximizing w_N, i.e., $x^* = f_N(d,S)$. Outcome x^* is uniquely defined by the definition of $\mathcal{B}_N$.

Let A be an affine transformation such that $A(d) = (0,0)$ and $A(x^*) = (1,1)$. Define $S_1 = A(S)$. Define S_2 as the set such that for every $(x_1,x_2) \in S_1$, both (x_1,x_2) and (x_2,x_1) are in S_2, i.e., $S_2 = S_1 \cup T(S_1)$. (This is the crucial difference from Nash's proof.)

Clearly S_1 and S_2 are in $\mathcal{B}_N$. Choosing an arbitrary $f\colon \mathcal{B}_N \Rightarrow \mathbb{R}^2$ satisfying N1 through N4, then by N1 (individual rationality) and N2 (symmetry), $f(0,S_2) = (1,1)$. Thus by N4 (independence), $f(0,S_1) = (1,1)$. By N3 (invariance), $f(0,S) = A^{-1}(1,1) = x^*$. Thus f and f_N coincide.

3) For convex S, the definitions of f_N and of Nash's solution are identical.

4) If $(d,S) \in \mathcal{B}'-\mathcal{B}_N$, then either no outcomes in S maximize w_N or more than one outcome does. The former situation is ruled out since S is compact and non-empty.

In the latter case choose two distinct outcomes in S maximizing w_N. For each, the argument of 2) can be repeated to show that it is the value of f(d,S). This is a contradiction so that f must be undefined for such a game (d,S). □

Proof of Theorem 2.

1) It may be verified directly that f_{KS} satisfies each axiom.

2) If $(d,S) \in \mathcal{B}_{KS}$, it can be assumed without loss of generality that $f_{KS}(d,S)$ is in $M_1(S)$, i.e., lies in the horizontal half-line containing the points of S maximizing w_{KS}.

Define x^* as the solution of (d,S), $x^* = f_{KS}(d,S)$ and let A be the affine transformation such that A(d) = (0,0) and $b_2(A(S)) = c_1(A(S)) = 1$. Define $S_1 = A(S)$ and $x^{*\prime}$ as the image of x^*, $x^{*\prime} = A(x^*)$.

Define S_2 as the strongly Pareto-optimal set of S. By the assumption that $x^* \in M_1(S)$ and by the formula for w_{KS} it follows directly that $x_1^{*\prime} \geq x_2^{*\prime}$. Define S_3 as identical to S_2 but with $(x_1^{*\prime}, x_2^{*\prime})$ replaced by $(x_2^{*\prime}, x_2^{*\prime})$. Define S_4 as the set of points (x_1, x_2) such that either (x_1, x_2) or (x_2, x_1) is in S_3, i.e., $S_4 = S_3 \cup T(S_3)$.

Since the set (d,S) is in $\mathcal{B}_{KS}$, it follows by the construction of S_1 to S_4 that $(0,S_1)$ to $(0,S_4)$ are in $\mathcal{B}_{KS}$.

Let f be some function f: $\mathcal{B}_{KS} \Rightarrow \mathbb{R}^2$ satisfying KS1, KS2 and KS3. By KS1 (symmetry), $f(0,S_4) = (x_2^{*\prime}, x_2^{*\prime})$. Set S_1 dominates S_2, S_2 dominates S_3, etc., as defined in the axiom KS3, so that by KS3 (monotonicity) $f(0,S_3) = (x_2^{*\prime}, x_2^{*\prime})$, $f(0,S_2) = f(0,S_1) = (x_1^{*\prime}, x_2^{*\prime})$. Since f satisfies KS2 (invariance) then $f(d,S) = A^{-1}[f(0,S)] = f_{KS}(d,S)$ and thus f_{KS} and f coincide.

3) For convex S, the equivalence of the two definitions can be shown by elementary geometry.

4) This can be shown analogously to Theorem 1 - 4). □

Proof of Theorem 3. The proof is identical to Theorem 2 with the exception that for Theorem 3, game (d,S) is normalized so that $b_2(S) = c_1(S) = 1$ and $b_1(S) = c_2(S) = 0$.

Theorem 4. (variable feasible set, exponential distribution)

For the following probability measure over bargaining games (d,S):

1) d = (0, 0).

