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OPTIMAL GROWTH WITH MANY CONSUMERS

by

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ABSTRACT

A method is described for constructing all Pareto-optimum resource allocations for a dynamic economy, under perfect certainty, in which both the technology and consumer preferences are recursive but preferences need not be additively separable over time. Optimum allocations are obtained through the study of an appropriate dynamic program, and the connection between optimum and equilibrium allocations is then used to interpret the optima as perfect foresight competitive equilibria. The main novelty of the paper is that consumers' relative weights in the objective function are among the state variables of the dynamic program, and the utilities "from now on" are among its control variables.

For an economy with one consumption good, sufficient conditions on the technology and preferences are given for the existence of a unique, interior, stationary distribution of consumption and wealth. These conditions are that the technology be <u>regular</u> in the sense of Burmeister [13], and that each consumer's preferences display <u>increasing impatience</u>. For a pure exchange economy with two consumers, sufficient conditions are given for the global asymptotic stability of the unique, interior stationary point. These are that each consumer display increasing impatience, and in addition that each view consumption today and stationary consumption from tomorrow on as non-inferior.

These conclusions are in contrast to the case where consumers' preferences are additively separable over time. As shown in Becker [2], such preferences lead asymptotically to corner allocations where the consumption and wealth of less patient consumers approach zero (if consumers have different discount rates), or else to a large multiplicity of stationary allocations (if all consumers have identical discount rates).

1. Introduction

This paper describes a method for constructing all Pareto-optimum resource allocations for a dynamic economy, under perfect certainty, in which both the technology and consumer preferences are recursive. Within the restrictions imposed by certainty and recursivity, the scope of the theory is quite broad: any finite number of capital goods may be considered, and any finite number of non-storable consumer goods. There is a finite number of consumers, each living an infinite number of periods or the entire life of the economy.

The basic idea of the paper is to obtain optimum allocations through the study of an appropriate dynamic program, and then to use the connections between optimum and equilibrium allocations to interpret the optima as perfect foresight competitive equilibria. The objective function of this dynamic program is a weighted sum of all agents' total discounted utilities, its state variables include the stocks of various kinds of capital goods with which the economy begins a period, and its control variables include current consumption and current investment. In addition, and this is the main novelty of the paper, the relative weights assigned to agents by the objective function are also state variables, and the utilities "from now on" to be given all of the agents are also control variables.

The paper is a direct descendant of the work on growth theory which developed from the contributions of Solow [26], Cass [14], Koopmans [20] and many others in the 1960s. It also rests heavily on the work of Bellman [3] and Debreu [16]. The earlier study of this problem that is closest to ours is Becker [2]. A main difference between Becker's treatment and ours is that we make essential use of the recursive but not time-additive preferences proposed by Koopmans, Diamond and Williamson [19], [21].

In the next section, some notation will be set and optimum and equilibrium allocations defined. Section 3 specifies the particular class of preferences that will be assumed. Section 4 similarly sets out the technology. Section 5 contains the study of the dynamic program referred to above and connects the solution of this program to optimum allocations in the sense of Section 2. Section 6 summarizes what can be said about the connections between optimum allocations as constructed in Section 5 and competitive equilibrium allocations.

Sections 7 and 8, proceeding under stronger assumptions than those used earlier, connect the dynamics of optimum allocations to the more familiar dynamics of optimum allocations in one-consumer economies. The Euler equations are set out in Section 7. Stationary solutions are studied in Section 8.

Sections 9 and 10 illustrate the use of results developed in the paper to the simple example of a two-agent, exchange economy. This one state-variable system serves as a kind of parallel to the one-agent, one capital good example on which is based such an embarrassingly large fraction of our capital-theoretic intuition. Section 11 contains concluding remarks.

2. Optimum and Equilibrium Allocations

The economy under study contains n consumers or agents, denoted $i = 1,\ldots,n$, each of whom lives for an infinity of periods, $t = 0,1,2,\ldots$ There are m consumption goods, $j = 1,2,\ldots,m$, all consumed or freely disposed of in the period in which they are produced. Let x_{ijt} be the quantity agent i consumes of good j in period t.

In order to keep bookkeeping issues within bounds, we will use omitted subscripts to indicate various arrays formed from these numbers x_{ijt}. Thus let x_{it} be the vector $(x_{i1t}, x_{i2t},\ldots,x_{imt})$ of goods consumed by agent i in period t, let $x_i = (x_{i0},x_{i1},x_{i2},\ldots)$ be the infinite sequence of agent i's consumptions, and let $x = (x_1,x_2,\ldots,x_n)$ be the complete listing of everyone's consumption of every good in every period. Similarly, x_t will denote a complete listing of everyone's consumption of each good in period t, x_{jt} a listing of the way good j is allocated over agents in period t, and so on. It will also be useful to have a shorthand notation for the economy's total production. Let $\bar{x}_{jt} = \Sigma_i x_{ijt}$ be the total amount of good j consumed in period t, and $\bar{x}_j$ be the sequence of productions of good j in all periods.

Consumptions x_{ijt} are nonnegative real numbers. The sequence x_{ij} of agent i's consumption of good j is taken to be an element of the nonnegative orthant of ℓ_∞ , the space of sequences $\{x_{ijt}\}$ with:

$$\|x_{ij}\| = \sup_t |x_{ijt}|$$

finite. We will drop the infinity subscript and call this orthant ℓ_+. Then the list of agent i's consumptions, x_i, is an element of $\ell_+^m = \ell_+ \times \ldots \times \ell_+$ (m times). The list of everyone's consumptions, the complete resource allocation, is in ℓ_+^{mn}.

The preferences of agent i are described by a function $u^i:\ell_+^m \to R$, restrictions on which will be developed in Section 3. Feasible allocations are allocations $x \in \ell_+^{mn}$ for which total consumptions $\bar{x} = \Sigma_i x_i = (\bar{x}_1,\ldots,\bar{x}_m)$ lie in a set $Y \subset \ell_+^m$, to be restricted in Section 4. With these conventions set, optimum and equilibrium allocations can be defined.

<u>Definition.</u> An allocation $x \in \ell_+^{mn}$ is <u>Pareto-optimal</u> if it is feasible and if there is no other feasible allocation x' with

$$u^i(x_i') \geqslant u^i(x_i), \qquad \text{for all } i,$$

and

$$u^i(x_i') > u^i(x_i), \qquad \text{for some } i.$$

Definition. A competitive equilibrium is a feasible allocation $x \in \ell_+^{mn}$ together with a price system $p \in \ell_+^m$ such that

(i) $$\Sigma_{jt}\, p_{jt}\bar{x}_{jt} < \infty, \qquad \text{for all } \bar{x} \in \ell_+^m,$$

(ii) for all i, and for any $x_i' \in \ell_+^m$

$$\Sigma_{jt}\, p_{jt}x'_{ijt} \leq \Sigma_{jt}\, p_{jt}x_{ijt},$$

implies

$$u^i(x_i') \leq u^i(x_i)$$

(iii) for all feasible x'

$$\Sigma_{jt}\, p_{jt}\bar{x}'_{jt} \leq \Sigma_{jt}\, p_{jt}\bar{x}_{jt}.$$

These definitions are standard, except perhaps for the restriction (i) in the definition of equilibrium to the effect that present values of all consumption streams must be finite at equilibrium prices. There are continuous linear functionals on ℓ_∞ which do not have this property, so our definition is more restrictive than that used, for example, in [16].

3. Preferences

This section deals with restrictions on the preference function u^i of a typical agent. Since only one agent will be discussed at a time, the superscript i on u^i and the subscript i on agent i's consumption x_i will temporarily be dropped.

In the theory of optimal growth it is convenient and usual to restrict preferences to assume the recursive and time additive form

$$u(x) = \Sigma_t \beta^t U(x_t) ,$$

where $0 < \beta < 1$ and where restrictions are imposed on the current-period utility function $U: R^m_+ \to R$. For reasons which by now are familiar, (and which in any case will be illustrated in Sections 7 and 9, below), this class of preferences proves to be too narrow to permit the development of an interesting theory of economies with many agents. Accordingly, we will follow Koopmans [19] and Koopmans, Diamond and Williamson [21] in using preferences in a broader, but still recursive, class, specified as follows.

For any sequence $x = (x_0, x_1, \ldots) \in \ell^m_+$ of consumptions, define ${}_tx \in \ell^m_+$ by ${}_tx = (x_t, x_{t+1}, \ldots)$. Thus ${}_0x = x$; ${}_1x$ means consuming x_1 today, x_2 tommorrow, and so on; and in general ${}_tx$ is the sequence beginning with x_t today, x_{t+1} tommorrow, and so on. Then [19], [21] are concerned with finding restrictions on preferences $u(x)$ sufficient to permit writing $u(x)$ as

$$u(x) = W(x_0, u({}_1x))$$

for all $x \in \ell^m_+$ for some function $W: R^m_+ \times R \to R$.[1] The argument of this paper proceeds in the reverse (and much simpler) direction, beginning with the

<u>Definition.</u> A function $W: R^{m+1}_+ \to R_+$ is an <u>aggregator function</u> if it is

W1: continuous and bounded,

W2: concave,

and satisfies

W3: $W(0,0) = 0$

W4: $(x,z) \leq (x',z')$ and $(x,z) \neq (x',z')$ implies $W(x,z) < W(x',z')$, and

W5: for all $x \in R_+^m$ and all $z, z' \in R_+$,

$$|W(x,z) - W(x,z')| \leq \beta|z - z'|, \quad \text{for some } 0 \leq \beta < 1.$$

Note that the restrictions W2 and W3 are cardinal in nature, so that the term aggregator function as used here is narrower than the ordinal interpretation used in [19] and [21].[2]

Next, let S be the space of functions $u: \ell_+^m \to R$ which are continuous and bounded in the norm

$$\|u\| = \sup_{x \in \ell_+^m} |u(x)|.$$

S is a complete, normed linear space. An aggregator function W defines an operator T_W: S → S by

$$(3.1) \qquad (T_W u)(x) = W(x_0, u({}_1x)).$$

In this paper, we will work with the class of preference functions u which are fixed points of T_W for some aggregator function W. The following result describes the main features of this class.

Theorem 1. For every aggregator function W there is a unique $u \in S$ satisfying $T_W u = u$, and for any $u_0 \in S$

$$(3.2) \qquad \|T_W^N u_0 - u\| \leq \beta^N \frac{B}{1-\beta}$$

where B is an upper bound for W. This function u satisfies:

(i) u is concave

(ii) $x' \geq x \implies u(x') \geq u(x)$

(iii) $u(0) = u(0,0,0,\ldots) = 0$

(iv) $\|u - u^N\| \leq \beta^N \|u\|$, where $u^N(x) \equiv u(x_1,\ldots,x_{N-1},0,0,\ldots)$.

Proof. For any $u,v \in S$, the fact that

$$\|T_W u - T_W v\| \leq \beta\|u-v\|$$

follows immediately from the definition of T_W in (3.1) and W5. Hence the existence of a unique fixed point u and (3.2) are implied by the contraction mapping theorem.

To prove the concavity of u we prove first that if $u \in S$ is concave, so is $T_W u$. Let $x,x' \in \ell_+^m$ be given, let $0 \leq \theta \leq 1$, and define $x^\theta = \theta x+(1-\theta)x'$. Then

$$\begin{aligned}\theta(T_W u)(x) + (1-\theta)(T_W u)(x') &= \theta W\big(x_o,u({}_1x)\big) + (1-\theta)W\big(x_o',u({}_1x')\big)\\ &\leq W\big(\theta x_o + (1-\theta)x_o',\ \theta u({}_1x) + (1-\theta)u({}_1x')\big)\\ &\leq W\big(x_o^\theta,\ u({}_1x^\theta)\big)\\ &= (T_W u)(x^\theta)\end{aligned}$$

where the first inequality follows from the concavity of W and the second from the assumed concavity of u and the fact (W4) that W is increasing in all arguments. Now choosing u_o to be concave and applying (3.2), (i) is proved.

To prove (ii), observe that by W4 T_W takes increasing functions into increasing functions, and apply (3.2) to an increasing u_o.

To prove (iii), use W3 plus (3.1).

