Discussion Paper No. 52

(TOPOLOGICAL) SEMIVECTOR SPACES: CONVEXITY AND FIXED POINT THEORY*

by

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September 30, 1973

*We would like to thank Profs. James R. Munkres (M.I.T.) and Alexander D. Wallace (U. Florida) for short-circuiting some early fallacious conjectures. We are indebted to Prof. M. P. Schutzenberger (U. Paris) for a remark which compelled us to seek and formulate the notion of a "left skew semifield" [Prakash & Sertel, 1972] as the general sort of object over which to define semivector spaces. To Prof. T. Bowman (U. Florida) we are grateful for his pointing out an error in an earlier draft of this paper.

We would like to acknowledge the support provided by Prof. Z. S. Zannetos (M.I.T.) through a NASA grant (NGL22-009-309) for the work of M.R.S. at the Management Information for Planning and Control Group, Alfred P. Sloan School of Management, M.I.T.

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1. INTRODUCTION AND PRELIMINARIES

In this paper, we first introduce the notion of a (topological) semivector space. Without speaking too roughly, (topological) semivector spaces are to (topological) semigroups as (topological) vector spaces are to (topological) groups. We extend the notion of convexity to semivector spaces and study consequences of "pointwise convexity" (i.e., convexity of singletons). This is done in Section 2, where a structure theorem (2.8) is obtained for pointwise convex semivector spaces and a sufficient condition is found (2.10 and 2.11) for the embeddability of such spaces in vector spaces. In Section 3, we identify and briefly study a hierarchy of local convexity properties in topological semivector spaces. Lastly, in Section 4, we establish fixed point properties for compact convex subsets when such sets are locally convex in one or another sense.

In general, we define a semivector space over a left skew semifield, by which we mean a bimonoid $<\Theta, +, .>$ in which $<\Theta, .>$ is a group with zero 0 distinct from its identity 1, $<\Theta, +>$ is a commutative semigroup with identity 0, and the (unitary) left action of $<\Theta, .>$ on $<\Theta, +>$ is homomorphic: $\alpha.(\beta + \gamma) = (\alpha . \beta) + (\alpha . \gamma)$. We term a left skew semifield $<\Theta, +, .>$ simply a <u>semifield</u> when the multiplication (.) is also commutative. The set R_+ of non-negative reals under the usual addition and multiplication will be referred to as the <u>usual real semifield</u>. <u>N. B</u>. Unless otherwise indicated, in this paper the set R of real numbers will carry its usual topology and R_+ will carry the subspace topology.

1.0 <u>Definition</u>: Let $<\Theta, +, .>$ be a left skew semifield, and $<S, \oplus>$ a commutative semigroup. Let $\Psi: \Theta \times S \to S$, where we denote $\Psi(\lambda, s) = \lambda s$, be a map satisfying

Axiom 1:	$\lambda(\mu s) = (\lambda . \mu)s$	(left action)
Axiom 2:	$s \in \Theta s$	(unitariness)
Axiom 3:	$\lambda(s \oplus t) = \lambda s \oplus \lambda t$	(homomorphism)

for all $\lambda, \mu \in \Theta$ and $s, t \in S$. S will be called a <u>semivector space over Θ</u>, <u>convex</u> iff Θ contains the usual real semifield. When S and Θ are topological spaces, S will be called a <u>topological</u> semivector space over Θ iff $\oplus$ and Ψ are continuous. <u>N. B</u>. Unless specifically indicated to be otherwise, a topological semivector space will be assumed to possess Hausdorff topology.

<u>Example 1</u>: Every left skew semifield is a semivector space over itself.

<u>Example 2</u>: A <u>cone</u> C in a real vector space V (i.e., a

subset $C \subset V$ with $C + C \subset C$ and, for each $\lambda \in R_+$, $\lambda C \subset C$) forms a (convex) semivector space over R_+. In particular, this is true if C is an ordered cone semigroup, i.e., its natural pre-order is, in fact, a partial order, which is equivalent to saying that $C \cap -C = \{\underline{0}\}$. (The cone semigroups studied by Keimel [4,5] and Lawson and Madison [6] are of this type.)

Example 3: Given a real vector space V, the set $[V]$ of nonempty subsets and the set $\mathcal{Q}[V]$ of nonempty convex subsets in V, each forms a convex semivector space (of which $\mathcal{Q}[V]$ is "pointwise convex" -- see 2.3). Radstrom [9] has shown that, when V is a normed space, the set $\mathcal{KQ}[V]$ of nonempty compact and convex subsets of V is (a topological semivector space, and) embeddable in a normed vector space. Then, $\mathcal{KQ}[V]$ is, of course, a cone, but not an ordered cone semigroup: $\mathcal{KQ}[V] = -(\mathcal{KQ}[V])$.

Given a semivector space S over Θ, for each $\lambda \in \Theta$, define the "λ-transition" $\Psi^\lambda: S \to S$ by $\Psi^\lambda(s) = \lambda s$. In view of the fact that $<\Theta, .>$ is a group with zero, it is easily seen that the second axiom of the definition (1.0) above is equivalent to the requirement that Ψ^1 be the identity map of S, i.e., that $1s = s$ should obtain for each $s \in S$. From Axiom 3 we see that each transition of S is an endomorphism of S. In fact, when $\lambda \neq 0$, Ψ^λ is an automorphism of S, so that $\lambda\omega = \omega$ if $\omega \in S$ is identity or zero in S $(\lambda \neq 0)$. When Ψ is continuous, then Ψ^λ for each $\lambda \neq 0$ is not only continuous but (since Ψ^μ for $\mu = \frac{1}{\lambda}$ is also continuous) it is furthermore both an open and a closed map. Then $\{\Psi^{\lambda} \mid \lambda \in \Theta\backslash\{0\}\}$ is a group of iseomorphisms of S. Of course, as in topological semigroups, "s-translation" $\oplus^{s}: S \to S$ (defined by $\oplus^{s}(t) = s \oplus t \quad (t \in S)$) in a topological semivector space S need not be open.