2) S contains m outcomes whose payoffs to the players are independent samplings from the unit exponential distribution, $F(x) = 1 - e^{-x}$.

Then the expected payoffs for each solution method are as follows (where X_1^*, X_2^* is the solution by each method).

(i) for f_{DICT}: $E(X_1^*) = \sum_{j=1}^{m} \frac{1}{j}$

$$\sim \log m + C \text{ as } m \to \infty$$

where C is Euler's constant, .577216⁻.

$$E(X_2^*) = 1$$

and $E[\mathrm{Min}(X_1^*, X_2^*)] = m/(m + 1)$

(ii) for f_{MXMN}: $E(X_i^*) = \frac{1}{2}\sum_{j=1}^{m} \frac{1}{j} + \frac{1}{2}$

$$\sim \frac{1}{2}(\log m + C + 1) \text{ as } m \to \infty$$

$$E[\mathrm{Min}(X_1^*, X_2^*)] = \frac{1}{2}\sum_{j=1}^{m} \frac{1}{j}$$

$$\sim \frac{1}{2}(\log m + C) \text{ as } m \to \infty$$

(iii) for f_{MXAV}: $E(X_i^*) = \frac{m}{2}\sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} \sum_{k=0}^{j} \binom{j}{k} \frac{(k+2)!}{(j+1)^{k+3}}$

$$E[\mathrm{Min}(X_1^*,\ X_2^*)] = \frac{m}{2}\sum_{j=0}^{m-1}(-1)^j\binom{m-1}{j}\sum_{k=0}^{j}\frac{j!}{k!(j+1)^{j-k+1}}\sum_{r=0}^{k}\binom{k}{r}(r+1)!(j+1)^{-r-2}$$

(iv) for F_{PO}: $E(X_2^*) = \sum_{\substack{p+q+r=m-1\\ p,q,r\geq 0}} A(p,q,r)\cdot B(p+q)\cdot C(q+r)$

$E[\mathrm{Min}(X_1^*,\ X_2^*)] =$

$$2\sum_{\substack{p+q+r=m-1\\ p,q,r\geq 0}} A(p,q,r)\sum_{i=0}^{p+q}(-1)^{i+q+r}\binom{p+q}{i}\frac{1}{i+r+1}\sum_{j=0}^{q+r}(-1)^j\binom{q+r}{j}\frac{1}{(1+m+r+i-j)^2}$$

where $A(p,q,r) = \frac{m!}{p!q!r!}\sum_{\substack{0\leq i\leq p\\ 0\leq j\leq r}}\frac{T(i,p)\ T(j,r)}{1+i+j}$

$$B(p+q) = \sum_{i=0}^{p+q}(-1)^{p+q+i}\binom{p+q}{i}\frac{1}{m-i}$$

$$C(q+r) = \sum_{i=0}^{q+r}(-1)^{q+r+i}\binom{q+r}{i}\frac{1}{(m-i)^2}$$

$T(0,0) = 1,\quad T(i,0) = T(0,p) = 0$ for $i,p \geq 1$

and $T(i,p) = \frac{p-1}{p}T(i,p-1) + \frac{1}{p}T(i-1,\ p-1)$

Lemma 1. Let X be a real-valued random variable with density function $f(x) = e^{-x}$ and let Y be a real-valued random variable. Then $E(X-Y)|X>Y) = 1$

Proof. Let $G(y)$ be the distribution function of Y.

$$E(X-Y|X>Y) = \int_{y=-\infty}^{\infty}\int_{x=0}^{\infty}(x-y)\frac{f(x)}{1-F(y)}\,dx\,dG(y)$$

Substituting $f(x) = e^{-x}$ and $F(y) = 1-e^{-y}$ and integrating,

$$E(X-Y|X>Y) = \int_{y=-\infty}^{\infty} 1\,dG(y) = 1.$$

□

Proof of Theorem 4.

i) Under the solution function f_{DICT}, player 1 will choose $X_1^* = \text{Max}\{X_{11},\ldots X_{1m}\}$.