To prove (iv), note that for any $x \in \ell_+^m$,

$$u^N(x) = u(x_0,x_1,\ldots,x_{N-1},0,0,\ldots)$$

$$= W(x_0,u(x_1,\ldots,x_{N-1},0,0,\ldots.))$$

$$= W(x_0,u^{N-1}({}_1x))$$

while

$$u(x) = W(x_0,u({}_1x)).$$

Hence by the contraction property

$$\|u-u^N\| < \beta\|u - u^{N-1}\|$$

$$\leq \beta^2\|u - u^{N-2}\|$$

$$\vdots$$

$$\leq \beta^N\|u - u^0\|.$$

Since $u^0(x) = u(0,0,\ldots) = 0$ by (iii), (iv) is proved.

4. Technology

This section deals with restrictions on the technology available for producing a sequence of goods $\bar{x} = (\bar{x}_1,\ldots,\bar{x}_m) = \{\bar{x}_t\}$. Since only the total production of each good, and not its distribution over agents, will be discussed in this section, the bar over $\bar{x}$ will temporarily be dropped.

The technology will be assumed recursive, in the sense that production possibilities in period t will be taken to be fully proscribed by a vector of capital stocks $k_t = (k_{1t},\ldots,k_{pt}) \in R_+^p$ on hand as of the beginning of t. These stocks determine the current consumption goods production x_t and the end-of-period (beginning-of-next-period) capital stocks k_{t+1} which are jointly producible, given k_t. Call this set $B(k_t)$, so that the technology is characterized by a correspondence B: $R_+^p \to R_+^m \times R_+^p$ taking points k_t into sets of feasible productions (x_t,k_{t+1}). With a given production correspondence B and a given initial capital vector k_o the set $Y(k_o)$ of feasible consumption goods sequences x is

$$(4.1) \qquad Y(k_o) = \{x \in \ell_+^m: (x_t, k_{t+1}) \in B(k_t),\ t = 0, 1, 2, \ldots, \text{ for some } k \in \ell_+^p,\ k_o \text{ given}\}.$$

The correspondence B will be assumed continuous (that is, upper-and lower-hemi-continuous[3]) and in addition to have the properties:

B1: for each k, B(k) is compact and convex,

B2: $(x,y) \in B(k)$ and $(x',y') \leq (x,y)$ implies $(x',y') \in B(k)$,

B3: $k' \leq k$ implies $B(k') \subseteq B(k)$,

B4: if $(x,y) \in B(k)$ and $(x',y') \in B(k')$, then

$$\big((\theta x + (1-\theta)x'), (\theta y + (1-\theta)y')\big) \in B(\theta k + (1-\theta)k'),$$

B5: the set

$$M = \{k \in R_+^p: (0,k) \in B(k)\}$$

has a non-empty interior,

B6: if k is an interior point of M, $(x,k) \in B(k)$ for some $x > 0$ (where $x > 0$ means $x_j > 0$ for $j = 1,\ldots,m$).

The restrictions B1 and B2 refer to production possibilities for consumption goods x and capital goods y for fixed beginning-of-period capital stocks k. They are standard, given free disposal. Assumptions B3 and B4 describe the way B(k) varies with k: B3 is a kind of monotonicity (or free disposal) assumption; B4 is a convexity assumption. The set M defined in B5 is the set of maintainable capital stock configurations. Assumption B6 requires that off the boundary of this maintainable set, it is possible both to maintain all capital stocks and to produce positive amounts of all consumptions goods.

The following describes the main implictions of this description of production possibilities for the set $Y(k_o)$ of feasible sequences of consumption goods production.

<u>Theorem 2.</u> Let the production correspondence B satisfy B1-B6 and let $Y(k_o)$ be defined by (4.1). Then Y is closed and convex, and if k_o is an interior point of M, $Y(k_o)$ has a non-empty interior, using the "sup" norm used in Section 3.

<u>Proof.</u> To prove closedness, consider the sets

$$C_N = \{x \in \ell_+^m,\ k \in \ell_+^p\colon\ (x_t, k_{t+1}) \in B(k_t),\quad t = 0,1,\ldots,N,\ k_o \text{ given}\}.$$

Then C_0 is closed by B1, and since

$$C_{N+1} = \{(x,k) \in C_N\colon\ (x_{N+1}, k_{N+2}) \in B(k_N)\},$$

C_{N+1} is closed if C_N is, by B1 and the continuity of B. Hence all C_N, $N = 0,1,\ldots,$ are closed. It follows that the projections Y_N of the sets C_N on the set ℓ_+^m of consumption sequences are closed, and hence that

$$Y = \bigcap_N Y_N$$

is closed.

To show that Y is convex, choose $x, x' \in Y$ and let k, k' be associated capital paths. Then from B4, if $0 \leq \theta \leq 1$,

$$\left(\theta x_t + (1-\theta)x_t',\ \theta k_{t+1} + (1-\theta)k_{t+1}'\right) \in B\left(\theta k_t + (1-\theta)k_t'\right),$$

for all t, so that $\theta x + (1-\theta)x' \in Y$.

If k_o is in the interior of M, B6 implies that $(x,k_o) \in B(k_o)$ for some $x > 0$. Then the sequence with consumption equal to $(1/2)x$ for all $t = 0,1,\ldots$ is an interior point of Y.

5. A Dynamic Program For Optimal Allocations

The general objective of this section is to formulate and study a dynamic program, the policy functions for which generate all Pareto-optimal allocations. Properties of the optimum value function for this program are established in several stages. We begin by defining the economy's utility possibility set and the support function for this set. Lemmas 1 and 2 establish properties of this set and its support function, and Lemma 3 shows how feasible utility allocations are represented in terms of the support function.

The dynamic program is then introduced. In Lemma 4 we show that the functional equation for the optimum value function of this program has exactly one bounded, continuous solution, and in Lemma 5 we show that the support function for the utility possibility set is this solution. In Lemma 6 properties of the associated optimum policy correspondence are established, and in Lemma 7 we show that an allocation is Pareto-optimal if and only if it is generated from this policy correspondence. The results of the section are summarized in Theorem 3.

The economy under study involves n agents with preference functions $u^1,\ldots u^n$ constructed from given aggregator functions $W^1,\ldots,W^n$ as described in section 3. Its technology is the set Y(k) constructed from a given initial capital vector k and a given production correspondence B as described in section 4. The problem of finding optimal allocations in this economy will be treated as one of constructing its utility possibility sets U(k), defined by

$$(5.1) \qquad U(k) = \{z \in R^n_+ : z_i = u^i(x_i), \quad i = 1,\ldots,n, \quad \text{for some } x \in Y(k)\}.$$

Thus U(k) contains the possible combinations of utility available to the n agents in the economy when the initial capital stock is k.

Define $v(k,\theta)$ to be the support function of U(k).

$$(5.2) \qquad v(k,\theta) \equiv \sup_{z \in U(k)} \Sigma_i \theta_i z_i.$$

Thus $v: R^p_+ \times I \to R_+$, where $I \equiv \{\theta \in R^n_+ : \Sigma_i \theta_i = 1\}$. Any allocation x attaining $v(k,\theta)$ will be Pareto-optimal, yielding utilities lying on the "northeast boundary" of the set U(k) defined in (5.1). From the definition of U, it follows that we can also write v as:

$$(5.3) \qquad v(k,\theta) \equiv \sup_{x \in Y(k)} \Sigma_i \theta_i u^i(x_i).$$

The next four lemmas describe the main features of U(k) and $v(k,\theta)$.

<u>Lemma 1.</u> For each $k \in R^p_+$, U(k) is compact and convex, and exhibits "free disposal" in the sense that $u \in U(k)$ and $0 \leq u' \leq u$ implies $u' \in U(k)$.

<u>Proof.</u> Free disposal is inherited from free disposal for the production set and the fact that the u^i are increasing. Convexity follows from the convexity of Y(k) and the concavity of the u^i. Boundedness follows from the boundedness of the u^i.

To prove that U(k) is closed, let $\{u_n\}$ be a sequence in U(k) with $u_n \to \bar{u}$. Let $x_n \in Y(k)$ attain u_n for $n = 1,2,\ldots$. The set Y(k) is not compact, but the sets

$$Y^N(k) = \{x \in Y(k): x_t = 0 \text{ for } t > N\}$$

of feasible, truncated alloctions are. For any feasible allocation x, let x^N

denote the projection of x onto $Y^N(k)$.

Since $Y^1(k)$ is compact, $\{x^1_n\}$ has a convergent subsequence, call it $\{x^1_{1n}\}$, with $\lim_{n\to\infty} x^1_{1n} = x^1$, and the utilities for the corresponding untruncated allocations satisfy $\lim_{n\to\infty} u(x_{1n}) = \bar{u}$. Continuing by induction, for any $N > 1$, since $Y^N(k)$ is compact, $\{x^N_{N-1,n}\}$ has a convergent subsequence, call it $\{x^N_{Nn}\}$, with the properties that:

(5.4) $\lim_{n\to\infty} u(x_{N,n}) = \bar{u},$

and

(5.5) $\lim_{n\to\infty} x^N_{N,n} = (x_1, x_2 \ldots, x_N, 0, 0 \ldots) = x^N$

(where convergence in (5.4) is in the Euclidean norm for R^n and in (5.5) is in the sup norm for ℓ^{mn}). This defines an allocation $x \in Y(k)$, the allocation such that x^N is given by (5.5) for all N. We need to show that $u(x) = \bar{u}$.

We have, for any N and n,

$$|u(x) - \bar{u}| \leq |u(x) - u(x^N)| + |u(x^N) - u(x^N_{N,n})|$$
$$+ |u(x^N_{N,n}) - u(x_{N,n})| + |u(x_{N,n}) - \bar{u}|.$$

By Theorem 1, part (iv) the first and third terms on the right can be made arbitrarily small by choosing N sufficiently large, independent of n. By (5.5) the second term can be made small, and by (5.4) the fourth, by a sufficiently large choice of n. Hence $u(x) = \bar{u}$, so that $\bar{u} \in U(k)$, and U(k) is closed. This completes the proof of the Lemma.

In view of Lemma 1, the suprema used in (5.2) and (5.3) to define the function $v(k,\theta)$ may be replaced with maxima. We have, next, Lemma 2. The value function $v(k,\theta)$ is bounded and continuous.

Proof. Boundedness is obvious. To prove continuity, let $Y^N(k)$ be the set of truncated allocations defined in the proof of Lemma 1, and let

$$v^N(k,\theta) \equiv \max_{x \in Y^N(k)} \Sigma_i \theta_i u^i(x_i).$$

Since $Y^N(k)$ is a compact set in R_+^{mnN}, v^N is continuous for each N by [6], p.116. For each N, $Y^N(k) \subseteq Y^{N+1}(k) \subseteq Y(k)$, so that $v^N \leq v^{N+1} \leq v$. Hence for each (k,θ), $v^N(k,\theta) \to v(k,\theta)$. We next show that this convergence is uniform.

For given (k,θ), let $\hat{x}$ attain $v(k,\theta)$. Then

$$\begin{aligned} v(k,\theta) - v^N(k,\theta) &\leq \Sigma_i \theta_i [u^i(\hat{x}_i) - u^i(\hat{x}_i^N)] \\ &\leq \max_i \beta_i^N \|u^i\| \end{aligned}$$

where the second inequality follows from Theorem 1, part (iv). Since $0 \leq \beta_i < 1$ for all i, the proof is complete.

Lemma 3. $u \in U(k)$ if and only if $u \geq 0$ and $v(k,\theta) - \theta \cdot u \geq 0$ for all $\theta \in I$.

Proof. That $u \in U(k)$ implies $v(k,\theta) - \theta \cdot u \geq 0$ is immediate from (5.2).

Suppose, for the converse, that $u \geq 0$, $v(k,\theta) - \theta \cdot u \geq 0$ for all $\theta \in I$, and $u \notin U(k)$. Since $U(k)$ is convex, it follows from the separation theorem for convex sets that for some $w \in R^n$, $w \neq 0$, $w \cdot u \geq w \cdot z$ for all $z \in U(k)$. Since $U(k)$ exhibits free disposal (Lemma 1), it follows that $w \geq 0$, and we may choose $w \in I$. Since $U(k)$ is closed, the inequality is strict: $w \cdot u > w \cdot z$. Now since $v(k,\theta) - \theta \cdot u \geq 0$ for all $\theta \in I$, $v(k,w) \geq w \cdot u > w \cdot z$ for all $z \in U(k)$, contradicting the fact that

$$v(k,w) = \max_{z \varepsilon U(k)} w \cdot z.$$

This completes the proof.

