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A STOCHASTIC CALCULUS MODEL OF CONTINUOUS TRADING: RETURN PROCESSES AND INVESTMENT PLANS

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Abstract. If the return process for a security is modeled as a semimartingale, then the price process is modeled as the exponential of the return process in the semimartingale sense. This relationship is examined in some detail, both for discrete-time as well as very general continuous-time stochastic process models. In addition, the notion of investment plans (which specify the relative portfolio proportions and thereby the return process for the portfolio) as models of investor decision making is introduced and compared with trading strategy models (which specify the number of shares of each security that are owned). The two models of investor decision making are shown to be equivalent in the case of discrete-time security processes, but an example is given of a continuous-time trading strategy for which there does not exist an equivalent investment plan.

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1. Introduction

This is the second sequel to the paper Harrison and Pliska [2], which presented a very general and comprehensive stochastic model of a frictionless security market with continuous trading. The first sequel Harrison and Pliska [3] dealt with complete markets. This one deals with several closely related matters involving return processes and investment plans.

Security prices are modeled in the literature as either price processes or return processes. Indeed, it is common to see both concepts used simultaneously, for one frequently sees authors implicitly using the return process when they write dS/S, where S is the price process. The paper [2] primarily dealt with price processes, although it did briefly introduce return processes. A primary objective of this paper is to expand on the role of return processes by explaining their relationship to price processes in the general, continuous setting. This is done partly by making an analogy with the discrete time, finite security market model described in [2, section 2]. Discrete time return processes were neither described nor analyzed in [2], so this is another topic covered here.

Just as there are two kinds of models of securities, there are also two kinds of models describing how an investor implements his decisions. The approach used exclusively in [2] involved trading strategies, which specify the number of units of each security owned at each point in time.

Garman and Ohlson [1], for example, is a recent reference using the trading strategy model.

An alternative model of investor decision making is the investment plan, which specifies the relative portfolio proportions, i.e., how the investor's wealth is divided among the various securities. Lee, Rao, and Auchmuty [5], for example, recently used this approach for their particular situation. A second primary objective of this paper is to describe investment plans in very general terms, both for the finite, discrete time model of [2, section 2] as well as the continuous trading model of [2, section 3]. It will be seen that investment plans are closely related to return processes. Moreover, it will be shown that the trading strategy and investment plan models of investor decision making are equivalent in the finite theory but not necessarily in the case of continuous trading: some contingent claims are attainable with a trading strategy but not with any investment plans.

2. Return Processes

After briefly reviewing the finite theory of [2, section 2], where time is discrete and the sample space is finite, the notion of return processes for this finite theory will be introduced. Attention will then turn to the relationship between continuous time price and return processes.

The probability space $(\Omega,\mathcal{J},P)$ is specified and fixed, and the sample space has a finite number of elements. A time horizon $T<\infty$ is specified, as is a filtration $\mathbb{F}=\{\mathcal{J}_0,\mathcal{J}_1,\ldots,\mathcal{J}_T\}$ that satisfies $\mathcal{J}_0=\{\emptyset,\Omega\}$ and $\mathcal{J}_T=\mathcal{J}$. There is also a $K+1$ dimensional stochastic process $S=\{S_t;\ t=0,1,\ldots,T\}$ with component processes $S^0,S^1,\ldots,S^K$. Here S is called the price process, because S_t^k is interpreted as the price at time t of security k. It is required that each component process S^k be strictly positive and adapted to $\mathbb{F}$ (which means S_t is $\mathcal{J}_t$ measurable for all t).

For any scalar-valued stochastic process $X=\{X_t;\ t=0,1,\ldots,T\}$ on $(\Omega,\mathcal{J},P)$, let $\Delta X=\{\Delta X_t;\ t=1,\ldots,T\}$ denote a second, corresponding process, called the difference process, defined by setting $\Delta X_t=X_t-X_{t-1}$ for $t=1,\ldots,T$. Note that $X_t=X_0+\Delta X_1+\ldots+\Delta X_t$ for $t=1,\ldots,T$. With this notation it is natural to call $\Delta S_t^k/S_{t-1}^k$ the rate of return earned by security k during period t. One therefore sets $R_0^k=0$ and

$$(1)\qquad \Delta R_t^k=\Delta S_t^k/S_{t-1}^k \quad \text{for} \quad t=1,\ldots,T.$$

and calls R^k the return process corresponding to price process component S^k. It may seem unnatural to call R^k rather than ΔR^k the return process, but the advantages of this system will be seen later when dealing with the continuous-time model. As usual, set $R=(R^0,R^1,\ldots,R^k)$.