2. CONVEXITY & POINTWISE CONVEXITY

The familiar notion of convexity for vector spaces extends naturally to semivector spaces:

2.0 Standing Notation: We denote the simplex $\{(\lambda_0, \ldots, \lambda_m) \in R_+^{m+1} \mid \sum_{i=0}^{m} \lambda_i = 1\}$ by Λ_m $(m = 0, 1, \ldots)$.

2.1 Definition: Let S be a convex semivector space. Given any two points $x, x' \in S$, their segment $[x:x']$ is defined to be $\{s = \lambda x \oplus \lambda' x' \mid (\lambda, \lambda') \in \Lambda_1\}$. A subset $T \subset S$ is called convex iff $[x:x'] \subset T$ whenever $x, x' \in T$.

Thus, what we call a convex semivector space (see 1.0), indeed checks to be convex accoring to the above definition.

The following are plain: if A is convex in a semivector space S, then $\mu A = \{\mu a \mid a \in A\}$ also is convex $(\mu \in R_+)$; if B, too, is convex in S, then so are $A \oplus B = \{a \oplus b \mid a \in A, b \in B\}$ and all convex combinations $\lambda A \oplus \lambda' B$ $((\lambda, \lambda') \in \Lambda_1)$. Of course, the intersection of any family of convex sets is convex. Again, as one is accustomed to in topological vector spaces, we have

2.2 Proposition: In convex topological (not necessarily Hausdorff) semivector spaces, topological closure (Cl) preserves convexity.

Proof: Let Q be convex in S, a convex topological semivector space. If $Q = \emptyset$ there is nothing to prove, so let q, q' be adherent points of Q. Suppose $\lambda q \oplus \lambda' q' = \bar{q} \notin \mathrm{Cl}(Q)$ for some $(\lambda, \lambda') \in \Lambda_1$. Then there exists a nbd V of $\bar{q}$ disjoint from $\mathrm{Cl}(Q)$. Then map $\Omega: S \times S \to S$, defined by $\Omega(x, x') = \lambda x \oplus \lambda' x'$, being continuous, there is a nbd $U \times U'$ of (q, q') such that $\Omega(U \times U') \subset V$. Since q and q' are adherent points of Q, there exists $(y, y') \in U \times U'$ such that $y, y' \in Q$. Then, by convexity of Q, $\Omega(y, y') \in Q$, a contradiction. Hence, $\bar{q} \in \mathrm{Cl}(Q)$ and $\mathrm{Cl}(Q)$ is convex, as to be shown.

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Not all aspects of convexity in (topological) vector spaces carry over, however, to (topological) semivector spaces. We give the following examples of convex (topological) semivector spaces so that we may discuss some of the pathologies which they allow.

<u>Example 4</u> ("Max Space"): Take $S = R_+$ with $s \oplus t = \mathrm{Max}(s, t)$, and define scalar multiplication $\Psi: R_+ \times S \to S$ as the usual multiplication. Then S is a convex semivector space, topological when R_+ has the usual topology. Here $[x:x'] = [\frac{xx'}{x + x'}, \mathrm{Max}(x, x')]$, unless $x = x' = 0$; and $[0:0] = \{0\}$.

<u>Example 5</u> ("Min Space"): In the above example define, instead, $s \oplus t = \mathrm{Min}(s, t)$, leaving all else as is. This too gives a convex semivector space, topological when R_+ has the usual topology. Here $[x:x'] = [0, \frac{xx'}{x + x'}]$, unless $x = x' = 0$;

and $[0:0] = \{0\}$.

Example 6: Take S to be the set $[R]$ of all nonempty subsets $A, B \subset R$ with $A \oplus B = \{a + b \mid a \in A, b \in B\}$ and $\lambda A = \{\lambda a \mid a \in A\}$, as usual $(\lambda \in R)$. This yields a convex semivector space, topological when all spaces are discrete. Here $[A:B] = \{\lambda A \oplus \lambda' B \mid (\lambda, \lambda') \in \Lambda_1\}$.

Example 7: In Example 6 set $0A = R$ for every $A \in [R]$, leaving everything else unchanged. Again, we obtain a convex semivector space, topological when all spaces are discrete. Here, however, $[A:B] = \{R\} \cup \{\lambda A \oplus \lambda' B \mid \lambda = (1 - \lambda') \in (0, 1)\}$.

The first "pathology" to note is that segments need not be convex (e.g., $[x:x']$ with $x, x' > 0$ in Max Space), even in a space where every segment $[x:x']$ owns its "generating points," x, x'. Segments are all convex in the spaces of Examples 5, 6 and 7. But even so, Examples 5 and 7 allow segments which own neither of their generating points: in Min Space consider any $[x:x']$ with $x, x' > 0$; and in Example 7, consider $[A:A]$ for any two-element set $A = \{a, a'\} \in [R]$. Finally, while in Example 6 segments are all convex and own their generating points, even then singleton sets need not be convex: $\{A\} \supset [A:A]$ iff $A \subset R$ is convex.

The discussion so far already motivates the following definition

2.3 Definition: Given a convex semivector space S and a subset

$T \subset S$, T will be said to be <u>pointwise convex</u> iff each $\{x\} \subset T$ is convex.