Let $X_{(m)}$ be the random variable Max $\{X_{11},\ldots X_{im}\}$. If an additional value, X_{1m} is added to a set of m-1, $\{X_{11},\ldots X_{1,m-1}\}$, the maximum of m, will be greater than the maximum of m-1 if and only if the new value is greater than $X_{(m-1)}$. If it is greater it will be greater by the amount $X_{1m} - X_{(m-1)}$. The expectations are related as follows:

$$E(X_{(m)}) = E(X_{(m-1)}) + Pr(X_{1m}>X_{(m-1)}) \cdot E(X_{1m}-X_{(m-1)}|X_{1m}>X_{(m-1)})$$

By symmetry $Pr(X_{1m}>X_{(m-1)}) = 1/m$. Then by Lemma 1

$$E(X_{(m)}) = E(X_{(m-1)}) + 1/m.$$
$$= \sum_{j=1}^{m} \frac{1}{j}$$

This is the first m terms of the harmonic series and is approaches by log m + C as $m \to \infty$, leading to the first two formulae in i).

Player 2's payoff X_2^* clearly has density function f(x) and thus expectation 1, leading to the third formula in i).

To show the last formula in i), $E[\text{Min}(X_1^*, X_2^*)]$ is

$$\int_{x=0}^{\infty} x[Pr(X_1^* \varepsilon (x,x+dx) \cap X_2^* > x) + Pr(X_2^* \varepsilon (x,x+dx) \cap X_1^* > x)]\, dx$$

Under the scheme f_{DICT}, X_1^* will be the maximum of m independent random variables, each distributed F(x). Also X_2^* will be independent of X_1^*, so that

$$E[\text{Min}(X_1^*,X_2^*)] = \int_{x=0}^{\infty} x\, [m\, f(x)\, F(x)^{m-1}(1-F(x)) + f(x)(1-F(x)^m)]\, dx$$

Substituting $f(x) = e^{-x}$, $F(x) = 1-e^{-x}$ and expanding the powers of F(x) as polynomials,

$$E[\mathrm{Min}(X_1^*,X_2^*)] = m\sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} \int_{x=0}^{\infty} xe^{-2x-jx}dx + \int_{x=0}^{\infty} xe^{-x}\, dx$$

$$- \sum_{j=0}^{m} (-1)^j \binom{m}{j} \int_{x=0}^{\infty} xe^{-x-jx}dx$$

Integrating, and combining the resulting series under a single index

$$E[\mathrm{Min}(X_1^*,\ X_2^*)] = \sum_{j=1}^{m} (-1)^{j-1} \binom{m}{j} \frac{1}{j+1}$$

The above sums to m/(m+1) (Gradshteyn and Rhyzik, 1965, 0•155•1), which is the final formula in i).

ii) Let Y_i be the random variable $\mathrm{Min}(X_{1i},X_{2i})$ and let g(y) be its density function. Then

$$g(y)dy = 2\ \Pr[X_{1i} \in (y,y + dy) \cap (X_{1i} < X_{2i})]$$

$$= 2\ f(y)(1 - F(y))\ dy$$

$$= 2e^{-2y}dy$$

Let $Z_{(m)}$ be the random variable $\mathrm{Max}(Y_1,\ldots Y_m)$. Following a method identical to Theorem 1, i), it can be shown that $E(Z_{(m)}) = \frac{1}{2}\sum\frac{1}{j}$.

The payoff pair with one member equal to $Z_{(m)}$ will be chosen as the solution. The player who does not receive $Z_{(m)}$ will receive a payoff at least as great of amount $Z'_{(m)}$.

By Lemma 1,

$$E(Z'_{(m)} - E(Z_{(m)}) = 1$$

The expected payoff of a player in the game is thus

$$E[(Z_{(m)} + Z'_{(m)})/2] = \frac{1}{2}\sum\frac{1}{j} + \frac{1}{2}.$$

which is the first formula of ii). The second formula for the asymptotic value of $E(X_i^*)$ follows as in i).

The third formula for $E[\mathrm{Min}(X_1^*, X_2^*)]$ was derived above as the value of $E(Z_{(m)})$.

iii) Define $Y_i = (X_{1i} + X_{2i})/2$ with density function $s(y)$. Then $s(y)$ has the gamma distribution:

$$s(y) = 4ye^{-2y}$$

The distribution function $S(y)$ is then $1 - 2ye^{-y} - e^{-y}$. The mean of the random variable $\mathrm{Max}[Y_1, \ldots Y_m]$ is then

$$\int_{y=0}^{\infty} y\ m\ s(y)[S(y)]^{m-1} dy$$

Substituting for $s(y)$ and $S(y)$ and integrating gives the first formula of iii).