Note that upon rearranging (1) and using it in an iterative manner one obtains

$$(2) \qquad S_t^k = S_0^k + \sum_{s=1}^{t} S_{s-1}^k \Delta R_s^k \quad \text{for} \quad t = 1,\ldots,T.$$

Thus the price process can be recovered from S_0 and the return process. To be more specific about this, as well as to introduce a concept that will arise in the continuous-time setting, it is convenient to present some additional notation. For any stochastic process $X = \{X_t;\ t = 0,1,\ldots,T\}$, let $\mathcal{E}(X) = \{\mathcal{E}_t(X);\ t = 0,1,\ldots,T\}$ denote a second corresponding process, called the exponential process, defined by setting $\mathcal{E}_0(X) = 1$ and

$$(3) \qquad \mathcal{E}_t(X) = \prod_{s=1}^{t} (1 + \Delta X_s) \quad \text{for} \quad t = 1,\ldots,T.$$

Thus if R^k is the return process corresponding to the price process component S^k, then

$$(4) \qquad S^k = S_0^k \mathcal{E}(R^k) \quad \text{for} \quad k = 0,1,\ldots,K.$$

In other words, the expression in (4) is a solution to equation (2).

An important issue is knowing what properties of a return process give rise via (2) or (4) to a price process. In other words, when is (4) an adapted, strictly positive process? Let $\mathcal{R}^+$ be the set of adapted, scalar-valued processes X such that $X_0 = 0$ and $1 + \Delta X_t > 0$ for all $t = 1,\ldots,T$. If S^k is a price process component, then its corresponding return process $R^k \in \mathcal{R}^+$. Conversely, if $X \in \mathcal{R}^+$, then $\mathcal{E}(X)$ is adapted and strictly positive.

We are now ready to turn to the case of continuous trading and the model detailed in Section 3 of [2]. The basic set-up is the same as in the finite theory, only Ω is not finite and the filtration $\mathbb{F}$ and all processes are defined for all $t \in [0,T]$. In particular, each component S^k of the price process is adapted and strictly positive. Moreover, by additional considerations detailed in [2], S^k is a semimartingale.

Now if the price process component S^k were a pure jump process, then with $\Delta S_t^k = S_t^k - S_{t-}^k$ one would have $\Delta R_t^k = \Delta S_t^k / S_{t-}^k$ as the continuous-time

counterpart of (1). But wanting to consider more general stochastic processes, it is necessary to define the return process R^k in terms of a stochastic integral, namely

$$(5) \qquad R_t^k = \int_0^t (1/S_{u-}^t) dS_u^k, \qquad 0 \le t \le T.$$

This is equation (4.7) in [2], and it reduces to $\Delta R_t^k = \Delta S_t^k / S_{t-}^k$ in the special case where S^k is a pure jump process.

Equation (5) is equivalent to the statement that $dS^k = S_-^k dR^k$, and this, in turn, is the same as

$$(6) \qquad S_t^k = S_0^k + \int_0^t S_{u-}^k dR_u^k, \qquad 0 \le t \le T,$$

which is the counterpart of (2). This is also the same as equation (4.1) of [2], although a minus sign there was inadvertantly omitted.

Notice that although equation (2) can be used in the finite case to derive the price process in terms of the return process, equation (6) can't be used directly for the same purpose. As explained in [2], however, if the semimartingale R^k is specified, then (6) always has a unique solution given by

$$(7) \qquad S^k = S_0^k \mathcal{E}(R^k),$$

where $\mathcal{E}(R^k)$ is now the exponential of R^k in the semimartingale sense. Clearly (7) is the same as (4), and the process $\mathcal{E}(R^k)$ is defined in a manner analogous to (3), as detailed in [2, eqn. (4.3)].