Given a convex semivector space S over a semifield Θ (containing R_+), the largest pointwise convex subset $T \subset S$ is evidently a semivector subspace: if $x, y \in T$, then $\lambda(x \oplus y) \oplus \lambda'(x \oplus y) = (\lambda x \oplus \lambda' x) \oplus (\lambda y \oplus \lambda' y) = x \oplus y$, and for any $\theta \in \Theta$, $\lambda(\theta x) \oplus \lambda'(\theta x) = (\lambda.\theta)x \oplus (\lambda'.\theta)x = (\theta.\lambda)x \oplus (\theta.\lambda')x = \theta(\lambda x \oplus \lambda' x) = \theta x$ $((\lambda, \lambda') \in \Lambda_1)$. From here on we will be concerned with semivector spaces which are pointwise convex.

Now, for a convex semivector space S to be pointwise convex is the same thing as for it to obey the distribution $(\lambda + \mu)x = \lambda x \oplus \mu x$ for $\lambda, \mu \in R_+$ $(x \in S)$. Clearly, if this distribution holds for the integers $\lambda, \mu \in R_+$, then it holds for the rationals $\lambda, \mu \in R_+$; thus, if furthermore S is topological, then the distributive formula extends for the whole of R_+. (In this context, it becomes interesting to note that if the mentioned distribution holds for the integers in R_+, then S is uniquely divisible: $\frac{1}{n} x$ is the n^{th} root of x (cf. [1]).)

In pointwise convex spaces, while segments $[x:x']$ and (by definition) singleton sets are always convex, segments may fail to own their generating points: in Example 8 below, the segment $[(0, 2):(1, 3)]$ fails to own the generating point $(0, 2)$.

<u>Example 8</u>: Take $S = \{0, 1\} \times R_+$ with $(\delta, x) \oplus (\delta', x') = (\mathrm{Max}(\delta, \delta'), x + y)$, and define scalar multiplication

Ψ: $R_+ \times S \to S$ by $\lambda(\delta, x) = (\delta, \lambda x)$ $(\delta, \delta' \in \{0, 1\};$ $x, x', \lambda \in R_+)$. S is a convex semivector space, topological when S has the product topology of $2 = \{0, 1\}$ discrete and R_+ usual.

Nevertheless, given a set $A \subset S$ in a pointwise convex semivector space S, its convex hull $\hat{A}$ (i.e., the smallest convex set containing A) -- and, hence, every convex set in S -- has the constructive characterization of

2.4 Proposition: Let S be a convex semivector space, and let $A \subset S$. Denote the set of finite subsets of A by $\mathcal{F}(A)$, and, for each $F = \{a_0, \ldots, a_m\} \in \mathcal{F}(A)$, define (the "closed simplex") $\sigma(F) = \{\lambda_0 a_0 \oplus \ldots \oplus \lambda_m a_m \mid (\lambda_0, \ldots, \lambda_m) \in \Lambda_m\}$. If S is pointwise convex, then $\hat{A} = \bigcup\{\sigma(F) \mid F \in \mathcal{F}(A)\}$.

Proof: Assume that S is pointwise convex. Denote $\Sigma(A) = \bigcup\{\sigma(F) \mid F \in \mathcal{F}(A)\}$. $A \subset \Sigma(A)$ is clear. To see the convexity of $\Sigma(A)$, take any $p, p' \in \Sigma(A)$ so that $p \in \sigma(F)$ and $p' \in \sigma(F')$ for some $F, F' \in \mathcal{F}(A)$. Take any $(\lambda, \lambda') \in \Lambda_1$ and check that $\lambda p \oplus \lambda' p' \in \sigma(F \cup F') \subset \Sigma(A)$. (Pointwise convexity is used to collect coefficients of elements, if any, in $F \cap F'$.) Taking any convex set $Q \supset A$, one shows $\sigma(F) \subset Q$ for all $F \in \mathcal{F}(A)$ by induction on the candinality of F. This then establishes $\Sigma(A) \subset Q$ and completes the proof.

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2.5 Remark: Defining the closed convex hull of a subset A in a convex topological semivector space S as the smallest closed convex subset of S containing A, from 2.2 we see that this is nothing but $Cl(\hat{A})$.

In a pointwise convex space S whose "origin" $0S$ is singleton, say $0S = \{e\}$, e is the identity of S: $e \oplus x = 0x \oplus 1x = (0 + 1)x = 1x = x$.

2.6 Proposition: A pointwise convex space S has a singleton origin iff $x, x' \in [x:x']$ for all $x, x' \in S$.

Proof: If the origin $0S$ is singleton, say $0S = \{e\}$, then e is the identity, and $0x \oplus 1x' = x'$, $1x \oplus 0x' = x \in [x:x']$, showing "only if." To see "if," take any $0x, 0x' \in 0S$ and note that their segment $[0x:0x'] = \{0x \oplus 0x'\}$ is singleton, so that $\{0x, 0x'\} \subset [0x:0x']$ implies $0x = 0x'$.

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Thus, among the convex semivector spaces, it is precisely the pointwise convex spaces with singleton origin that are free of all the "pathologies" discussed after Example 7. Furthermore, in such spaces the characterization of convex sets is the familiar one which obtains in real vector spaces:

2.7 Proposition: Let S be a pointwise convex semivector space, and let $A \subset S$. For each $F = \{a_0, \ldots, a_m\} \in \mathscr{F}(A)$ (see 2.4), define (the "open simplex") $\sigma^*(F) = \{\lambda_0 a_0 \oplus \ldots \oplus \lambda_m a_m$

$\mid (\lambda_0, \ldots, \lambda_m) \in \Lambda_m$ with $\lambda_i > 0 \quad (i = 0, \ldots, m)\}$. If OS is singleton, then $\hat{A} = \{\sigma^*(F) \mid F \in \mathcal{F}(A)\}$.