Next we will show that $v(k,\theta)$ is the unique solution of the functional equation:

$$\hat{v}(k,\theta) = \max_{\substack{(x,y)\varepsilon B(k) \\ z \geq 0}} \Sigma_i \theta_i W^i(x_i,z_i) \tag{5.6}$$

subject to:

$$\min_{w \varepsilon I} \hat{v}(y,w) - w \cdot z \geq 0. \tag{5.7}$$

The idea in (5.6) is to view the problem of choosing an optimal allocation for given capital stocks k and vector of weights θ, as one of choosing a feasible current period alloction (x,y) of consumption and capital goods, <u>and</u> a vector z of "utilities-from-tomorrow-on" subject to the constraint that these utilities be attainable given the capital accumulation decision, i.e., that z satisfies (5.7). The weights w that attain the minimum in (5.7) will then be the new weights used in selecting tomorrow's allocation, and so on, *ad infinitum*. Further properties of $\hat{v}(k,\theta)$ will be established by studying this functional equation, and optimal resource allocations will be generated from the policy correspondence associated with it. First we will show that (5.6) - (5.7) has a unique continuous, bounded solution (Lemma 4), and that $v(k,\theta)$ as defined in (5.2) is this solution (Lemma 5).

<u>Lemma 4</u>. There is exactly one continuous, bounded function $\hat{v}$: $R_+^p \times I \rightarrow R$ satisfying (5.6). This solution $\hat{v}$ is increasing and concave in k and convex in θ.

Proof. Let F be the Banach space of continuous, bounded functions $f: R^p_+ \times I \to R$, with norm

$$\|f\| = \sup_{k,\theta} |f(k,\theta)|.$$

Let T be the operator on F defined by the right-hand side of (5.6). The set of (x,y,z) values satisfying $(x,y) \varepsilon B(k)$, $z \geq 0$, and (5.7) is compact, and the function to be maximized in applying T is continuous, so Tf is well defined for each $f \varepsilon F$. Since B and f are continuous, it follows from ([6], p.116) that Tf is continuous. It is evidently bounded, so that $T: F \to F$.

We next show that T is a contraction mapping, by showing that it satisfies the hypotheses of ([8], Theorem 5) or that

(5.8) for all $f, f' \varepsilon F$, $f \leq f'$ implies $Tf \leq Tf'$,

and

(5.9) for any $f \varepsilon F$ and any constant $a > 0$,

$$T(f + a) \leq Tf + \beta a \text{ for some } \beta \varepsilon [0,1).$$

That the monotonicity condition (5.8) is satisfied follows from (5.7): increasing f everywhere enlarges the feasible (x,y,z) set.

To verify (5.9), let $f \varepsilon F$, $k \varepsilon R^p_+$, $\theta \varepsilon I$, be given, and choose $a > 0$. Let (x^a, y^a, z^a) attain $T(f + a)$ at (k,θ). Define the sets U, U^a and B as follows.

$$U = \{z \varepsilon R^n_+ : f(y^a, w) - w \cdot z \geq 0 \text{ for all } w \varepsilon I\}$$

$$U^a = \{z \varepsilon R^n_+ : f(y^a, w) + a - w \cdot z \geq 0 \text{ for all } w \varepsilon I\}$$

$$B = \{z \varepsilon R^n_+ : z \leq z' + a \text{ for some } z' \varepsilon U\}.$$

We show that, as illustrated in Figure 1, $U^a = B$.

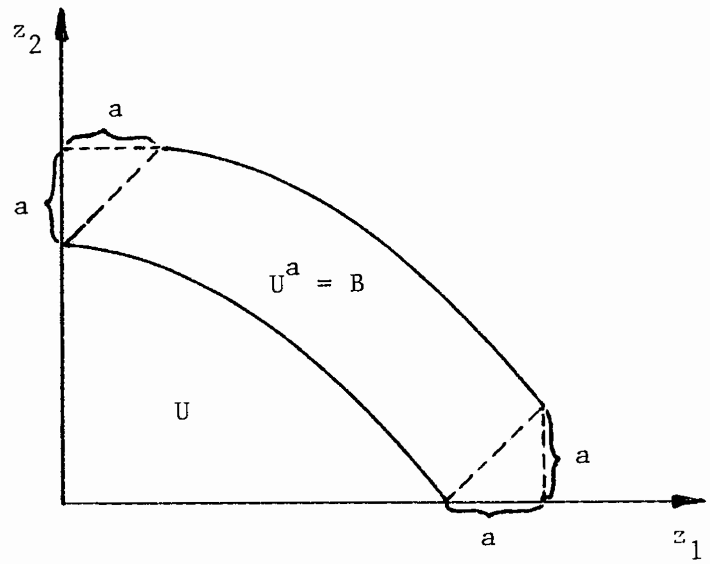

Figure 1

Since $\Sigma_i w_i = 1$, it is clear that $z' \in U$ implies $z' + a \in U^a$, so that $B \subseteq U^a$. Suppose $z \in U^a$ and $z \notin B$. It follows that if $z' \in R^n_+$ satisfies $z' + a \geq z$, then $z' \notin U$. Hence there exists $w \in I$, such that $f(y^a,w) - w\, z' < 0$. This in turn implies $f(y^a,w) - w\,(z-a) < 0$, or $z \notin U^a$: a contradiction. Hence $U^a = B$.

If (x^a,y^a,z^a) attains $T(f+a)$, then $z^a \in U^a = B$, so that $z^a \geq z' + a$ for some $z' \in U$. Then $(x^a,y^a,z'+a)$ also attains $T(f+a)$, since the W^i are increasing functions of their last argument. Then since (x^a,y^a,z') is feasible for the problem $(Tf)(k,\theta)$,

$$(Tf)(k,\theta) \geq \Sigma_i \theta_i W^i(x^a_i, z'_i),$$

from which it follows that

$$\begin{aligned}(T(f+a))(k,\theta) - (Tf)(k,\theta) &\leq \Sigma_i \theta_i [W\,(x_i, z'_i + a) - W\,(x_i, z'_i)] \\ &\leq \Sigma_i \theta_i \beta_i a \\ &< \max_i \beta_i a \;<\; a\end{aligned}$$

where the last inequality follows from property W5 of the functions W^i. Since this conclusion holds for all choices of f,k and θ, (5.9) is verified.

Thus T is a contraction, and it follows from the contraction mapping theorem that there exists a unique solution $\hat{v}$ to f = Tf, and that

$$(5.10) \qquad \lim_{n\to\infty} \|T^n f_o - \hat{v}\| = 0, \quad \text{for all } f_o \in F.$$

To prove that $\hat{v}$ is increasing in k, recall (B3) that $k' \leq k$ implies $B(k') \subseteq B(k)$.

To prove that $\hat{v}$ is concave in k and convex in θ, we show that if f ε F has these properties, so does Tf. Then choosing f_o to be concave in k and convex in θ it will follow from (5.10) that the solution $\hat{v}$ has these properties.

The proof that T takes concave-convex functions into concave-convex functions is a standard argument (see, for example, [22], p. 1433) utilizing the concavity of $W^1,\ldots,W^n$ (W2), the non-increasing returns to k of B(k) (B4), the fact that each W^i is an increasing function of its last argument, and the convexity of the (x,y,z) set defined by (x,y) ε B(k), z ⩾ 0, and (5.7). To verify this last fact, let (x^i,y^i,z^i), i = 1,2, lie in this set and let $(x^\lambda,y^\lambda,z^\lambda)$ be a convex combination. Then $(x^\lambda,y^\lambda) \in B(k)$ by B1 and

$$\begin{aligned}\min_{w\in I} \hat{v}(y^\lambda,w) - w\cdot z^\lambda \;&\geq\; \min_{w\in I} \left(\lambda\hat{v}(y^1,w) + (1-\lambda)\hat{v}(y^2,w) - w\cdot z^\lambda\right)\\ &\geq \lambda \min_{w\in I} \left(\hat{v}(y^1,w) - w\cdot z^1\right) + (1-\lambda)\min_{w\in I}\left(\hat{v}(y^2,w) - w\cdot z^2\right)\\ &\geq 0,\end{aligned}$$

so that (y^λ,z^λ) satisfies (5.7).

This completes the proof of Lemma 4.

In view of Lemma 4, the following lemma establishes that $v = \hat{v}$, or that $v(k,\theta)$ as defined in (5.2) is the unique continuous, bounded solution of the functional equation (5.6)-(5.7).

Lemma 5. The function $v: R^P_+ \times I \to R_+$ defined in (5.2) satisfies (5.6)-(5.7).

Proof. We have:

$$
\begin{aligned}
v(k,\theta) &= \max_{z \in U(k)} \Sigma_i \theta_i z_i \\
&= \max_{x \in Y(k)} \Sigma_i \theta_i u^i(x_i). \\
&= \max_{x \in Y(k)} \Sigma_i \theta_i W^i(x_{io}, u^i({}_1x_i)) \\
&= \max_{\substack{(x_o,y) \in B(k) \\ z \in U(y)}} \Sigma_i \theta_i W^i(x_{io}, z_i) \\
&= \max_{\substack{(x_o,y) \in B(k) \\ z \geq 0}} \Sigma_i \theta_i W^i(x_{io}, z_i)
\end{aligned}
$$

subject to (5.7), where the second line follows from the definition of U(k), the third from the definition of u^i in terms of W^i in Section 3, the fourth from the definition of Y(k) in Section 4, and the last from Lemma 3.

This completes the proof of Lemma 5.

In view of Lemma 4 the maximum problem on the right-hand side of (5.6) involves maximizing a continuous, concave function over a compact, convex set. Hence for each fixed (k,θ), $v(k,\theta)$ is attained by some $(x,y) \in B(k)$ and $z \geq 0$ satisfying (5.7). Given such an (x,y,z) choice, the minimum problem in

(5.7) involves minimizing a continuous function over the compact set I and the minimum is hence attained at some $w \in I$. Call the set of such maximizing-minimizing choices the values of a <u>policy correspondence</u> G taking states (k,θ) in $R_+^p \times I$ into choices (x,y,z,w) in $R_+^{mn} \times R_+^p \times R_+^n \times I$. The main properties of this correspondence are described in

<u>Lemma 6</u>. The policy correspondence $G(k,\theta)$ is lower hemi-continuous and convex-valued.

<u>Proof</u>. Lower hemi-continuity is from ([6], p.116). Convexity follows from the concavity of $W^1,\ldots,W^n$, the concavity of v in k, the convexity of I and the convexity of v in θ.

Given an initial capital vector k_o and a vector of weights θ_o, any sequence $\{x_t,y_t,z_t,w_t\}_{t=0}^{\infty}$ satisfying:

(5.11) $$(x_{t+1},y_{t+1},z_{t+1},w_{t+1}) \in G(k_t,\theta_t), \quad t = 0,1,2,\ldots,$$

(5.12) $$k_{t+1} = y_t, \quad t = 0,1,2,\ldots,$$

(5.13) $$\theta_{t+1} = w_t, \quad t = 0,1,2,\ldots,$$

defines a set of resource allocations $x = \{x_t\} \in \ell_+^{mn}$. Moreover, the sequence $\{z_t\}$ defines for each period t, utilities to be enjoyed from t on.

It is clear from their construction that these allocations are feasible, given k_o. We will refer to these allocations and their associated utility paths as being <u>generated from</u> (k_o,θ_o) by the policy correspondence G.

We have, finally,

<u>Lemma 7.</u> An allocation attains $v(k_o,\theta_o)$, if and only if is generated from (k_o,θ_o) by the policy correspondence G.

<u>Proof.</u> Let $x \in \ell_+^{mn}$ be any allocation generated from (k_o,θ_o) by G, let k be an associated capital path, and let $z = \{z_t\} \in \ell_+^n$, $z_t = \left(u^1({}_tx_1),\ldots,u^n({}_tx_n)\right)$ be the associated path of utilities.

If x does not attain $v(k_o,\theta_o)$, then some other feasible allocation, $\hat{x}$ does. Let $\hat{k}$ and $\hat{z}$ denote the capital and utility paths associated with this allocation. Then

$$v(k_o,\theta_o) = \Sigma_i\theta_{oi}W^i(\hat{x}_{oi},\hat{z}_{1i}) > \Sigma_i\theta_{oi}W^i(x_{oi},z_{1i}) . \tag{5.14}$$

Now since $\hat{x}$ is feasible, $(\hat{x}_o,\hat{k}_1) \in B(k_o)$ and $\hat{z}_1 \in U(\hat{k}_1)$. By Lemma 3, this last fact implies $v(\hat{k}_1,w) - w\cdot\hat{z}_1 \geq 0$, for all $w \in I$. Hence (5.14) contradicts (5.6). Since this argument holds whether or not $\hat{x}$ is generated by G, it proves both that all allocations so generated are Pareto-optimal and that no other allocations are.