If S^k is a component of a price process, then its return process $R^k \in \mathcal{R}^+$, where now in the continuous case $\mathcal{R}^+$ is the set of semimartingales X such that $X_0 = 0$ and $1 + \Delta X > 0$ (where $\Delta X \equiv X - X_-$). Conversely, if $X \in \mathcal{R}^+$, then $\mathcal{E}(X)$ is a strictly positive semimartingale.

A multidimensional diffusion model, a point process model, and other examples of return processes are provided in [2].

3. Investment Plans in the Finite Theory

Trading strategies in the finite theory of [2, Section 2] are defined in terms of predictable processes. Briefly, a trading strategy is a predictable vector process $\phi = \{\phi_t;\ t = 1,\ldots,T\}$ with components $\phi^0,\phi^1,\ldots,\phi^K$. Predictable means ϕ_t is $\mathcal{F}_{t-1}$ measurable for all t. Interpret ϕ_t^k as the quantity of security k held by the investor between times t - 1 and t.

A trading strategy ϕ is self-financing if

$$\phi_t S_t = \phi_{t+1} S_t, \qquad t = 1,\ldots,T-1,$$

where $\phi_t S_t$ denotes the scalar product of the vectors ϕ_t and S_t. This means that no funds are added to or withdrawn from the value of the portfolio at any of the times $t = 1,\ldots,T-1$.

Corresponding to each trading strategy ϕ is a process $V(\phi)$ defined by

$$V_t(\phi) = \begin{cases} \phi_t S_t, & t = 1,\ldots,T \\ \phi_1 S_0, & t = 0. \end{cases}$$

We call $V(\phi)$ the value process for ϕ, since $V_t(\phi)$ represents the market value of the portfolio held just before time t transactions.

A trading strategy ϕ is called admissible if it is self-financing and $V(\phi) \geq 0$. The requirement that $V(\phi) \geq 0$ prohibits short sales that can put the investor into a position of debt. Let Φ denote the set of all admissible trading strategies.

The purpose of this section is to describe for the finite theory an alternative, equivalent scheme for modeling how the investor makes his investment decisions. The next section will then describe the analogous scheme for general, continuous trading models.

Let $\mathcal{P}$ be the set of adapted, scalar-valued processes X such that $X_0 = 0$ and $1 + \Delta X_t \geq 0$ for all $t = 1,\ldots,T$, and let $\mathcal{G}(X)$ be as in (3). Define an

investment plan as a predictable $K+1$ dimensional process $\theta = \{\theta_t;\ t = 1,\ldots,T\}$ whose components $\theta^0, \theta^1, \ldots, \theta^K$ satisfy

$$\theta_t^0 + \theta_t^1 + \ldots + \theta_t^K = 1, \quad t = 1,\ldots,T, \tag{8}$$

and for which $\rho(\theta) \in \mathcal{R}$, where the stochastic process $\rho(\theta)$ is defined in terms of the return process R by $\rho_0(\theta) = 0$ and

$$\rho_t(\theta) = \sum_{s=1}^{t} \theta_s \Delta R_s = \sum_{s=1}^{t} \sum_{k=0}^{K} \theta_s^k \Delta R_s^k, \quad t = 1,\ldots,T. \tag{9}$$

Let Θ be the set of all such investment plans. As will shortly be seen, θ_t^k should be interpreted as the fraction of his wealth that an investor puts into security k to be carried forward from time $t-1$. Hence $\Delta\rho_t(\theta)$ represents the overall rate of return that he earns on his portfolio during the period between $t-1$ and t, although this interpretation must be taken with a grain of salt, since individual components θ_t^k can be negative or greater than one.

Call $\rho(\theta)$ the return process for plan θ, and define a corresponding value process

$$U(\theta) = \mathcal{S}(\rho(\theta)).$$

Interpret $U_t(\theta)$ as the wealth at time t of an investor who starts with one dollar and follows the investment plan θ. The restriction $\rho(\theta) \in \mathcal{R}$ means that in each period the investor can do no worse than lose all his wealth.