Proof: Define $\Sigma^*(A) = \cup\{\sigma^*(F) \mid F \in \mathcal{F}(A)\}$. Now, recalling the proof of 2.4, $A \subset \Sigma^*(A) \subset \Sigma(A)$ is clear. Thus, any convex set $Q \supset A$ contains $\Sigma^*(A)$ as $Q \supset \Sigma(A)$ already holds according to 2.4. This leaves only the convexity of $\Sigma^*(A)$ to show. For this, we assume $OS = e$ and take any $p, p' \in \Sigma^*(A)$, so that $p \in \sigma^*(F)$ and $p' \in \sigma^*(F')$ for some $F, F' \in \mathcal{F}(A)$. Taking any convex combination $\bar{p} = \lambda p \oplus \lambda' p'$ with $(\lambda, \lambda') \in \Lambda_1$, we show that $\bar{p} \in \sigma^*(F \cup F')$ when $\lambda \in (0, 1)$ just as in the proof of 2.4. Then, as $OS = e$, we see that $\bar{p} = p$ when $\lambda = 1$ and $\bar{p} = p'$ when $\lambda = 0$, and conclude that $\Sigma^*(A)$ is convex, finishing the proof.

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The following theorem gives the structure of pointwise convex semivector spaces in terms of semivector subspaces having singleton origins.

2.8 THEOREM: Let S be a pointwise convex semivector space and define the (equivalence) relation $\mathcal{S} \subset S \times S$ by $(x, y) \in \mathcal{S}$ iff $Ox = Oy$. Then each equivalence class $S_e \in S/\mathcal{S}$ (where $OS_e = \{e\}$) is itself a (pointwise convex) semivector space with e as its identity, and $S/\mathcal{S}$ is an "unscaled" semivector space of idempotents. (Thus, $S/\mathcal{S}$ is a semilattice.)

Proof: By definition, for each equivalence class S_e, $0S_e$ is singleton, say, $0S_e = \{e\}$. Since S is pointwise convex, e is identity for S_e. Now S_e is a semivector space, since $0x = e$ implies $0(\lambda x) = e$ and, if $0y = e$ too, then $0(x \oplus y) = 0x \oplus 0y = e \oplus e = e$. This much also shows that $S/\mathcal{S}$ consists of idempotents which are "unscaled." But $\mathcal{S}$ is a semigroup congruence: $0x = 0y$ implies that $0(x \oplus z) = 0x \oplus 0z = 0y \oplus 0z = 0(y \oplus z)$ for each $z \in S$ $(x, y \in S)$. Thus, $S/\mathcal{S}$ is commutative semigroup of idempotents, i.e., a semilattice. Since it is "unscaled," calling it a semivector space is another way of looking at it.

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Given a pointwise convex semivector space S, R_+x for each $x \in S$ is an ordered cone semigroup (cf. Example 2), so that S can always be expressed as a union of ordered cone semigroups; if, furthermore, S has singleton origin $0S = \{e\}$, then S is a union of ordered cone semigroups with common identity e. All this should not, however, lead one to believe that S is, therefore, embeddable in a group, since the following is a counter-example.

Example 9: Take $X = R_+ \times \{0\}$ and $Y = \{0\} \times R_+$, each with usual coordinatewise addition, and extend this to $\oplus$ on $S = X \cup Y$ by equating $(a, 0) \oplus (0, b)$ to $(0, a + b)$ if $b \neq 0$ and to $(a, 0)$ if $b = 0$. (Thus, $(0, 0)$ is identity.) Define scalar multiplication $\Psi: R_+ \times S \to S$ by $\lambda(a, b) = (\lambda a, \lambda b)$.

Then S is a non-cancellative pointwise convex semivector space with $0S = \{(0, 0)\}$. [N.B. It follows from 2.10 below that there is no Hausdorff topology for S which yields it operations continuous!]

In fact, for a convex semivector space S, consider the properties (1) S is (Hausdorff) topological, (2) S is pointwise convex, (3) $0S = \{e\}$. Although 2.10 shows the sufficiency of the conjunction (1) & (2) & (3) to yield S cancellative, no two of these properties by themselves make S cancellative: Example 9 shows the insufficiency of (2) & (3); Example 4 shows the insufficiency of (1) & (3); and the next proposition shows the insufficiency of (1) & (2).

2.9 <u>Proposition</u>: Let S be a cancellative semivector space. If $x = x \oplus 0x$ holds for each $x \in S$ (e.g., if S is pointwise convex), then the origin $0S$ consists of exactly one element, which acts as the identity element.

<u>Proof</u>: Assume the hypothesis and let $x, y \in S$. Then $(x \oplus y) \oplus 0(x \oplus y) = x \oplus y = x \oplus (y \oplus 0y) = (x \oplus y) \oplus 0y$, so that cancellation gives $0(x \oplus y) = 0y$. Similarly, $0(x \oplus y) = 0x$, showing that $0x = 0y$ and that this unique element acts as identity.

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2.10 <u>THEOREM</u>: Let S be a pointwise convex topological semivector space with the origin $0S = \{e\}$. Then S is cancellative [and e is identity].

Proof: That e is identity is a consequence of the formula $x = (0 + 1)x = 0x \oplus x$. To see cancellation, take any $x, y, z \in S$, and suppose $x \oplus y = x \oplus z$. Then $x \oplus y \oplus y = x \oplus z \oplus y = x \oplus z \oplus z$. By pointwise convexity, we can write this as $x \oplus 2y = x \oplus 2z$, and repeating the argument, we have $x \oplus ny = x \oplus nz$ for every positive integer n, so that $\frac{1}{n}x \oplus y = \frac{1}{n}x \oplus z$. As $n \to \infty$, $\frac{1}{n}$ converges to 0, so that the continuity of the operations with S Hausdorff gives $0x \oplus y = 0x \oplus z$ in the limit. $0x$ being identity, we have $y = z$, so that S is cancellative.