To show the second formula of iii) let the joint distribution of (X_1^*, X_2^*) be $g(x,y)$. Then

$$g(x,y) = \Pr[(X_1^*, X_2^*) \in (x, x+dx) \times (y, y+dy)]$$

If the solution function f_{MXAV} is used,

$$g(x,y) = m \Pr[(X_{11}, X_{21}) \in (x, x+dx) \times (y, y+dy)$$

$$\cap\ (X_{1i} + X_{2i}) < x+y, \text{ for } i=2 \text{ to } m]$$

$$= m\ e^{-x}\ e^{-y}\ dx\ dy[1 - (x+y+1)e^{-x-y}]^{m-1}$$

Then

$$E[\mathrm{Min}(X_1^*, X_2^*)] = 2 \int_{y=0}^{\infty} y \Pr[X_2^* \in (y, y+dy) \cap X_1^* > y]$$

$$= 2\int_{y=0}^{\infty} y \int_{x=y}^{\infty} g(x,y)\ dx\ dy$$

Substituting for $g(x,y)$, expanding the factor $[1 - (x+y+1)e^{-x-y}]^{m-1}$ as a polynomial and integrating gives the second formula of iii).

iv) Let the possible outcomes in S be $\{A_1, \ldots A_m\}$. Let $\overline{Q}(A_i)$ be a vector-valued random variable giving the number of outcomes in the four quadrants centered on A_i, that is, $\overline{Q}(A_i) = (p,q,r,s)$ means that p outcomes lie northwest of a_i in the two-dimensional utility plane, q lie southwest, r lie southeast and s lie northeast. The payoffs associated with A_i are (X_{1i}, X_{2i}). As before let the coordinates of the outcome chosen by F_{PO} be (X_1^*, X_2^*). Then

$$E(X_1^*) = \sum_{i=1}^{m} \int_{x=0}^{\infty} x \int_{y=0}^{\infty} \Pr[(X_{1i},X_{2i}) \in (x,x+dx)\times(y,y+dy)]\ dy\ dx$$

Since the outcomes are identically and independently distributed

$$E(X_1^*) = \int_{x=0}^{\infty} x \int_{y=0}^{\infty} m \Pr\ [F_{PO} = A_1 \cap (X_{11}, X_{21}) \in (x,x+dx)\times(y,y+dy)]\ dy\ dx$$

Now if $\overline{Q}(A_i) = (p,q,r,s)$, A_1 cannot be chosen unless $s = 0$. Thus

$$\Pr[F_{PO} = A_1 \cap (X_{11},X_{21}) \in (x,x+dx)\times(y,y+dy)]$$

$$= \Pr[F_{PO} = A_1 \mid (X_{11},X_{21}) \in (x,x+dx)\times(y,y+dy)]$$

$$\cdot \Pr[(X_{11},X_{21}) \in (x,x+dx)\times(y,y+dy)]$$

$$= \sum_{\substack{p,q,r \geq 0 \\ p+q+r=m-1}} \Pr[F_{PO} = A_1 \mid \overline{Q}(a_1) = (p,q,r,0) \cap (X_{11},X_{21}) \in (x,x+dx)\times(y,y+dy)]$$

$$\cdot \Pr[\overline{Q}(A_1) = (p,q,r,0) \mid (X_{11},X_{21}) \in (x,x+dx)\times(y,y+dy)]$$

$$\cdot \Pr[(X_{11},X_{21}) \in (x,x+dx)\times(y,y+dy)] \qquad (1)$$

To calculate the first factor in the expression above, note that the solution function F_{PO} is such that the likelihood that $F_{PO} = a_1$ depends only on $\overline{Q}(a_1)$, so that the first factor equals $\Pr[F_{PO} = a_1 | \overline{Q}(a) = (p,q,r,0)]$.

Let $T(i,p)$ be the probability that of the p outcomes northwest of a_1, exactly i are Pareto-optimal. The corresponding probability for the r points southeast of a_1 will be $T(j,r)$. These functions are well-defined and satisfy the recursive equation given in Theorem 1,(iv), (O'Neill, 1981).