It will be useful for later reference to sum up the main results of this section in:

<u>Theorem 3.</u> For any initial capital vector k, the utility possibility set U(k) has the properties listed in Lemma 1. Its support function $v(k,\theta)$ is continuous and bounded, and is the unique solution with these properties to the functional equation (5.6)-(5.7). The allocations generated by the policy correspondence G of this dynamic program are Pareto-optimal, and all Pareto-optimal allocations are so generated.

6. Optimality and Equilibrium

One of the reasons we are interested in being able to construct Pareto-optimal resource allocations (though certainly not the only reason) is that under certain conditions they are equivalent to the set of competitive equilibrium allocations. We mean here "competitive equilibrium" in the complete market sense of Section 2, but in recursive systems it is frequently the case that such equilibria may be interpreted as perfect foresight equilibria with a complete set of "spot" markets operating at each date and a very limited set of securities linking spot markets at different dates.

The economy under study is a special case of those studied in [16], [24], [7], provided the initial capital stock k_o is in the interior of the maintainable set M defined in B5. That this is so is the content of Theorems 1 and 2 of Sections 3 and 4. From Theorem 1 of [16], then, every equilibrium allocation is a Pareto-optimum. From Theorem 2 of [16] and Theorem 1 of [24], for every Pareto-optimal allocation x^o there exists a price system $p \in \ell_+^m$ such that (ii) and (iii) of the definition of competitive equilibrium are satisfied, and such that $u^i(x_i) \geq u^i(x_i^o)$ implies

$$(6.1) \qquad \Sigma_t\, p_t \cdot x_{it} \geq \Sigma_t\, p_t \cdot x_{it}^o .$$

Following [16] (Remark, p. 591) if $\Sigma_t\, p_t \cdot x_{it}^o > 0$ for all i, this can be strengthened to condition (ii) in the definition of equilibrium.

Finally, if an equilibrium is defined by allocating to trader i a share α_i of the total value of consumption, with $\Sigma_i\, \alpha_i = 1$ and $\alpha_i \geq 0$, all i, and with the budget constraint in (ii) replaced by:

$$(6.2) \qquad \Sigma_t\, p_t \cdot x_{it} \leq \alpha_i\, \Sigma_t\, p_t \cdot \bar{x}_t, \qquad i = 1,\ldots,n,$$

then the existence of an equilibrium for each α follows from Theorems 1 and 3 of [7].

Since all Pareto-optimal allocations can be constructed by the method described in Section 5, there is a sense, then, in which all competitive equilibria can be constructed too. For two reasons, however, this sense is a rather limited one. First, given a Pareto-optimal allocation x, one needs to be able to calculate the price system(s) p that support it. Second, even if one can calculate p and hence also the wealth distribution

$$\alpha_i = [\Sigma_t\, p\cdot\bar{x}_t]^{-1}\, \Sigma_t\, p_t\cdot x_{it}, \qquad i = 1,\ldots,n,$$

corresponding to x, it will not in general be true that this will offer a method (other than trial-and-error) for finding an equilibrium for a <u>given</u> wealth distribution α'. This latter capability is what is typically meant by the ability to construct equilibria.

We will not pursue either of these difficult issues at this level of generality, but both will come up again in the context of a specific example, in Section 10.

7. Dynamics

In the preceding sections it was shown that the optimal policy correspondences for the dynamic program (5.6) generate all the Pareto-optimum allocations for the economy under study. In this and in the remaining sections of the paper, various aspects of the dynamics of this economy will be examined. As the reader familiar with the dynamics of recursive, one-consumer economies with heterogeneous capital will recognize, however, the present model is too generally formulated to yield useful qualitative results on either the uniqueness or the stability of stationary optima (or equilibria).[4] Accordingly, our first task in the present section will be to specialize the model to the point where its first-order conditions describe its motion, in a way that it may be easily compared to the more familiar equations of motion for the one-consumer case. With this done, the parallels and differences between the dynamics of this model and earlier ones can be exhibited in as clear a way as possible. Stationary solutions will be examined, under these same restrictions, in the section following, and a much more specialized example will be studied in sections 9 and 10.

The new restrictions to be imposed have to do with the number of consumption goods, and with the strict concavity and differentiability of both the technology and preferences. We begin with the technology.

In order to make it easier to focus on dynamic issues, we will restrict attention to economies with one consumer good, so that $x_{it} \varepsilon R_+$. Define the function $F: R^p_+ \times R^p_+ \rightarrow R$ by:

(7.1) $$F(y,k) = \max_{(x,y)\varepsilon B(k)} x \, .$$

Thus $F(y,k)$ is the maximum output of consumption good consistent with an initial vector of capital k and a terminal vector y. It follows from B1-B4 via standard arguments (see, for example, [28]) that F is continuous, decreasing in y, increasing in k, and concave in (y,k). To the restrictions B1-B6 we add, for the remainder of the paper, the assumption that F is continuously differentiable.

Analogously, to assumptions W1-W5 on preferences we add the assumptions that W^i is continuously differentiable, strictly concave, and satisfies the Inada condition

(7.2) $$\lim_{x\rightarrow 0} W^i_1(x,z) = \infty \, , \qquad \text{for all } z, \ i = 1,\ldots,n.$$

With this additional structure we have, first,

Lemma 8. The value function $v(k,\theta)$ is strictly concave in k and strictly convex in θ.

Proof. Strict concavity in k is an immediate consequence of the strict concavity of the W^i, the concavity of v in k (Lemma 4), and the fact that $v = Tv$.

To show that v is strictly convex in θ, choose $\theta^o, \theta^1 \varepsilon I$, let θ^λ be a convex combination, and let (x^r, y^r, z^r) attain $v(k,\theta^r)$, $r = 0,1,\lambda$. Given the differentiability of W^i and (7.2), a necessary condition on (x^r, y^r, z^r) is

$$\theta_i^r W_1^i(x_i^r, z_i^r) = \theta_j^r W_1^j(x_j^r, z_j^r) , \qquad r = 0,1,\lambda, \text{ for all } i,j.$$

Since $W_1^i > 0$, it follows that if $\theta^o \neq \theta^1$, $(x^o, z^o) \neq (x^1, z^1)$ and hence that $(x^\lambda, z^\lambda) \neq (x^r, z^r)$, for $r = 0$ or 1 or both. Then if $0 < \lambda < 1$,

$$\begin{aligned} v(k,\theta^\lambda) &= \Sigma_i \theta_i^\lambda W^i(x_i^\lambda, z_i^\lambda) \\ &= \lambda\, \Sigma_i \theta_i^o W^i(x_i^\lambda, z_i^\lambda) + (1-\lambda)\, \Sigma_i \theta_i^1 W^i(x_i^\lambda, z_i^\lambda) \\ &< \lambda\, v(k,\theta^o) + (1-\lambda)\, v(k,\theta^1) \end{aligned}$$

as was to be shown.

In view of Lemma 8, the right hand side of (5.6) is attained by a unique (x,y,z) value for each (k,θ), and the minimum problem (5.7) is solved by a unique w value for each z. Hence the policy correspondence G will be a continuous function of (k,θ), and we refer to it hereafter as the policy function for (5.6)-(5.7). Let G^x, G^y, G^z, and G^w denote the projections of G onto the consumption, capital, and utility spaces, and n-simplex, respectively, with components $(G_1^x,\ldots,G_n^x)$, etc.

Next, we establish

Lemma 9. If $\theta \gg 0$, then for all $k \geq 0$, v is continuously differentiable at (k,θ), with derivatives given by

$$v_j(k,\theta) = \theta_i W_1^i(G_i^x(k,\theta),G_i^z(k,\theta))\ F_{p+j}(G^y(k,\theta),k)\ , \tag{7.3}$$

$$j = 1,\ldots,p,\ i = 1,\ldots,n,$$

and

$$v_{p+i}(k,\theta) = W^i(G_i^x(k,\theta),G_i^z(k,\theta))\ , \qquad i = 1,\ldots,n. \tag{7.4}$$

Proof. (7.4) is obvious, given the continuity of W^i and G. We verify (7.3) by paralleling exactly the proof of Theorem 1 of [4]. In view of condition (7.2), $\theta \gg 0$ implies $G_i^x(k_o,\theta) > 0$, for all i. Then for some neighborhood of k_o,

$$x(k) = G_i^x(k_o,\theta) + F(G^y(k_o,\theta),k) - F(G^y(k_o,\theta),k_o)$$

will be positive. Define R(k) on such a neighborhood by

$$R(k) = v(k_o,\theta) + \theta_i[W^i(x(k),G_i^z(k_o,\theta)) - W^i(G_i^x(k_o,\theta),G_i^z(k_o,\theta))].$$

Then $R(k_o) = v(k_o,\theta)$, and since v is the optimum value function, $R(k) \leq v(k,\theta)$ for all k. Moreover, R(k) is differentiable and concave, since W^i and F have these properties. Then by [4], Lemma 1, $v(k,\theta)$ is differentiable at (k_o,θ) and its derivatives at this point are those of R(k). Then (7.3) follows from the definitions of R(k) and x(k).

With these preliminaries completed, the unique solutions to the maximum and minimum problems (5.6) and (5.7) are given by the first-order conditions for these problems, obtained as follows. The Lagrangean corresponding to (5.6)-(5.7) is

$$L = \Sigma_i \theta_i W^i(x_i, z_i) + \lambda[F(y,k) - \Sigma_i x_i] + \mu \min_{w \varepsilon I} [v(y,w) - w \cdot z] .$$

The first-order conditions are:

(7.5) $\quad 0 = \theta_i W^i_1(x_i, z_i) - \lambda \quad , \qquad i = 1,\ldots,n,$

(7.6) $\quad 0 = \theta_i W^i_2(x_i, z_i) - \mu w_i, \quad , \qquad i = 1,\ldots,n,$

(7.7) $\quad 0 = \lambda\, F_j(y,k) + \mu v_j(y,w) \; , \qquad j = 1,\ldots,p,$

(7.8) $\quad 0 = F(y,k) - \Sigma_i x_i \quad ,$

(7.9) $\quad 0 = 1 - \Sigma_i w_i \quad ,$

(7.10) $\quad 0 = v_{p+i}(y,w) - z_i \quad , \qquad i = 1,\ldots,n,$

(7.11) $\quad 0 = v(y,w) - w \cdot z \quad .$

Equations (7.5)-(7.11) are $3n + p + 3$ equations in the "unknowns" x, y, z, w, λ and μ. Since $v(y,w)$, viewed as a function of $w \in R^n_+$, is homogeneous of degree one in w, it follows from Euler's theorem that (7.11) is redundant, given (7.9)-(7.10).

There are three interrelated uses to which (7.5)-(7.10) may be put. If one actually knew the functions W^i and F and wished to calculate the Pareto-optimal program for a given (k_o, θ_o) then the function $v(k,\theta)$ would be tabulated using the constructive methods of Section 5, and its derivatives could be taken as known, too, to any desired degree of accuracy. Then for each (k,θ), (7.5)-(7.10) would be solved for (x,y,z,w,λ,μ), and (x,y,z,w) would be the value of the policy function G at (k,θ). Having so tabulated the function G, the system could be "run" from any initial (k_o,θ_o) and its dynamics explored in as much detail as desired (and affordable).

For other purposes one may be interested in such qualitative questions as whether the system described by G has a stationary point, whether it is unique, and how it varies with given changes in tastes and technology. These issues will be studied in some detail in the next section.