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2.11 Remark: Let S be a cancellative pointwise convex semivector space over R_+. Then S is embeddable in a real vector space, in standard fashion. Equipping $S \times S$ with coordinatewise addition $(a, b) \oplus (c, d) = (a \oplus c, b \oplus d)$, the relation $\mathcal{S} \subset (S \times S) \times (S \times S)$ defined by $(a, b) \mathcal{S} (c, d)$ iff $a \oplus d = b \oplus c$ is a congruence and $G = (S \times S)/\mathcal{S}$ is a group. Define scalar multiplication Ψ^*: $R \times G \to G$ by setting $\Psi^*(\lambda, (a, b))$ equal to $(\lambda a, \lambda b)$ if $\lambda \geq 0$, and equal to $(|\lambda|b, |\lambda|a)$ if $\lambda < 0$. Then G is a real vector space, and sending $x \in S$ to the equivalence class of $(2x, x)$ embeds the semivector space S into the real vector space G.

2.12 Remark: The example of the half plane underlines the fact that a pointwise convex semivector space, even when embeddable in a vector space, and hence a cone, need not be an ordered cone semigroup.

2.13 Remark: Combining 2.8 and 2.10, we see that every pointwise convex (Hausdorff) topological semivector space is a semilattice union of cones.

2.14 Proposition: Let X be a compact and convex set in a pointwise convex topological semivector space S with $0X = \{e\}$. Then $X \oplus a$ for some $a \in S$ is embeddable in a topological vector space.

Proof: Clearly, X lies in the semivector subspace $S_e = \{s \in S \mid 0s = e\}$ for which e is identity (see 2.8). Thus, by 2.10, S_e is cancellative and is embeddable (see 2.11) in a vector space. If $X \oplus X = S$, then S is a compact cancellative topological semigroup, thus a topological group, hence a real topological vector space, so that we just take $a = e$. If $X \oplus X$ is a proper subset of S, then either $X \oplus X$ has an element y such that $e \notin y \oplus X \oplus X$, in which case we set $a = \frac{1}{2}y$; or $X \oplus X$ has no such element, in which case we set $a = \frac{1}{2}y'$ for any $y' \in S \backslash (X \oplus X)$. In either case, $e \notin X \oplus a \oplus X \oplus a$, so that the compact and convex $X \oplus a$ admits a projective ordered cone semigroup $C = \Psi(R_+ \times (X \oplus a))$ through itself. Clearly, C is a locally compact ordered cone semigroup, so that, using the Lawson and Madison embedding theorem [6: Theorem 3.2], C is embeddable in a topological vector space.

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3. LOCAL CONVEXITY

We just saw (2.14) that a compact and convex set X in a pointwise convex Hausdorff topological semivector space S with singleton origin OS can always be embedded through an affine homeomorphism in a topological vector space. If X were known to be embeddable in a locally convex vector space, then one of the benefits of such an embedding would have been that a fixed point theorem (FPT) of Fan [2] would assure X to possess the fixed point property (FPP). Note here that the local convexity of X is a necessary condition for its embeddability in a locally convex vector space. Indeed, there is a whole hierarchy of local convexity properties which X would necessarily have to satisfy if it were so embeddable: Given a subset X in a convex topological semivector space, we consider the following

3.0 Hierarchy of Local Convexity Properties

0. For any $x \in X$ and any nbd V of x, there exists a nbd U of x such that $U \subset V$ with $U \cap X$ convex.

1. There exists a quasi-univormity $\mathcal{S} = \{E_\alpha \subset X \times X \mid \alpha \in A\}$ of X inducing its subspace topology, such that, for each $E_\alpha \in \mathcal{S}$, there exists a closed $E_\beta \in \mathcal{S}$ with $E_\beta \subset E_\alpha$ and $E_\beta(x)$ convex for each $x \in X$.

2. There exists a quasi-uniformity $\mathcal{S} = \{E_\alpha \subset X \times X \mid \alpha \in A\}$

of X inducing its subspace topology, such that, for each $E_\alpha \in \mathcal{E}$, there exists a closed $E_\beta \in \mathcal{E}$ with $E_\beta \subset E_\alpha$ and $E_\beta(K)$ convex for each compact and convex subset $K \subset X$.

3. X is convex and there exists a uniformity $\mathcal{E} = \{E_\alpha \subset X \times X \mid \alpha \in A\}$ of X inducing its subspace topology, such that, for each $E_\alpha \in \mathcal{E}$, there exists a convex $E_\beta \in \mathcal{E}$ with $E_\beta \subset E_\alpha$.

X will be called <u>$0°/1°/2°/3°$ locally convex (l.c.)</u> accordingly as it satisfies 0/1/2/3 among these properties. (Thus, $0°$ local convexity is the familiar local convexity.) While for real topological vector spaces X all four of these properties are equivalent, in general we are able to assert only the following

3.1 <u>Proposition</u>: Given a subset X of a convex topological semivector space,

1. If X is $1°$ l.c., then it is $0°$ l.c.;

2. If X is $2°$ l.c. and pointwise convex, then it is $1°$ l.c.; and

3. If X is $3°$ l.c., then it is $2°$ l.c.

In anticipation of the fixed point results of the next section, we recognize an interest in the question of when some of the implications of 3.1 can be reversed. In particular, we pose two embedding problems.

Problem 1: Let S be a pointwise convex topological semivector space with singleton origin. If S is n° locally convex, is S embeddable in a locally convex topological vector space $(n = 0, 1, 2, 3)$? (cf. 6.7 of Keimel [5])

Problem 2: Let X be a compact and convex subset of a real topological vector space. If X is n° locally convex, is X embeddable in a locally convex topological vector space $(n = 0, 1, 2, 3)$? (cf. 6.8 of Keimel [5])

(Also see the problems listed by Lawson and Madison [6] for locally compact cone semigroups.) While we are unable to solve these problems here, in the next section we obtain a number of FPT's for compact convex subsets with various degrees of local convexity in pointwise convex topological semivector spaces. Toward that, the rest of this section collects some basic facts relating to local convexity.