For each admissible trading strategy $\phi \in \Phi$, define the stopping time

$$\tau(\phi) = \begin{cases} \inf\{t \le T:\ V_t(\phi) = 0\}, & \text{if } V_t(\phi) = 0 \text{ for some } t, \\ T, & \text{otherwise.} \end{cases}$$

The following Proposition establishes the correspondence between Φ and Θ. Its hypothesis that the model be viable means there are no arbitrage opportunities; this condition was extensively examined in [2].

Proposition. Let the model be viable. Suppose $\phi \in \Phi$, and let $v = V_0(\phi)$ and

$$\theta_t^k = \begin{cases} \phi_t^k S_{t-1}^k / V_{t-1}(\phi) & \text{if } 1 \le t \le \tau(\phi) \\ 1_{(k=0)} & \text{if } \tau(\phi) < t \le T \end{cases}$$

for $k = 0,1,\ldots,K$. Then $\theta \in \Theta$ and $V(\phi) = v\,U(\theta)$. Conversely, suppose $\theta \in \Theta$ and the initial wealth $v \ge 0$, and let

$$\phi_t^k = v\theta_t^k U_{t-1}(\theta)/S_{t-1}^k \quad \text{for} \quad t = 1,\ldots,T$$

and $k = 0,1,\ldots,K$. Then $\phi \in \Phi$ and $V(\phi) = v\,U(\theta)$.

Proof. For the first part, it is clear that θ is predictable. By the definition of $V(\phi)$ and the fact that ϕ is self-financing, it follows that $\theta_t^0 + \theta_t^1 + \ldots + \theta_t^K = 1$ for all t. Moreover, for $t \le \tau(\phi)$,

$$\begin{aligned} 1 + \Delta\rho_t(\theta) &= 1 + \theta_t\,\Delta R_t = 1 + \sum_k \phi_t^k S_{t-1}^k \Delta R_t^k / V_{t-1}(\phi) \\ &= 1 + \sum_k \phi_t^k \Delta S_t^k / V_{t-1}(\phi) = 1 + \sum_k \phi_t^k (S_t^k - S_{t-1}^k)/V_{t-1}(\phi) \\ &= 1 + (V_t(\phi) - V_{t-1}(\phi))/V_{t-1}(\phi) = V_t(\phi)/V_{t-1}(\phi), \end{aligned}$$

whereas for $\tau(\phi) < t \le T$,

$$1 + \Delta\rho_t(\theta) = 1 + \theta_t \Delta R_t = 1 + \Delta R_t^0.$$

Hence $\rho(\theta) \in \mathcal{R}$, in which case $\theta \in \Theta$. We show $V(\phi) = v\,U(\theta)$ by induction. If $V_{t-1}(\phi) = v\,U_{t-1}(\theta)$ and $t \le \tau(\phi)$, then by (3) and the above equation

$$v\,U_t(\theta) = V_{t-1}(\phi)(1 + \Delta\rho_t(\theta)) = V_t(\phi).$$

Thus $\tau(\phi) < T$ and (3) imply $U_t(\theta) = 0$ for all $t = \tau(\phi), \tau(\phi)+1, \ldots, T$. But viability and [2, Proposition (2.8)] imply $V_t(\phi) = 0$ for the same t, so $V(\phi) = v\,U(\theta)$.

For the second part of the Proposition, it is clear that ϕ is predictable. Since $\theta_t^0 + \theta_t^1 + \ldots + \theta_t^K = 1$, one has

$$V_t(\phi) = \sum_k \phi_t^k S_t^k = v\, U_{t-1}(\theta) \sum_k \theta_t^k S_t^k / S_{t-1}^k$$

$$= v\, U_{t-1}(\theta) \sum_k \theta_t^k (1 + \Delta R_t^k)$$

$$= v\, U_{t-1}(\theta)(1 + \Delta \rho_t(\theta)) = v\, U_t(\theta).$$

Hence $V(\phi) \geq 0$, since $\rho(\theta) \in \mathcal{R}$ implies $U(\theta) \geq 0$. Finally,

$$\phi_{t+1} S_t = v\, U_t(\theta) \sum_k S_t^k \theta_{t+1}^k / S_t^k = V_t(\phi),$$

so ϕ is self-financing, $\phi \in \Phi$, and this proof is completed.