Finally, given the existence of a stationary point, one is interested in its local and global stability. Qualitative study of this question is often more conveniently carried out with the system (7.5)-(7.10) stated in Euler equation form. We next give this re-statement, in order to facilitate comparison with the more familiar one-agent economy. Using the derivatives of the maximized objective function given by (7.3)-(7.4), and paralleling (7.5)-(7.10) line by line, we find that

(7.12) $$0 = \theta_{it}W_1^i(x_{it},z_{i,t+1}) - \lambda_t \quad , \ i = 1,\ldots,n,$$

(7.13) $$0 = \theta_{it}W_2^i(x_{it},z_{i,t+1}) - \mu_t\theta_{i,t+1} \quad , \ i = 1,\ldots,n,$$

(7.14) $$0 = \lambda_t F_j(k_{t+1},k_t) + \mu_t\nu_{j,t+1} \quad , \ j = 1,\ldots,p,$$

(7.15) $$0 = F(k_{t+1},k_t) - \Sigma_i x_{it} \quad ,$$

(7.16) $$0 = 1 - \Sigma_i\theta_{i,t+1} \quad ,$$

(7.17) $$z_{it} = W^i(x_{it},z_{i,t+1}) \quad , \ i = 1,\ldots,n,$$

(7.18) $$\nu_{jt} = \lambda_t F_{p+j}(k_{t+1},k_t) \quad , \ j = 1,\ldots,p.$$

In transcribing (7.5)-(7.10) into the form (7.12)-(7.18), k,θ,x,z,λ and μ have been given time subscripts; y and w have become k_{t+1} and θ_{t+1}; and the value function derivative $\partial v(k,\theta)/\partial k$ has become υ_t. These 3n + 2p + 2 equations may be viewed as a dynamic system in the n + 2 "control variables" x_t, λ_t and μ_t, the n + p "state variables" k_t and θ_t, and the n + p "co-state variables" ν_t and z_t. To study the dynamics of the system in a neighborhood of a stationary point, then, one would eliminate the control variables and examine the 2(n + p) characteristic roots of the system in $(k_t, \theta_t, \nu_t, z_t)$. Since $\Sigma_i \theta_{i,t+1} = 1$, one of these roots is zero. Hence the condition for local saddlepoint stability is that n + p - 1 non-zero roots lie within the unit circle and n + p without. Such a study would not, at this level of generality, be informative. An example will be examined in Section 9.

The advantage of studing the dynamics of this model in the form (7.12)-(7.18) lies in the fact that these Euler equations do not involve the value functions or its derivatives in any way. The only functions whose "shapes" matter are those describing preferences W^i and the technology F, which are those on which we place restrictions directly (and hence are as well informed about as we wish to be).

On the other side, it is clear that information has been lost in moving from (7.5)-(7.10) to (7.12)-(7.18), so that the latter system will--for given (k_o, θ_o)-- have many more solutions than will the former: One for each possible initial configuration of the state <u>and</u> co-state variables. This situation is sometimes termed a "problem" and it has even been suggested that the relative paucity of solutions to (7.5)-(7.10) is a kind of defect of looking at things in a recursive way, or a sweeping under the rug of important economic issues. Section 5 of this paper shows that just the opposite is true. One of its implications is that there is (apart from the problems raised by "flats" in objective functions that can arise in any programming problem) only one Pareto-optimal way for the system to evolve from an initial vector k of capitals and allocation of utility weights θ_o. Thus, only for the "right" initial values for the costate variables $\left(\nu_o = \partial v(k_o,\theta_o)/\partial k_o\right.$ and $\left.z_o = \partial v(k_o,\theta_o)/\partial\theta_o\right)$ is a solution of (7.12)-(7.18) also a solution of the original resource allocation problem.

It may be instructive to consider (7.12)-(7.18) for the particular case where the aggregator functions W^i assume the time-additive form: $W^i(x,u) = U_i(x) + \beta_i u$, for some $0 < \beta_i < 1$. Then (7.13) reduces to

(7.19) $$\theta_{it}\beta_i = \mu_i\theta_{i,t+1} ,$$

since in the time-additive case $W_2^i = \beta_i$. Then for any two consumers i,j

(7.20) $$\frac{\theta_{i,t+1}}{\theta_{j,t+1}} = \frac{\beta_i}{\beta_j}\frac{\theta_{i,t}}{\theta_{j,t}} .$$

Consider the possible dynamics under (7.20). If $\beta_i < \beta_j$ (so that consumer i is more "impatient" than consumer j), the relative weight θ_{it} on consumer i's utility will, from (7.20), converge to zero at a geometric rate. Then from (7.12), his consumption must also converge to zero. This description of the relative positions of consumers i and j can simply be reversed if $\beta_i > \beta_j$. Thus the only dynamics consistent with (7.20) require that the consumption of all but the most patient (highest β) consumers converge to zero.

If $\beta_i = \beta_j$ for all consumers, this conclusion is avoided, but in an equally drastic way. In this case, <u>all</u> non-negative constant weights $\theta\varepsilon I$ satisfy (7.19), which is to say that <u>any</u> initial distribution of utility will be stationary. If either of these strong conclusions arose from some <u>economic</u> feature of the model they would be of some interest, but in fact they are simply transparent consequences of time-additivity, an assumption of convenience for which no one has ever claimed an economic rationale. This is exactly why it seems necessary to use a broader class of preferences in a study of dynamics with consumer heterogeneity.

Notice also that with time-additive preferences with common discount factor β, equations (7.12),(7.14),(7.15) and (7.18) are, given an initial (and hence permanent) set of weights θ, a dynamic system in k_t, υ_t, x_t and λ_t alone. Equation (7.17) continues to describe the evolution of utilities over time, but there is no interaction between the dynamics of capital accumulation and utility. In the more general case under study there are rich possibilities for substitution or complementarity relationships among current and future consumptions, which can interact with intertemporal possibilities in production in complicated ways.

8. Stationary Points

In this section we will study stationary points of the system described in Section 7, that is, vectors (x^*,k^*,z^*,θ^*) satisfying:

$$(x^*,k^*,z^*,\theta^*) = G(k^*,\theta^*) \ ,$$

$$z_i^* = W^i(x_i^*,z_i^*) \ , \qquad i = 1,\ldots,n.$$

From (7.12)-(7.18), we see that any interior stationary solution must satisfy:

(8.1) $$0 = \theta_i^* W_1^i(x_i^*,z_i^*) - \lambda^* \quad , \ i = 1,\ldots,n,$$

(8.2) $$0 = \theta_i^*[W_2^i(x_i^*,z_i^*) - \mu^*] \quad , \ i = 1,\ldots,n,$$

(8.3) $$0 = \lambda^*[F_j(k^*,k^*) - \mu^* F_{p+j}(k^*,k^*)] \quad , \ j = 1,\ldots,p,$$

(8.4) $$0 = F(k^*,k^*) - \Sigma_i x_i^* \quad ,$$

(8.5) $$0 = 1 - \Sigma_i \theta_i^* \quad ,$$

(8.6) $$0 = W^i(x_i^*, z_i^*) - z_i^* \quad , \; i = 1,\ldots,n.$$

We will look for solutions of this system in the following way. Given μ, (8.2) and (8.6) may (perhaps) have a unique solution $(x,z) = (D^i(\mu), Z^i(\mu))$ for $\theta > 0$. Letting $D(\mu) \equiv \Sigma_i D^i(\mu)$, we can view $D(\mu)$ as a "demand curve" for total stationary consumption as a function of the stationary interest factor μ ($\mu = 1/(1+r)$, where r is the stationary interest rate). Similarly, given μ, (8.3) may have a unique solution $K(\mu)$.
Letting $S(\mu) \equiv F(K(\mu),K(\mu))$, we can view $S(\mu)$ as a "supply curve" for total stationary consumption. The market-clearing condition (8.4) is then equivalent to finding the interest factor μ^* where the "demand curve" and "supply curve" intersect, $D(\mu^*) = S(\mu^*)$. The terms "demand curve" and "supply curve" are enclosed in quotation marks to indicate their rather illegitimate use in this steady state context: obviously there is not really a market for "stationary consumption."

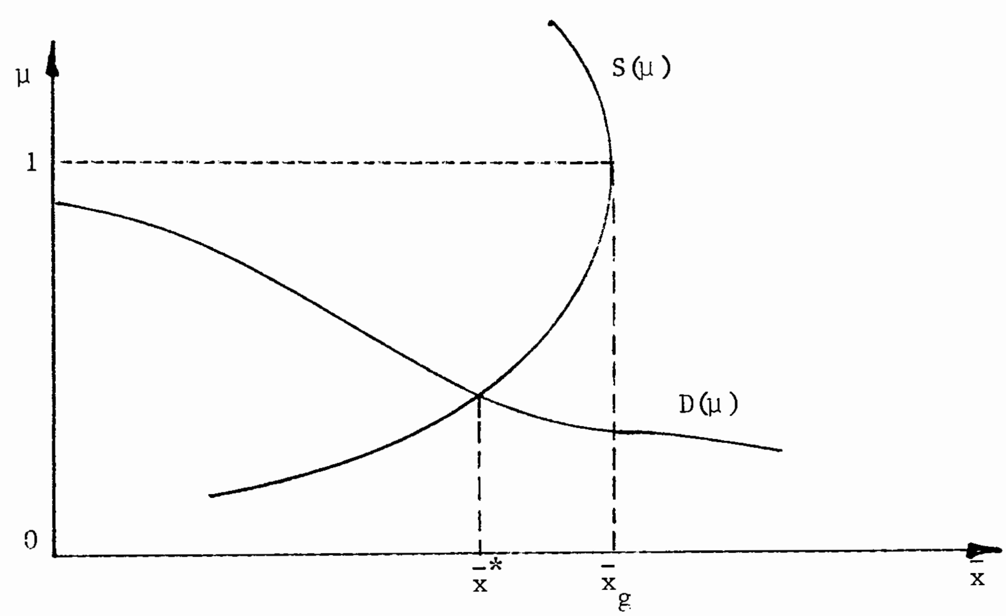

Figure 2

Now an economist with any sense of decency would surely want these curves to look as drawn in Figure 2, so that they intersect exactly once, or so that there exists a unique stationary point to the system (8.1)-(8.6). It is an unfortunate but well established fact that this need not be so. One is led, then, to the study of conditions on the functions F and $W^1,\ldots,W^n$, in addition to those already imposed, under which Figure 2 is an accurate description. This study occupies the remainder of this section.

For the standard one-sector (that is, one capital good) model, we know that S(R) is as drawn in Figure 2. Thus, if

$$F(y,k) = f(k) + (1 - \delta)k - y \ ,$$

where f is increasing and strictly concave and δ is the rate of depreciation, (8.3) reduces to

(8.7) $$0 = \lambda^* \cdot [f'(k^*) + 1 - \delta] - \mu^*,$$

and (8.4) becomes

(8.8) $$0 = f(k^*) - \delta k^* - \Sigma_i x_i^* \ .$$

Then the solution function $x^* = S(\mu^*)$ obtained from (8.7) and (8.8) has the depicted shape with $\bar{x}_g$ on the diagram being the maximal sustainable or "golden rule" consumption level.

With many capital goods there are many more possibilities and the best one can do is to find useful sufficient conditions for $S(\mu)$ to take this form. This problem has been extensively studied by Burmeister and Brock ([10], [11], [13]). See [13] for a good summary and an exposition of the idea of regularity of a multi-sector economy, which is one sufficient condition for $S'(\mu) > 0$ for $\mu < 1$.[5] This matter will not be pursued further here.

On the demand side matters are much simpler. Since $D(\mu) = \Sigma_i D^i(\mu)$, a sufficient condition for D to be nonincreasing is that D^i, $i = 1,\ldots,n$, be so. Dropping the i, we will consider a representative consumer.

Property W5 insures that we can define the function $\phi: R_+ \to R_+$ by:

(8.9) $\qquad \phi(x) \equiv W(x,\phi(x)), \qquad$ for all $x \geqslant 0$.

Therefore, D,Z satisfy (8.2) and (8.6) for all μ, for $\theta_i > 0$, if and only if:

(8.10) $\qquad W_2\big(D(\mu),\phi(D(\mu))\big) = \mu , \qquad$ for all $\mu > 0$,

(8.11) $\qquad Z(\mu) = \phi(D(\mu)), \qquad$ for all $\mu > 0$.

It is clear from (8.10) that $D(\mu)$ is nonincreasing in μ if and only if $W_2(x,\phi(x))$ is nonincreasing in x. Hence D is nonincreasing if and only if the consumer's subjective time discount factor is a nonincreasing function of steady state consumption.

This condition is most clearly interpreted by studying the function $\Psi: R_+^2 \to R_+$ defined by:

(8.12) $\qquad \Psi(x,\hat{x}) \equiv W(x,\phi(\hat{x}))$.