If there are i and j Pareto-optimal points in the two quadrants specified above, then outcome a_1 will be chosen with probability $\frac{1}{i+j+1}$, given it is Pareto-optimal. Otherwise it will be chosen with probability 0.

Thus

$$\Pr[F_{PO} = a_1 \mid \bar{Q}(a) = (p,q,r,0)]$$

$$= \sum_{\substack{0 \le i \le p \\ 0 \le j \le r}} T(p,i)\, T(r,j)\, \frac{1}{i+j+1} \qquad (2)$$

The second factor in (1) is calculated as follows. It is the probability that p <u>specified</u> outcomes lie northwest of a_1, and q <u>specified</u> outcomes lie southwest of a_1, etc., all of this times the number of ways of assigning the m-1 outcomes to the three distinct cells, lying northwest, southwest and southeast of a_i.

Thus it is

$$\frac{(m-1)!}{p!q!r!} F(x)^p\, (1-F(y))^p\, F(x)^q\, F(y)^q\, (1-F(x))^r\, F(y)^r \qquad (3)$$

where $F(x) = 1-e^{-y}$, and $F(y) = 1-e^{-y}$

The third factor in (1) is clearly

$$f(x)\, f(y)\, dx\, dy \qquad (4)$$

where $f(x) = e^{-x}$, $f(y) = e^{-y}$.

Performing these substitutions (2), (3), and (4) in (1) and integrating gives the first formula of Theorem 4,iv).

To find an expression for $E[\mathrm{Min}(X_1^*,X_2^*)]$ using the method F_{PO}, we substitute (2), (3) and (4) in (1). This gives an expression for the joint density of (X_1^*,X_2^*). Let this be $g(x,y)$. Then

$$E[\mathrm{Min}(X_1^*,X_2^*)] = 2\int_{y=0}^{\infty} y \int_{x=y}^{\infty} g(x,y)\,dx\,dy$$

Performing this integration gives the second formula of iv). □

Theorem 5. (variable feasible set, uniform distribution)

Under the assumptions of Theorem 4 but with $F(x) = 2x$, $0 \le x \le 1$, the uniform distribution on $[0,1]$,

i) for f_{DICT}:

$$E(X_1^*) = m/(m+1)$$
$$E(X_2^*) = 1/2$$
$$E[\mathrm{Min}(X_1^*,X_2^*)] = \frac{m(m+3)}{2(m+1)(m+2)}$$

ii) for f_{MXAV}: $E(X_i^*) =$

$$\frac{m}{2m+1}\,\frac{1}{2^m} + 4m\sum_{j=0}^{m-1}(-1)^j\binom{m-1}{j}\frac{j+2}{(j+1)(2j+3)}\,2^{-j-3}$$

$$E[\mathrm{Min}(X_1^*,X_2^*)] = 2m\sum_{j=0}^{m-1}\left(-\frac{1}{2}\right)^j\binom{m-1}{j}\sum_{k=0}^{2j}\frac{1}{k+1}\binom{2j}{k}\sum_{l=0}^{k+2}(-1)^{\ell}\binom{k+2}{l}\frac{1-\left(\frac{1}{2}\right)^{2j-k+\ell+1}}{2j-k+\ell+1}$$

iii) for f_{MXMN}

$$E(X_i^*) = \frac{m}{2}\sum_{j=0}^{m-1}(-1)^{m-1-j}\,2^j\binom{m-1}{j}\left[\frac{2}{2m-j}+\frac{1}{2m-1-j}-\frac{3}{2m-j+1}\right]$$

$$E[\mathrm{Min}(X_1^*,X_2^*)] = m\sum_{j=0}^{m-1}\binom{m-1}{j}(-1)^{m-1-j}\frac{2^{j+1}}{(2m-j)(2m-j+1)}$$

iv) for F_{PO}

$$E(X_i^*) = 2\sum_{\substack{p+q+r=m-1\\ p,q,r\ge 0}} A(p,q,r)\left(\sum_{i=0}^{m-1-p-q}(-1)^j\binom{m-1-p-q}{i}\frac{1}{p+q+i+1}\right)\left(\sum_{i=0}^{m-1-q-r}(-1)^i\binom{m-1-q-r}{i}\frac{1}{q+r+i+2}\right)$$

$$E[\mathrm{Min}(X_1^*,X_2^*)] = 2 \sum_{\substack{p+q+r=m-1\\ p,q,r \ge 0}} A(p,q,r) \sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{1}{p+q+i+1} \sum_{j=0}^{p} (-1)^j \binom{p}{j} \left(\frac{1}{q+r+j+2} - \frac{1}{p+2q+r+i+j+3}\right)$$

where A is given in Theorem 4.