3.2 Proposition: Every 0° l.c. set with T_1 topology is pointwise convex.

Proof: Let X be a 0° l.c. T_1 space, and suppose $x \in X$. As X is 0° l.c., there is a local base $\mathcal{B} = \{B_\alpha \mid \alpha \in A\}$ at x consisting of convex nbds. Thus, $x \in B = \bigcap_A B_\alpha$, and B is convex. In fact, $B = \{x\}$, since X is T_1.

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Of course, all the local convexity properties 0°-3° are inherited by relative topologies on convex subsets. We now turn to some basic facts relating local convexity properties of Cartesian products with those of their factor spaces.

Given a family $\{S_\alpha \mid \alpha \in A\}$ of topological semivector spaces over Θ, we equip $S = \prod_A S_\alpha$ with the product topology and define its operations coordinatewise as follows:

$$\{s_\alpha\}_{\alpha\in A} \oplus \{t_\alpha\}_{\alpha\in A} = \{s_\alpha \oplus_\alpha t_\alpha\}_{\alpha\in A},$$

$$\lambda\{s_\alpha\}_{\alpha\in A} = \{\lambda s_\alpha\}_{\alpha\in A},$$

where $\oplus_\alpha$ stands for the semigroup operation of S_α and $s_\alpha, t_\alpha \in S_\alpha$ are generic $(\alpha \in A)$. Clearly, S is then a topological semivector space over Θ. We call it the <u>product</u> of $\{S_\alpha \mid \alpha \in A\}$. Evidently, a set $X \subset S = \prod_A S_\alpha$ is convex [resp. pointwise convex] only if [resp. iff] each projection $X_\alpha = \pi_{S_\alpha}(X) \subset S_\alpha$ is so.

3.3 <u>Proposition</u>: The product of a family of sets is 0° l.c. [resp. 3° l.c.] iff each of the factors is 0° l.c. [resp. 3° l.c.].

3.4 <u>Proposition</u>: Let $\{X_\alpha \mid \alpha \in A\}$ be a family of 2° l.c. sets of which all but finitely many are convex, and let $\mathcal{S}$ be a quasi-uniformity inducing the product topology on $X = \prod_A X_\alpha$. Then, given any $F \in \mathcal{S}$, there exists a closed $E \in \mathcal{S}$ such that $E \subset F$ and $E(K)$ is convex whenever K is the product

$K = \prod_A K_\alpha$ of compact and convex subsets $K_\alpha \subset X_\alpha$.

Proof: Contained in F, find a basic $H \in \mathcal{S}$ which restricts a finite set $N \subset A$ of coordinates, including (w.l.g.) the set $M \subset A$ of indices m for which X_m is not convex. Now

$$H = \prod_N H_n \times \prod_{A \setminus N} (X_\alpha \times X_\alpha),$$

where H_n belongs to the quasi-uniformity $\mathcal{S}_n$ of X_n $(n \in N)$. For each $n \in N$, using the 2° l.c. of X_n, find a closed $E_n \in \mathcal{S}_n$ such that $E_n \subset H_n$ with $E_n(K_n)$ convex for each compact and convex $K_n \subset X_n$. Write

$$E = \prod_N E_n \times \prod_{A \setminus N} (X_\alpha \times X_\alpha).$$

#

3.5 Lemma: The product of a family of 1° l.c. sets is 1° l.c., if all but a finite number of the factors are convex.

Proof: Imitate the last proof.

#

3.6 Proposition: Let $S = \prod_A S_\alpha$ be the product of a family of convex topological semivector spaces, and let $X \subset S$ be compact. If the projection $X_\alpha = \pi_{S_\alpha}(X)$ of X into S_α is Hausdorff, then X_α is $1^\circ/2^\circ$ l.c., accordingly as X is.

4. FIXED POINT AND MINMAX THEOREMS

The sets for which we seek to establish fixed point properties here are all compact and convex and can be seen, through 2.14, to be embeddable in a topological vector space. The sets in question also enjoy one or another kind of local convexity, and it is this that allows us -- regardless of whether the set can be embedded in a locally convex topological vector space (cf. Problem 1) -- to demonstrate the fixed point properties we aim at.

Given topological spaces X and Y and a mapping f of X into the set of nonempty subsets of Y, when we say that f is upper semi-continuous (usc), we will mean that, for each $x \in X$, given any nbd $V \subset Y$ of $f(x)$, there exists a nbd U of X such that $f(u) \subset V$ for each $u \in U$. For the composition of two binary relations $F \subset A \times B$ and $E \subset C \times D$, we will write $E \circ F$ for the set (binary relation)

$$\{(a, d) \mid \exists x \in B \cap C \text{ such that } (a, x) \in F \text{ and } (x, d) \in E\}.$$

4.1 FIXED POINT THEOREM I: Given a pointwise convex topological semivector space S, let $X = \{\lambda_0 a_0 \oplus \ldots \oplus \lambda_m a_m \mid \lambda = (\lambda_0, \ldots, \lambda_m) \in \Lambda_m\}$ be the "closed simplex" of a finite set $\{a_0, \ldots, a_m\} \subset S$. Then X has the FPP for upper semi-continuous point-to-set transformations $f\colon X \to \mathcal{CL}[X]$ into its nonempty, closed and convex subsets.