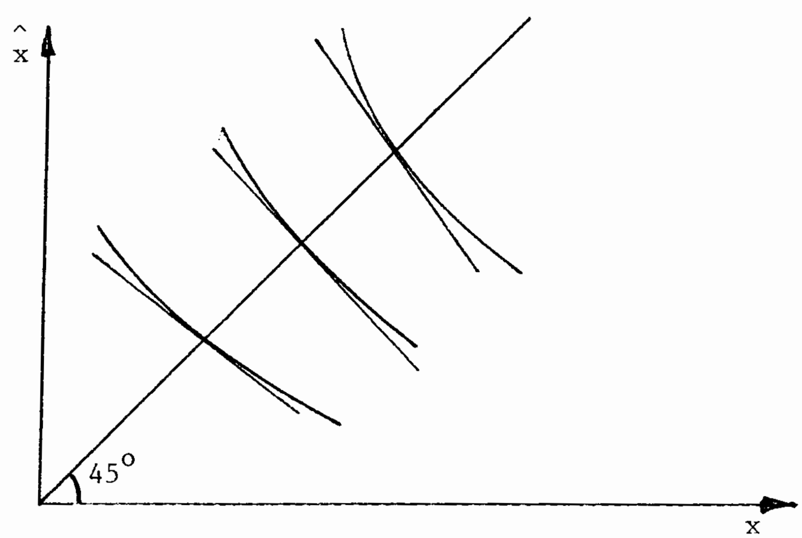

Figure 3

Level sets of the function Ψ, shown in Figure 3, are indifference curves between levels of consumption x today and stationary levels of consumption $\hat{x}$ from tomorrow on. Since $\Psi(x,x) = \phi(x)$,

$$(8.13) \qquad \phi'(x) = \Psi_1(x,x) + \Psi_2(x,x) , \qquad \text{for all } x \geq 0.$$

Hence, differentiating (8.12) with respect to $\hat{x}$, using (8.13), and evaluating at $x = \hat{x}$, we find that:

$$W_2(x,\phi(x)) = \Psi_2(x,x)/\big(\Psi_1(x,x) + \Psi_2(x,x)\big) = [1 + \Psi_1(x,x)/\Psi_2(x,x)]^{-1}.$$

Therefore, D is nonincreasing in μ if and only if $\Psi_1(x,x)/\Psi_2(x,x)$ is nondecreasing in x, i.e., if and only if:

$$(8.14) \qquad W_2(x,\phi(x)) \text{ is nondecreasing in } x.$$

Diagrammatically, the required condition is that the indifference curves in Figure 3 get steeper farther out along the 45^o line, or that the consumer becomes more impatient (at the margin) at higher levels of stationary consumption.

Given these two conditions of regularity of the technology and increasing impatience of preferences, stationary states are described by:

Theorem 4. Let the technology F be regular in the sense of Burmeister [13], and for $i = 1,\ldots,n$, let W^i satisfy W1-W5, satisfy (7.2), and be strictly concave. Assume further that $W_2^i\big(x,\phi^i(x)\big)$ is nonincreasing in x, for all $x \geq 0$, where $\phi^i(x)$ is as defined in (8.9). Then there exists a stationary state (x^*,k^*,z^*,θ^*) satisfying (8.1)-(8.6), with associated multipliers μ^* and λ^*. Moreover,

$$(8.15) \qquad \max_i \underline{\beta}^i \equiv \underline{\beta} < \mu^* < \bar{\beta} \equiv \max_i \bar{\beta}^i,$$

where

$$\bar{\beta}^i \equiv \lim_{x\to 0} W_2^i(x,\phi^i(x)), \qquad i = 1,\ldots,n,$$

$$\underline{\beta}^i \equiv \lim_{x\to\infty} W_2^i(x,\phi^i(x)), \qquad i = 1,\ldots,n.$$

If $W_2^i(x,\phi^i(x))$ is strictly decreasing in x, for all $x \geq 0$, for $i = 1,\ldots,n$, then the equilibrium is unique. Moreover, $\theta_i^* > 0$ if and only if $\bar{\beta}_i > \mu^*$.

Proof. Since F is regular, $S(\mu)$ is strictly increasing on the range $0 \leq \mu \leq 1$, with $S(0) = \underline{x} \geq 0$, $S(1) = \bar{x}_g < \infty$, as shown in Figure 4. Under the given assumptions on the W^i's, $D(\mu)$ is defined on the range $\underline{\beta} < \mu \leq 1$. It is non-increasing in this range, with

$$D(\mu) = 0 , \qquad \bar{\beta} \leq \mu \leq 1 ,$$

$$\lim_{\mu\to\underline{\beta}} D(\mu) = \infty .$$

As shown in Figure 4, $S(\mu)$ and $D(\mu)$ must cross exactly once, and at the point of intersection (8.15) holds.

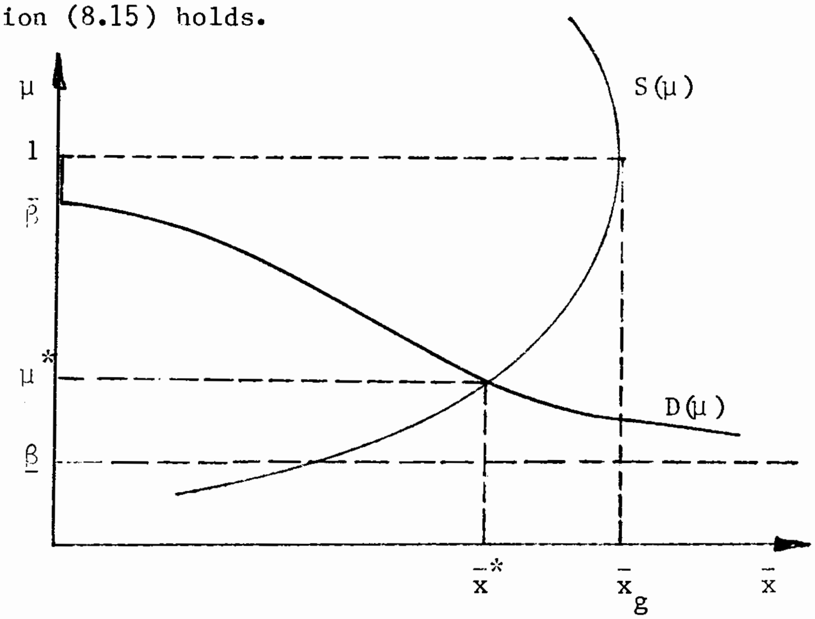

Figure 4

If $W_2^i(x,\phi(x))$ is strictly decreasing in x, for all $x \geq 0$, for i =

$1,\ldots,n$, then $D^i(\mu)$ is strictly decreasing for all i. Thus, given μ^* the uniqueness of the allocation $(x_1^*,\ldots,x_n^*)$ follows as shown on Figure 5.

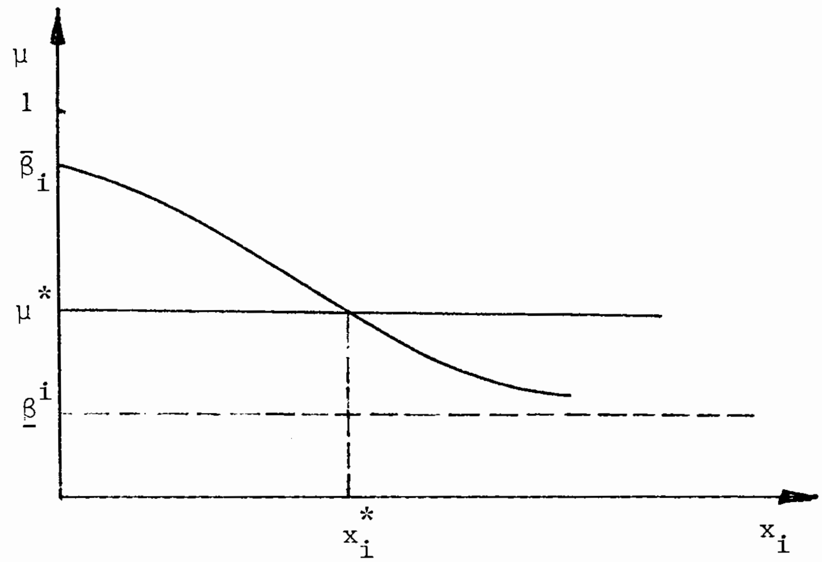

Figure 5

The uniqueness of $(z_1^*,\ldots,z_n^*)$ and $(\theta_1^*,\ldots,\theta_n^*)$ is then immediate. If $\bar{\beta}_i > \mu^*$ holds, then $x_i^* > 0$, and conversely, as shown in Figure 5. Then it follows from nonsatiation that $\theta_i^* > 0$ if $x_i^* > 0$.

9. A Two-Agent Exchange Economy: Stability

Some of the ideas of the preceding sections can be illustrated by their application to the study of a simple example: a two-agent pure exchange system. Let there be two consumers, with contiuously differentiable, strictly concave preferences W^1 and W^2 satisfying assumptions W1-W5 of section 3 and condition (7.2) of section 7. Let there be a single, non-storable consumption good x, the endowment of which is fixed at the constant level $\bar{x} > 0$, so that the technological assumptions of section 4 are trivially satisfied. In the notation used earlier, then, $n = 2$, $m = 1$, and $p = 0$.

Under these restrictions, it has been shown that a continuous optimal policy function exists, giving the current period goods allocation (x_1,x_2), the end-of-period-utility allocation (z_1,z_2), and the beginning-of-next-period weights (w_1,w_2) as functions of the beginning of period weights (θ_1,θ_2). For this two-agent case, it is convenient to let $\theta_1 = \theta$, $\theta_2 = 1-\theta$, and denote the optimal policy functions by $G^x(\theta,1-\theta) \equiv \left(x(\theta),\bar{x}-x(\theta)\right)$, $G^z(\theta,1-\theta) \equiv \left(z_1(\theta),z_2(\theta)\right)$, and $G^w(\theta,1-\theta) \equiv \left(w(\theta),1-w(\theta)\right)$. Then $x\colon [0,1] \to [0,\bar{x}]$; $z_i\colon [0,1] \to [0,\phi^i(\bar{x})]$, where ϕ^i is the function defined in (8.9); and $w\colon [0,1] \to [0,1]$. The motion of θ_t, the one state variable of the system, is described by:

$$(9.1) \qquad \theta_{t+1} = w(\theta_t), \qquad \theta_o \text{ given,}$$

and the Pareto-optimal resource allocation, for given θ_o, is the consumption sequence described by:

$$(9.2) \qquad x_t = x(\theta_t) \quad , \qquad t = 0,1,2,\dots .$$

The objective of this section is to characterize the behavior of these optimal allocations via the study of the policy functions. Clearly, $x(0) = 0$, $x(1) = \bar{x}$ and from (7.4), $0 < x(\theta) < \bar{x}$ if $0 < \theta < 1$. Similarly, $z_1(0) = 0$, $z_1(1) = \phi^1(\bar{x})$, $z_2(0) = \phi^2(\bar{x})$ and $z_2(1) = 0$. Further information on the policy functions can be obtained from the first-order conditions (7.5)-(7.11), suitably specialized to the present case. Two of these are:

$$(9.3) \qquad \theta\, W_1^1(x(\theta),z_1(\theta)) = (1-\theta)\, W_1^2(\bar{x} - x(\theta),z_2(\theta)),$$

and

$$(9.4) \qquad \frac{\theta}{w(\theta)}\, W_2^1(x(\theta),z_1(\theta)) = \frac{1-\theta}{1-w(\theta)}\, W_2^2(\bar{x} - x(\theta),z_2(\theta)).$$

Where convenient, these derivatives of W^i will be abbreviated to $W_j^i(\theta)$.

We will impose two additional restrictions on preferences: the increasing marginal impatience assumption (8.14), and the assumption that both x and z are non-inferior "goods" for both consumers, or that for i = 1,2,

$$(9.5) \qquad x < \hat{x} \text{ and } z > \hat{z} \Rightarrow \frac{W_1^i(x,z)}{W_2^i(x,z)} > \frac{W_1^i(\hat{x},\hat{z})}{W_2^i(\hat{x},\hat{z})} .$$

Under (8.14), it follows from Theorem 4 of the preceding section that there is at most one stationary point $(x^*,z_1^*,z_2^*,\theta^*)$ with $0 < \theta^* < 1$. We will show, in Theorem 5, that if such a stationary point exists, it is stable for all $\theta_0 \varepsilon (0,1)$ and convergence is monotone, provided (8.14) and (9.5) hold.

This demonstration will rely heavily on Figure 6, which illustrates a single period's allocation of goods and utilities-from-tomorrow-on between the two agents. The lower box is an indifference map for agent 1, with goods consumption on the horizontal axis and utility z_1 on the vertical. The upper box contains the preferences and goods-utility possibilities for agent 2, with the origin at the upper right, goods measured right to left, and utilities measured top to bottom. This display is as close a replica of an Edgeworth box as can be obtained for this economy, since the second "good", utility, is not measured in a common unit.