The proof is almost identical to that of Theorem 4, and will be omitted.

Theorem 6. (feasible set in the unit disc, variable disagreement point)

For the bargaining game (d,S), let the disagreement point be determined by sampling from a uniform distribution over the unit disc $x_1^2 + x_2^2 \le 1$, and let the feasible set S be those points in the disc that are individually rational, i.e., $x_1 \ge d$, and $x_2 \ge d_2$. Then the expected values and expected minimum of the two values are as follows,

i) for f_{DICT}:

$$E(X_1^*) = \frac{1}{2} + \frac{4}{3\pi}$$

$$E(X_2^*) = \frac{2}{\pi} - \frac{1}{2}$$

$$E[\mathrm{Min}(X_1^*,X_2^*)] = \frac{23}{6\pi} - \frac{11}{3\sqrt{2}\pi} - \frac{1}{4}$$

ii) for f_{NASH}:

$$E(X_i^*) = \frac{32}{15\pi}$$

$$E[\mathrm{Min}(X_1^*,X_2^*)] = \frac{32-28\sqrt{2}}{15\pi}$$

iii) for F_{PO}:

$$E(X_i^*) = \frac{2\pi + 8/3}{\pi^2 + 4}$$

$$E[\mathrm{Min}(X_1^*,X_2^*)] = \frac{\pi(4-2\sqrt{2}) + 2\sqrt{2}/3}{\pi^2 + 4}$$

iv) for f_{MXAV} and f_{MXMN}:

$$E(X_i^*) = 1/\sqrt{2}$$

$$E[\mathrm{Min}(X_1^*,X_2^*)] = 1/\sqrt{2}$$

Proof: i) for f_{DICT}:

Suppose the disagreement point has coordinates (s,t). Then player 1's solution payoff, X_1^*, will be maximal given that player 2's payoff, X_2^*, is greater than t, and (X_1^*, X_2^*) is in the unit disc, that is,

$$X_1^* = 1 \quad \text{if } t < 0$$

$$= \sqrt{1 - t^2} \quad \text{if } t \geq 0.$$

Thus, letting the random variable yielding the disagreement point be (S,T)

$$E(X_1^*) = \int_{t=-1}^{0} Pr(T = t) \times 1 + \int_{t=0}^{1} Pr(T = t) \sqrt{1 - t^2}\, dt$$

$$= \int_{-1}^{0} \frac{2}{\pi}\sqrt{1 - t^2}\, dt + \int_{0}^{1} (1 - t^2)\, dt$$

$$= 1/2 + 4/3\pi$$

This is the first formula in i).

Concerning player 2's payoff, X_2^*, under f_{DICT},

$$X_2^* = 0 \quad \text{if } t < 0$$

$$X_2^* = t \quad \text{if } t \geq 0$$

$$E(X_2^*) = \int_{t=-1}^{0} Pr(T - t) \times 0 + \int_{t=0}^{1} Pr(T - t) + dt$$

$$= \int_{0}^{1} \frac{2t}{\pi\sqrt{1-t^2}}\, dt$$

$$= 2/\pi - 1/2$$

The second formula in i).

To calculate $E[Min(X_1^*, X_2^*)]$ under f_{DICT}, three cases are possible:

1) if $t < 0$, then $X_1^* = 1$, $X_2^* = 0$, and $Min(X_1^*, X_2^*) = X_2^* = 0$

2) if $0 \leq t < 1/\sqrt{2}$, then $X_1^* = \sqrt{1-t^2}$, $X_2^* = t$, and $Min(X_1^*, X_2^*) = X_2^* = t$

3) if $1/\sqrt{2} \leq t$, then $X_1^* = \sqrt{1-t^2}$, $X_2^* = t$ and $Min(X_1^*, X_2^*) = X_1^* = \sqrt{1-t^2}$

Therefore,

$$E[Min(X_1^*, X_2^*)] = \int_{t=-1}^{0} 0\, dt + \int_{t=0}^{\frac{1}{\sqrt{2}}} \frac{2}{\pi} \frac{t}{\sqrt{1-t^2}}\, dt + \int_{t=\frac{1}{\sqrt{2}}}^{1} \frac{2}{\pi} \frac{\sqrt{1-t^2}}{\sqrt{1-t^2}}\, dt$$

$$= 23/6\pi - 11/3\sqrt{2}\pi - 1/4$$

the third formula in i).