Proof: Denoting $0(a_0 \oplus \ldots \oplus a_m) = e$, $0X = \{e\}$, so that

2.14 applies and there is an embedding $X \overset{h}{\to} X'$ of X in a real topological vector space. Now X' is the closed simplex $\sigma(\{h(a_0 \oplus e), \ldots, h(a_m \oplus e)\})$, so it lies in a Euclidean space, and Kakutani's FPT (4.2) applies.

#

4.2 Corollary (Kakutani's Fixed Point Theorem [3]): Let $f\colon X \to \mathcal{C}\mathcal{Q}[X]$ be an upper semi-continuous transformation of an n-dimensional closed simplex $X \subset R^{n+1}$ into $\mathcal{C}\mathcal{Q}[X]$. Then there exists a (fixed) point $x^* \in X$ such that $x^* \in f(x^*)$.

4.3 FIXED POINT THEOREM II: Let S be a convex (Hausdorff) topological semivector space, and let $X \subset S$ be a nonempty convex subset. If X is compact and 1° l.c. with $0X$ singleton, then X has the FPP for continuous transformations $f\colon X \to X$.

Proof: Assume that X satisfies the hypothesis. By 3.1 and 3.2, X is pointwise convex, so, recalling the discussion following 2.3, X lies in some pointwise convex semivector subspace of S; and, as $0X$ is singleton, by 2.8 we may assume this subspace to have singleton origin. Thus, w.l.g., we assume that S is pointwise convex with $0S = \{e\}$.

Since X is compact, there exists a unique uniformity on X compatible with its subspace topology. Since X is 1° l.c., we assume that $\{E_\alpha \subset X \times X \mid \alpha \in A\}$ is a fundamental system of closed entourages of this uniformity such that

$E_\alpha(x)$ is (closed and) convex for all $x \in X$. Let $f: X \to X$ be continuous. Define $Y_\alpha = \{x \mid x \in E_\alpha(f(x))\}$. We will show that Y_α is nonempty and closed for each $\alpha \in A$. Then, as the intersection of any finite collection of Y_α's is nonempty, compactness of X will imply that $\bigcap_{\alpha \in A} Y_\alpha \neq \emptyset$, thus proving the theorem, for $x^* \in \bigcap_{\alpha \in A} Y_\alpha$ implies $x^* = f(x^*)$.

To show that Y_α is nonempty, let $\{D_\alpha \subset X \times X \mid \alpha \in A\}$ be a family of open symmetric entourages such that $D_\alpha \subset E_\alpha$ $(\alpha \in A)$. Thus, for any given $\alpha \in A$, $\{D_\alpha(x) \mid x \in X\}$ is an open cover of X, so that there exist $x_0, \ldots, x_n \in X$ with $X \subset \bigcup_{i=0}^{n} D_\alpha(x_i)$. Denote the closed convex hull $\{p = \lambda_0 x_0 \oplus \ldots \oplus \lambda_n x_n \mid \lambda = (\lambda_0, \ldots, \lambda_n) \in \Lambda_n\}$ of $\{x_0, \ldots, x_n\}$ by P. As $0S = \{e\}$, $P \supset \{x_0, \ldots, x_n\}$. Define the map F_α on P by $F_\alpha(p) = E_\alpha(f(p)) \cap P$. Then, by symmetricity of $D_\alpha \subset E_\alpha$, for all $p \in P$, $F_\alpha(p)$ is nonempty; clearly, it is also closed and convex. Thus F_α maps P into $\mathcal{C}\mathcal{Q}[P]$. Denoting the graph of $E_\alpha \circ f$ by G_α, the graph of F_α is simply $\Gamma_\alpha = G_\alpha \cap (P \times P)$. Since E_α is usc (by the closedness of E_α in the compact $X \times X$) and since f is continuous, $E_\alpha \circ f$ is usc, i.e., G_α is closed, as X is regular (in fact, compact). Hence, Γ_α is closed and, by compactness of P, F_α is usc. Thus, by 4.1, there exists $p \in F_\alpha(p)$, i.e., $p \in Y_\alpha$, showing that Y_α is nonempty. Y_α is obviously closed, since it is nothing but the projection

$\pi_X(G_\alpha \cap \Delta)$ of the compact set $G_\alpha \cap \Delta$, where Δ is the diagonal in $X \times X$. This completes the proof.

#

4.4 <u>Corollary</u> (<u>Tychonoff's Fixed Point Theorem</u> [10]): Let $f: X \to X$ be a continuous transformation of a nonempty compact and convex subset X of a locally convex (Hausdorff) topological vector space. Then there exists a (fixed) point $x^* \in X$ such that $x^* = f(x^*)$.

4.5 <u>FIXED POINT THEOREM III</u>: Let $\{X_\alpha \subset S_\alpha \mid \alpha \in A\}$ be a nonempty family, where, for each $\alpha \in A$, S_α and X_α satisfy the hypothesis of 4.3. Let $\{f_\alpha: X \to X_\alpha \mid \alpha \in A\}$ be a family of continuous functions on $X = \prod_A X_\alpha$, and define $F: X \to X$ by $F(x) = \{f_\alpha(x)\}_{\alpha\in A}$. Then there exists a (fixed) point $x^* \in X$ such that $x^* = F(x^*)$.

<u>Proof</u>: Clearly, the topological semivector space $S = \prod_A S_\alpha$ is Hausdorff, and $X \subset S$ is nonempty, compact and convex with $0X$ singleton. Since each X_α is 1° l.c., so is X (see Lemma 3.5). Furthermore, F is continuous, as each f_α is so. Hence, the result follows readily by application of 4.3.

#

4.6 <u>FIXED POINT THEOREM IV</u>: Let S be a convex (Hausdorff) topological semivector space, and let $X \subset S$ be a nonempty convex subset. If X is compact, pointwise convex and

2° l.c., with $0X$ singleton, then X has the FPP for upper semi-continuous transformations $f\colon X \to \mathcal{CL}[X]$.