The curve $(x,\phi^1(x))$ in the lower box pairs each possible allocation x to agent 1 with the utility $\phi^1(x)$ agent 1 would enjoy if he were to consume this amount forever. Its counterpart in the upper box is the curve $(\bar{x}-x,\phi^2(\bar{x}-x))$. From the properties of ϕ^i, these curves are continuous, increasing, and touch the corners of their respective boxes, as drawn.

The curve $(x(\theta),z_1(\theta))$ in the lower box gives the Pareto-optimal allocations of goods and utility to agent 1, traced out as θ varies from 0 to 1. Its counterpart in the upper box is $(\bar{x}-x(\theta),z_2(\theta))$. Along these curves, which together are the counterparts to the contract curve in an Edgeworth box, the tangency conditions (9.3) and (9.4) hold. It follows from Lemma 6 that these curves are continuous.

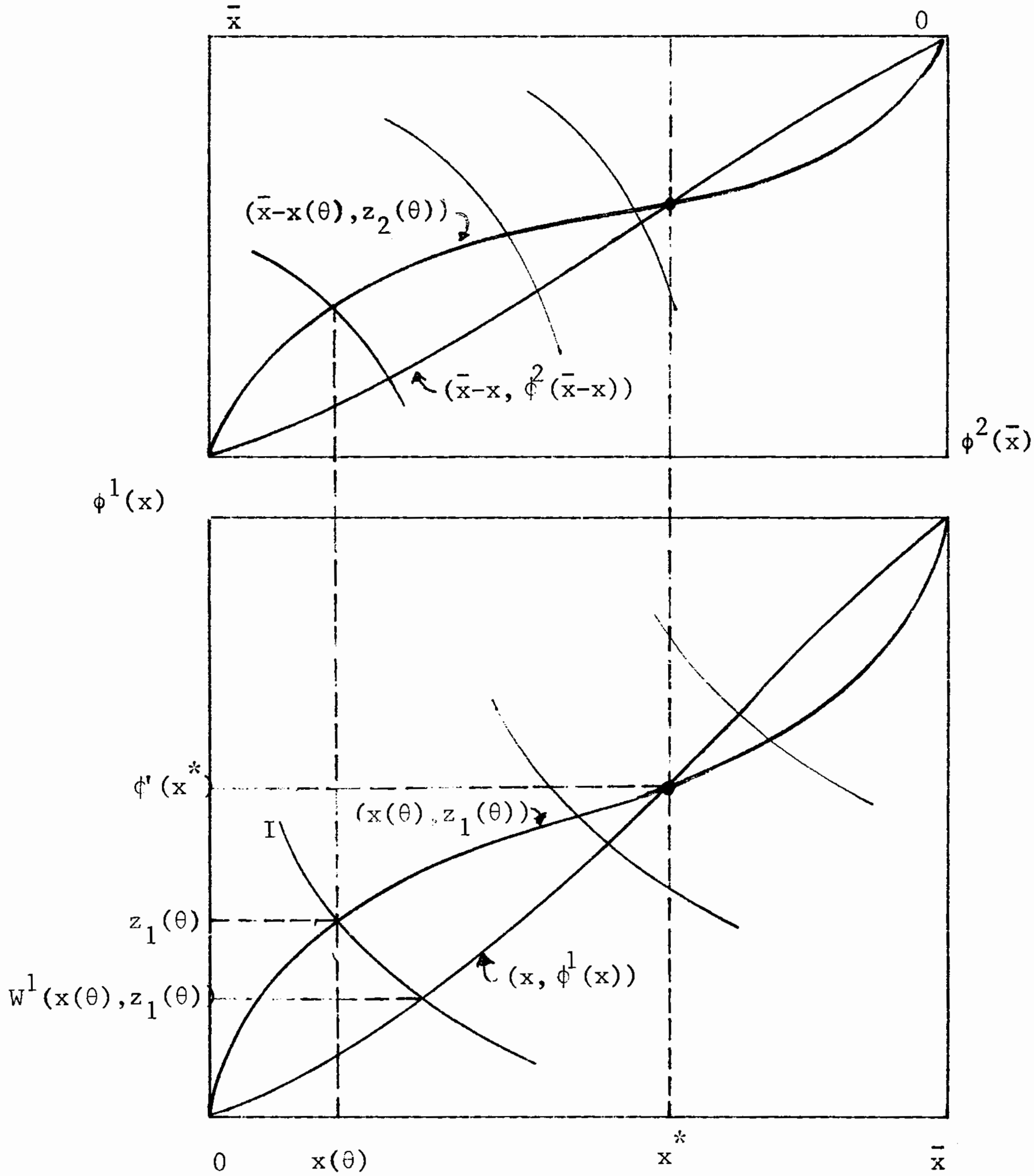

Figure 6

The curves $(x,\phi^1(x))$ and $(x(\theta),z_1(\theta))$ meet at the corners of the box, as drawn. From Theorem 4, and assumption (8.14), they coincide at exactly one other point, labelled $(x^*,\phi^1(x^*))$. There are no other stationary points. These observations have their obvious counterparts in the upper box. Because the goods allocation must sum to $\bar{x}$, the curve $(x(\theta),z_1(\theta))$ lies to the left (right) of $(x,\phi^1(x))$ if and only if $(\bar{x}-x(\theta),z_2(\theta))$ lies to the left (right) of $(\bar{x}-x,\phi^2(\bar{x}-x)$. In Lemma 10, we verify that $(x(\theta),z_1(\theta))$ is monotonically increasing, as drawn. In Lemma 11, we show that it cuts the curve $(x,\phi^1(x))$ from above at $(x^*,\phi^1(x^*))$, as drawn. The stability result in Theorem 5 will then be an easy consequence of these two facts.

Lemma 10. Under the non-inferiority condition (9.5), both $x(\theta)$ and $z_1(\theta)$ are increasing functions of θ.

Proof. If the assertion is false, at least one of the two situations depicted in Figure 7 must obtain. That is, for some weights θ_a and θ_b, both $x(\theta_a) < x(\theta_b)$ and $z_1(\theta_a) > z_1(\theta_b)$ must hold.

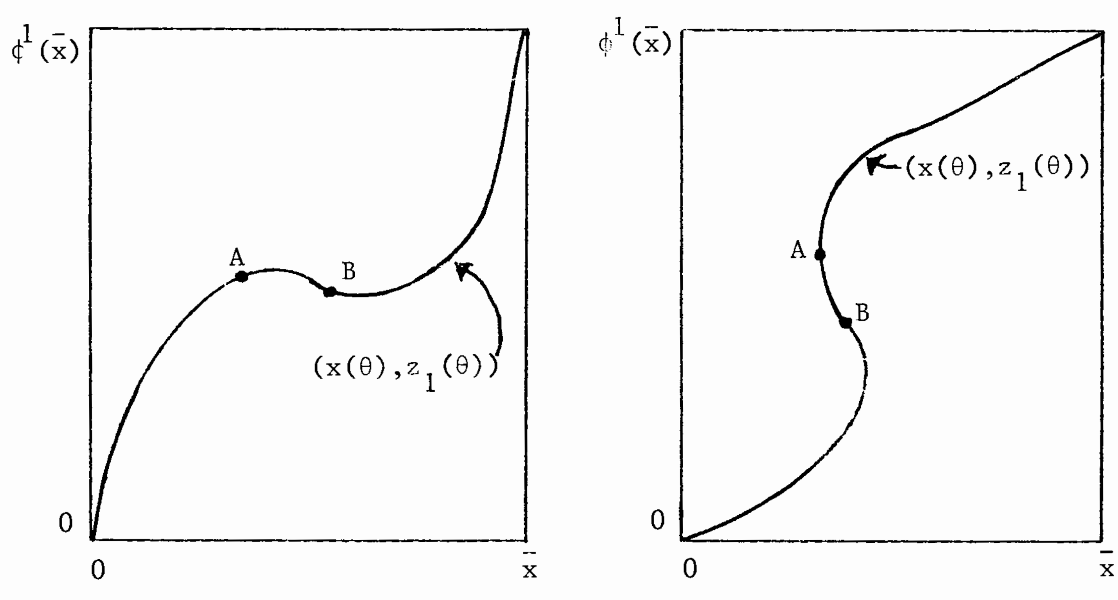

Figure 7

Then $\bar{x} - x(\theta_a) > \bar{x} - x(\theta_b)$, and since $z_1(\theta_a) > z_1(\theta_b)$ if and only if

$w(\theta_a) > w(\theta_b)$, it follows that $z_2(\theta_a) < z_2(\theta_b)$. Then from (9.5), both

$$(9.6) \qquad \frac{W_1^1(\theta_a)}{W_2^1(\theta_a)} > \frac{W_1^1(\theta_b)}{W_2^1(\theta_b)}$$

and

$$(9.7) \qquad \frac{W_1^2(\theta_a)}{W_1^2(\theta_a)} < \frac{W_1^2(\theta_b)}{W_2^2(\theta_b)}$$

must hold. From the two tangency conditions (9.3) and (9.4),

$$(9.8) \qquad w(\theta_k)\,\frac{W_1^1(\theta_k)}{W_2^1(\theta_k)} = (1-w(\theta_k))\,\frac{W_1^2(\theta_k)}{W_2^2(\theta_k)}$$

for k= a,b. If (9.6)-(9.8) hold simultaneously, then $w(\theta_a) < w(\theta_b)$. This contradicts $z_1(\theta_a) > z_1(\theta_b)$ and thus completes the proof.

Lemma 11. Under the increasing marginal impatience assumption (8.14), $0 < \theta < \theta^*$ implies $z_1(\theta) > \phi^1(x(\theta))$ and $\theta^* < \theta < 1$ implies $z_1(\theta) < \phi^1(x(\theta))$.

Proof. If the assertion is false, the situation depicted in Figure 8 must obtain. That is, for some (and hence all) $\theta \in (0,\theta^*)$, it must be the case that $z_1(\theta) < \phi^1(x(\theta))$. From condition (8.14), $W_2^1(x,\phi^1(x))$ is declining as x increases, so that since $\theta < \theta^*$

$$(9.9) \qquad W_2^1(x(\theta),\phi^1(x(\theta))) > W_2^1(x^*,(\phi^1(x^*)).$$

By concavity, W_2^1 is a decreasing function of its second argument so that $z_1(\theta) < \phi^1(x(\theta))$ also implies

(9.10) $$W_2^1(x(\theta),z_1(\theta)) > W_2^1(x(\theta),\phi^1(x(\theta)) .$$

Combining (9.9) and (9.10) gives

(9.11) $$W_2^1(x(\theta),z_1(\theta)) > W_2^1(x^*,\phi^1(x^*)) .$$

Since agent 2's situation is entirely symmetric, the same reasoning yields:

(9.12) $$W_2^2(\bar{x}-x(\theta),z_2(\theta)) < W_2^2(\bar{x}-x^*,\phi^2(\bar{x}-x^*)) .$$

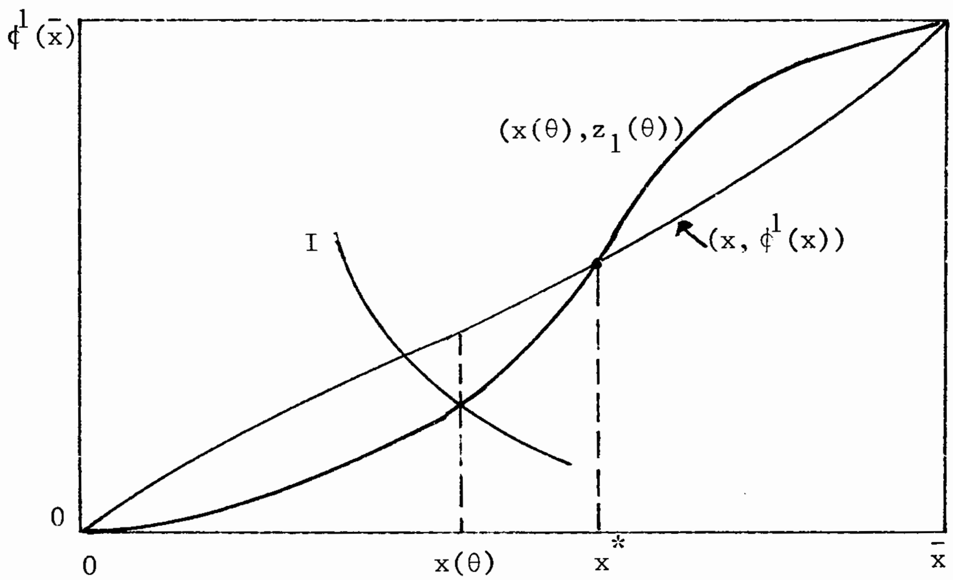

Figure 8

The tangency condition (9.4) holds at $(x(\theta),z_1(\theta)),(\bar{x}-x(\theta),z_2(\theta))$ and at $(x^*,\phi^1(x^*)),(\bar{x}-x^*,\phi^2(\bar{x}-x^*))$, so that from (9.4), (9.10) and (9.11) it follows that

$$\frac{1-\theta}{1-w(\theta)} W_2^2(\theta) = \frac{\theta}{w(\theta)} W_2^1(\theta)$$

$$> \frac{\theta}{w(\theta)} W_2^i(\theta^*) \qquad (i=1,2)$$

$$> \frac{\theta}{w(\theta)} W_2^2(\theta) .$$

This in turn implies

(9.13) $$w(\theta) > \theta .$$

To see that (9.13) is a contradiction, refer again to Figure 8. The indifference curve I of $W^1(x,z)$ that passes through $(x(\theta),z_1(\theta))$ intersects the curve $(x,\phi^1(x))$ at some point $(\hat{x},\hat{z})$ with $\hat{x} < x(\theta)$ and $\hat{z} > z_1(\theta)$. By the definition of the curve $(x,\phi^1(x))$, it follows that $\hat{z} = W^1(\hat{x},\hat{z})$, which in turn must equal $W^1(x(\theta),z_1(\theta))$. Hence $z_1(\theta) < W^1(x(\theta),z_1(\theta))$. This conflicts with (9.13), and so completes the proof.