As a step in calculating formula ii) the expected value of the Nash solution, we will calculate for a general point on the Pareto-optimal set, the set of disagreement points d, which yield that solution. The Nash solution has the attractive property that this set is always a straight line segment.

For a Pareto-optimal point $(x, \sqrt{1-x^2})$, $x \in [0,1]$, the set of disagreement points which would result in that solution is a line segment of slope $x/\sqrt{1-x^2}$, extending from the solution point $(x, \sqrt{1-x^2})$, to the far side of the unit disc. This length of the segment can be calculated to be $4x\sqrt{1-x^2}$.

In a similar manner it can be shown that the set of disagreement points yielding a solution between $(x, \sqrt{1-x^2})$ and $(x + dx, \sqrt{1 - (x+dx)^2})$ is a thin trapezoid, with height 2x dx near the Pareto-optimal set, 6x dx on the opposite boundary of the disc. Thus player 1's expectation is

$$\int_{x=0}^{1} x \left(\frac{1}{\pi} \times \text{Area of trapezoid}\right)$$

$$= \int_{x=0}^{1} \frac{x}{\pi} (4x)\left(x\sqrt{1-x^2}\right) dx$$

$$= 32/15\pi$$

which is the first formula of iii).

The minimum value of X_1^*, X_2^* under f_N can be calculated:

$$E[\mathrm{Min}(X_1^*, X_2^*)] = X_1^* \quad \text{if } X_1^* \le 1/\sqrt{2}$$

$$= X_2^* \quad \text{if } X_1^* \ge 1/\sqrt{2}$$

$$= \int_{x=0}^{1/\sqrt{2}} x \, \frac{1}{\pi} (4x)\left(x\sqrt{1-x^2}\right) dx$$

$$+ \int_{x=1/\sqrt{2}}^{1} \left(\sqrt{1-x^2}\right) \frac{1}{\pi} (4x)\left(x\sqrt{1-x^2}\right) dx$$

$$= (32 - 28\sqrt{2})/15\pi$$

verifying the second formula in iii).

Concerning F_{PO}, formulae iii), the probability of player 1 receiving a payoff in $(x,x+dx)$, is proportional to the length of the arc of the Pareto-optimal surface lying between x and x+dx and also proportional to the area of the disc lying below and to the left of the point $(x, \sqrt{1-x^2})$.

The length of the arc is $dx/\sqrt{1-x^2}$ and the area below and left of the point is $\pi/2 + 2x\sqrt{1-x^2}$. To determine the constant of proportionality we use the fact that the total probability must sum to 1.

$$k \int_0^1 (\pi/2 + 2x \sqrt{1-x^2}) \;/\; \sqrt{1-x^2}\; dx = 1$$

which yields $k = 4/(\pi^2 + 4)$.

To find the average payoff

$$E(X_1^*) = k \int_0^1 x \; (\pi/2 + 2\,x\sqrt{1-x^2}) \;/\; \sqrt{1-x^2}\; dx$$

$$= (2\pi + 8/3)/(\pi^2 + 4)$$

verifying the first formula in iv).

To find the expectation of the minimum payoff

$$E[\mathrm{Min}(X_1^*,X_2^*)] =$$

$$k \int_0^{1/\sqrt{2}} x \;(\pi/2 + 2x\sqrt{1-x^2}\; dx$$

$$+ k \int_{1/\sqrt{2}}^{1} \sqrt{1-x^2}\; (\pi/2 + 2\,x\sqrt{1-x^2}) / \sqrt{1-x^2}\; dx$$

$$= [\pi(4 - 2\sqrt{2}) + 2\sqrt{2}/3]/(\pi^2 + 4)$$

which is the final formula of iii). □

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