Proof: Assuming that X satisfies the hypothesis, as in the proof of 4.3, it suffices to show that the sets $Y_\alpha = \{x \mid x \in E_\alpha(f(x))\}$ are nonempty and closed, where, in this case, $\{E_\alpha \mid \alpha \in A\}$ is a findamental system of closed entourages of the space X such that $E_\alpha(K)$ is (closed and) convex for each nonempty, compact and convex subset $K \subset X$. The proof is the same as that of 4.3, except that appeal is now made to the upper semi-continuity, rather than the continuity, of f.

#

4.7 Corollary (Fan's Fixed Point Theorem [2]): Let X be nonempty, compact and convex in a locally convex Hausdorff topological vector space, and let $f\colon X \to \mathcal{CL}[X]$ be an upper semi-continuous transformation. Then there exists a (fixed) point $x^* \in X$ such that $x^* \in f(x^*)$.

4.8 FIXED POINT THEOREM V: Let $\{X_\alpha \subset S_\alpha \mid \alpha \in A\}$ be a nonempty family, where, for each $\alpha \in A$, S_α and X_α satisfy the hypothesis of 4.6; and let $\{f_a\colon X \to \mathcal{CL}[X_\alpha] \mid \alpha \in A\}$ be a family of upper semicontinuous transformations, where $X = \prod_A X_\alpha$. Define $F\colon X \to \mathcal{CL}[X]$ by $F(x) = \prod_A f_\alpha(x)$ $(x \in X)$. Then there exists a (fixed) point $x \in X$ such that $x^* \in F(x^*)$.

Proof: Clearly, $S = \prod_A S_\alpha$ is a convex Hausdorff topological semivector space; $X \subset S$ is nonempty, compact, convex and pointwise convex with $0X$ singleton; and F is easily seen to be upper semicontinuous. Although X need not be 2° l.c., by the 2° local convexity of each X_α, the uniformity on X allows a fundamental system $\{E_i \mid i \in I\}$ of closed entourages such that, whenever K is the product $K = \prod_A K_\alpha$ of compact and convex subsets $K_\alpha \subset X_\alpha$, each $E_i(K)$ is closed and convex (see Prop. 3.6). Notice that $F(x)$ is such a product of compact and convex sets $f_\alpha(x) \subset X_\alpha$. Thus, as in 4.6, defining $Y_i = \{x \mid x \in E_i(F(x))\}$, it is clear that Y_i is nonempty and closed for each $i \in I$, implying that $\bigcap_I Y_i \neq \emptyset$ and proving the theorem.

#

The minmax theorems below are fairly straightforward consequences of 4.1, 4.5 and 4.8. We record them for their interest in economic theory, optimization and game theory.

4.9 MINMAX THEOREMS: Let S_1 and S_2 be convex topological semivector spaces, let $X_1 \subset S_1$ and $X_2 \subset S_2$ with $X = X_1 \times X_2 \neq \emptyset$, and let $u: X \to R$ be a continuous real valued function. Define

$$f_1(x_2) = \{x_1 \in X_1 \mid u(x_1, x_2) = \operatorname*{Max}_{y \in X_1} u(y, x_2)\} \quad (x_2 \in X_2),$$

$$f_2(x_1) = \{x_2 \in X_2 \mid u(x_1, x_2) = \operatorname*{Min}_{z \in X_2} u(x_1, z)\} \quad (x_1 \in X_1).$$

Each of the following is a sufficient condition for the equation

$$\underset{X_2}{\text{Min}}\ \underset{X_1}{\text{Max}}\ u(x_1, x_2) = \underset{X_1}{\text{Max}}\ \underset{X_2}{\text{Min}}\ u(x_1, x_2).$$

(1) $f_1(x_2)$ and $f_2(x_1)$ are nonemtpy, closed and convex for each $x_1 \in X_1$ and $x_2 \in X_2$, while

(a) S_1 and S_2 are pointwise convex, and the sets X_1 and X_2 are "closed simplices" $X_1 = \sigma(A_1)$ and $X_2 = \sigma(A_2)$ of finite sets $A_1 \subset S_1$ and $A_2 \subset S_2$, respectively; <u>or</u>

(b) X_1 and X_2 are compact, convex, pointwise convex and 2° l.c. with $0X_1$ and $0X_2$ both singleton.

(2) X_1 and X_2 are compact, convex and 1° l.c. with $0X_1$ and $0X_2$ both singleton, and $f_1(x_2)$ and $f_2(x_1)$ are singleton for each $x_1 \in X_1$ and $x_2 \in X_2$.

<u>Proof</u> (Sketch): The inequality $\underset{X_1}{\text{Max}}\ u(\cdot, x_2) \geq \underset{X_2}{\text{Min}}\ \underset{X_1}{\text{Max}}\ u(\cdot, \cdot) \geq \underset{X_1}{\text{Max}}\ \underset{X_2}{\text{Min}}\ u(\cdot, \cdot) \geq \underset{X_2}{\text{Min}}\ u(x_1, \cdot)$ is always true, leaving only the reverse inequality to show. To do this for (1), use 4.1 in case (a), and 4.8 in case (b), suitably modifying the argument

<u>ad</u> (2): Assuming (2), check that the functions f_1 and f_2 are continuous, so that the function $F: X \to X$ defined by

$F(x_1, x_2) = (f_1(x_2), \quad f_2(x_1))$ is continuous. Then by 4.5, there exists an $x^* \in X$ such that $x^* = (x_1^*, x_2^*) = F(x^*)$. Hence, $\underset{X_1}{\text{Max}}\ u(\cdot, x_2^*) = \underset{X_2}{\text{Min}}\ u(x_1^*, \cdot)$, thus proving the desired equality.

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