We may now sum up in:

<u>Theorem 5</u>. For the two-agent economy under study, described by W^1, W^2 and $\bar{x}$, restricted by the non-inferiority assumption (9.5) and the increasing marginal impatience assumption (8.14), the solution $\{\theta_t\}$ to the difference equation (9.1) converges monotonically to θ^* for all initial $\theta_0 \in (0,1)$.

<u>Proof</u>. The accuracy of the salient features of Figure 6 have been verified in Lemmas 10 and 11. For a point $x(\theta)$ to the left of θ^* then, one may reason along the indifference curve I in Figure 6 to prove that $z_1(\theta) > W^1(x(\theta),z_1(\theta))$, and hence that $w(\theta) > \theta$, in exactly the manner used to conclude the proof of Lemma 11. Since the situation is exactly reversed for $x(\theta) > x^*$, the proof is complete.

It is instructive to compare the global stability result in Theorem 5 to the comparable result in the Cass [14] - Koopmans [20] one-consumer, one capital good economy. Both systems involve a single state variable only, and in both cases the proof of global stability exploits this one-dimensional character in essential ways, with arguments that have no obvious counterparts in higher dimensional systems. In the Cass-Koopmans case, the argument rests entirely on diminishing returns to the one state variable in the system. In the present model, the same role is played by the assumption of increasing marginal impatience, a kind of diminishing returns to the accumulation of individual wealth, as wealth increases.

In the Cass-Koopmans model, diminishing returns occur in two, mutually reinforcing places: the strict concavity of current period preferences and the concavity of the production function. It is fairly evident, then, that conditions found sufficient for global stability in these models are stronger than necessary, or that, for example, stability might occur under increasing returns in production if preferences were concave enough to offset this effect. Similarly, in the present system, both agents are assumed to have increasing marginal impatience. It must surely be the case that one could obtain convergence to an interior stationary point if one consumer's preferences failed to exhibit this property, provided the other's had it in a "strong enough" way to be offsetting.

10. A Two-Agent Exchange Economy: Equilibria

For the specific economy analyzed in the preceding section, we have seen that for each utility weight $\theta_o \in [0,1]$ assigned to agent 1 there is exactly one Pareto-optimal allocation $\{x_t(\theta_o)\}$, say, of consumption to agent 1, and the behavior through time of these allocations has been fully characterized under the restrictions on preferences (8.14) and (9.5). In this section, these allocations will be reinterpreted as "perfect foresight" or (in this context, equivalently) "rational expectations" equilibria.

As observed in Section 6, it follows from Theorem 2 of [16] and Theorem 1 of [24] that for each of these allocations $\{x_t(\theta_o)\}$ there is a price sequence $\{p_t(\theta_o)\}$ with $\Sigma_t p_t(\theta_o) < \infty$ such that $\{x_t(\theta_o)\}$ is cost-minimizing at these prices for agent 1 and $\{\bar{x} - x_t(\theta_o)\}$ is cost-minimizing for agent 2. In particular, then, for any $t \geq 0$, $(x_t(\theta_o), x_{t+1}(\theta_o))$ solves:

$$(10.1) \qquad \min_{x_t, x_{t+1}} \; p_t(\theta_o)x_t + p_{t+1}(\theta_o)x_{t+1}$$

subject to

$$(10.2) \qquad W^1[x_t, W^1(x_{t+1}, u^1({}_{t+2}x(\theta_o)))] \geq u^1({}_t x(\theta_o)) \ ,$$

where, as in Section 3, u^1 denotes the preference function over infinite sequences induced by W^1 and where ${}_t x(\theta_o) = (x_t(\theta_o), x_{t+1}(\theta_o), \ldots)$.

If $\theta_o \in (0,1)$, then $0 < x_t(\theta_o) < \bar{x}$, for all t, as in the preceding section. Since W^1 is strictly increasing in both arguments, it follows that $p_t(\theta_o) > 0$ for all t for each $\theta_o \in (0,1)$. If both consumers are initially positively endowed, then, the conditions of the Remark on p. 591 of [16] are satisfied and $\{x_t(\theta_o), p_t(\theta_o)\}$ is in fact a competitive equilibrium.

These equilibrium prices are readily calculated from the first-order conditions for the problem (10.1). In terms of the optimal policy functions $x(\theta)$, $z_1(\theta)$, and $w(\theta)$ used in Section 9, define $q(\theta)\colon [0,1] \to R_+$ by:

$$(10.3) \qquad q(\theta) = \frac{W_2^1\,[x(\theta), z_1(\theta)]\; W_2^1[x(w(\theta)), z_1(w(\theta))]}{W_1^1[x(\theta), z_1(\theta)]} \ .$$

Then given the normalization $p_o(\theta_o) = 1$, the equilibrium prices $p_t(\theta_o)$ are uniquely given by the difference equation (9.1), the initial value θ_o, and the difference equation:

$$(10.4) \qquad p_{t+1}(\theta_o) = q(\theta_t)\, p_t(\theta_o) \;.$$

Obviously, this construction could as well have been based on the marginal conditions obtaining for agent 2.

One may think of these equilibrium prices as being established at time 0, in a single, grand clearing of a market for infinite sequences of consumptions. Alternatively, one may think of $q(\theta)$ as the spot price of a one-period, goods-denominated bond, established in an infinite sequence of temporary equilibria in which agents have rational expectations. The two interpretations are interchangeable, though they would cease to be if one were to try to supplement them with a tatonnement-type stability theory.

Given an initial θ and the price-quantity behavior described in (9.1), (9.2), (10.3) and (10.4), the relative wealth of agent 1 is given by

$$(10.5) \qquad \alpha(\theta) = [\bar{x}\, \Sigma_t p_t(\theta))]^{-1}\, \Sigma_t p_t(\theta) x_t(\theta) \quad ,$$

and of agent 2, $1 - \alpha(\theta)$. This function α: $[0,1] \to [0,1]$ is, being composed of continuous functions, continuous. By [7], Theorems 1 and 3, it is onto, an observation that also follows, in this more specific setting, from continuity and the facts $\alpha(0) = 0$ and $\alpha(1) = 1$. It need not, however, be monotonic. A possible graph of α is Figure 9.

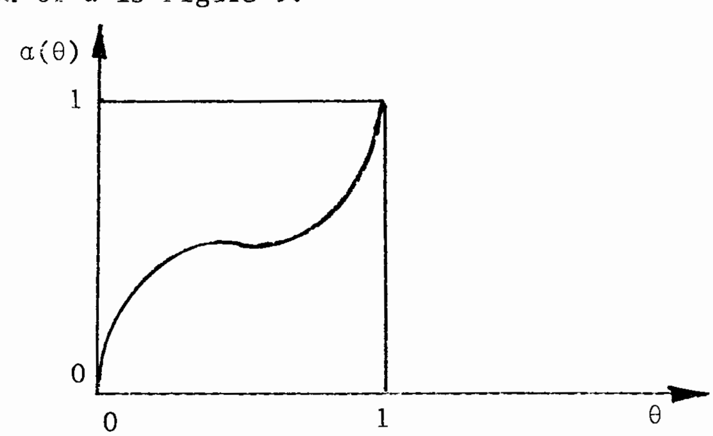

Figure 9

To each utility weight θ to agent 1, then, there is a unique wealth share $\alpha(\theta)$ under which he will obtain this utility in a perfect foresight equilibrium, and a unique consumption path that will deliver him this utility. For a given wealth level α, however, there may be several equilibrium allocations with different associated paths of equilibrium prices.

As remarked in Section 6, it remains dubious whether one wishes to view this as a method for constructing equilibria. Certainly, one could tabulate the function $\alpha(\theta)$ to any degree of fineness, read off the value(s) of θ corresponding to a particular α, and obtain approximations as close as desired. In the present, one-dimensional context this would be entirely practical method (though as is the case with most algorithms, we imagine examples could be devised to embarrass it badly). In more dimensions, such a virtually complete enumeration of possibilities would rapidly become impractical.

11. Concluding Comments

Optimum growth theory is useful in qualitatively characterizing simple systems and in providing constructive methods for calculating solutions to more complex ones because it is so arranged as to produce solutions taking the form of a system of autonomous difference equations. This usefulness is, for some purposes, enhanced because of the intimate connections between optimal and competitive equilibrium allocations, so that theories constructed for normative purposes can turn out to be useful for positive purposes, and conversely.

On the other side, to attain this usefulness, growth theory has utilized many "assumptions of convenience" that preclude its applicability to interesting and easily imaginable general equilibrium systems. As we read

Koopmans [19], that paper is directed at the question: How far can the assumptions of convenience of growth theory be relaxed, without losing the convenience? This paper has been an effort to push a step further, to the study of economies with heterogeneous agents--economies that do not seem analyzable in an interesting way under the limits imposed by the assumption of time-additive preferences. This inquiry has surely added to the confirmation provided by [18], [1], [9], [23] and others that Koopmans' instincts were accurate: time-additivity is neither a desirable nor an analytically necessary property to impose on preferences. The formal structure of the dynamic systems with many agents discussed in earlier sections is in most essentials identical to that of one-agent systems. This observation does not make the difficult study of such systems any easier, but we hope it may make the returns from success in such studies more evident.

The hypothesis of increasing marginal impatience, illustrated in Figure 3, appears to be an essential component that any theory within the class considered in this paper must possess if it is to generate dynamics under which wealth distributions converge to determinate, stationary equilibria in which all agents have positive wealth and consumption levels. Although we pay lip service to the idea that our theories should have content, its emergence in fact tends to be unsettling. It remains to be seen whether this addition to the list of ways in which diminishing returns is required to produce equilibria that remain away from corners will, or should, be accepted as being as "natural" as its predecessors.

NOTES

[1]Actually, [19] and [21] consider aggregator functions of the slightly more specialized form $u(x) = W(v(x_o), u(_1x))$ where $v: R_+^m \to R$ and $W: R_+ \times R \to R$.

[2]A broader class of aggregator functions than that used here could evidently be obtained by considering any function $W: R_+^m \times R \to R$ with the property that, after a monotone transformation of utilities, the transformed function obeys W1-W5. That is, let $H: R \to R$ be continuous and strictly increasing and, given $W: R_+^m \times R \to R$ define $\tilde{W}_H: R_+^m \times R \to R$ by

$$\tilde{W}_H(x,z) = H[W(x,H^{-1}(z))] .$$

Then our theory applies to any aggregator function W such that $\tilde{W}_H$ satisfies W1-W5, for some continuous increasing H.

[3]This terminology is from Hildenbrand [17], Part 1.

[4]See [12], [11], [14], [27].

[5]An alternative condition for the uniqueness of stationary points in one-agent systems, which is easier to verify in some applications, is the "non-vanishing Jacobian" condition of Brock [10]. See also [5].

[6]An evidently closely related condition was used by Uzawa [27], and in many subsequent applications of his continuous-time formulation.

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