The first part of the Proposition justifies the interpretation of investment plans given earlier. In brief, θ specifies the investor's relative portfolio proportions in dollar terms, with the choice of θ_t for $t > \tau$, i.e., after all is lost, being arbitrary. The second part of the Proposition shows how an admissible trading strategy ϕ can be recovered from an investment plan when the initial investment $v = V_0(\phi)$ is specified. Observe that the restriction $\sum \theta^k = 1$ in the definition of Θ corresponds to the self-financing condition, while the requirement $\rho(\theta) \in \mathcal{R}$ corresponds to $V(\phi) \geq 0$. In summary, in the finite theory, investment plans and trading strategies are equivalent models of investor decision making.

4. Investment Plans in the Continuous Theory

Trading strategies in the continuous theory (see Section 3 of [2]) are defined in terms of the discounted price process $Z = (Z^1, \ldots, Z^K)$, where $Z^k = \beta S^k$ for $k = 1, \ldots, K$, $\beta \equiv 1/S^0$, and, without loss of generality, $\beta_0 = 1$. It is assumed the model is viable, i.e., there exists a reference measure

P^*, which is a probability measure on $(\Omega,\mathcal{F})$ equivalent to P under which Z is a martingale. For convenience also assume (as was done throughout [2]) that S^0 is continuous and VF (of finite variation).

Let L(Z) denote the set of all vector valued, predictable processes that are integrable with respect to the semimartingale Z (see Jacod (4, p. 52) for details about L(Z)). An <u>admissible trading strategy</u> is any vector valued, predictable stochastic process $\phi = (\phi^0,\phi^1,\ldots,\phi^K) = \{\phi_t;\ 0 \le t \le T\}$ such that

(10a) $\quad (\phi^1,\ldots,\phi^K) \in L(Z)$

(10b) $\quad V^*(\phi) \ge 0$, where $V^*(\phi) = \beta\,\phi\,S = \beta \sum_{k=0}^{K} \phi^k S^k$,

(10c) $\quad V^*(\phi) = V_0^*(\phi) + G^*(\phi)$, where

$$G^*(\phi) = \int \phi\, d\, Z = \sum_{k=1}^{K} \int \phi^k\, d\, Z^k, \text{ and}$$

(10d) $\quad V^*(\phi)$ is a martingale under P^*.

Here ϕ_t^k represents the number of shares or units of security k held by the investor at time t, $V^*(\phi)$, the <u>discounted value process</u>, represents the discounted value of the portfolio, and $G^*(\phi)$, the <u>discounted gains process</u>, represents the discounted net profit or loss earned by the investments. Thus (ii) says admissible trading strategies cannot permit the value of the portfolio to become negative, (iii) says that all changes in the value of the portfolio are due to the investment rather than due to infusion or withdrawal of funds, and (iv) serves to rule out certain foolish strategies that throw away money, as discussed in detail in [2].

The purpose of this section is to describe an alternative model of how the investor makes decisions, a model based on relative portfolio proportions, as in the finite theory. For the continuous theory define an

investment plan to be a $K+1$ dimensional predictable process $\theta = \{\theta_t;\ 0 \le t \le T\}$ whose components $\theta^0, \theta^1, \ldots, \theta^K$ satisfy

$$\theta_t^0 + \theta_t^1 + \ldots + \theta_t^K = 1, \qquad 0 \le t \le T; \tag{11}$$

this is, of course, the same as condition (8). In addition, an investment plan θ must satisfy $\rho(\theta) \in \mathcal{R}$, just as in the finite theory, where the stochastic process $\rho(\theta)$ is now defined, analogously to (9), by

$$\rho_t(\theta) = \int_0^t \theta_s \, dR_s = \sum_{k=0}^{K} \int_0^t \theta_s^k \, dR_s^k, \qquad 0 \le t \le T, \tag{12}$$

and $\mathcal{R}$ is the set of semimartingales X such that $X_0 = 0$ and $1 + \Delta X \ge 0$. Thus $\rho(\theta)$ should be interpreted as the return process for the overall portfolio, and the components of an investment plan θ should be interpreted as relative portfolio proportions. These "fractions" may exceed one or be negative, just as in the finite theory.

In view of this definition of investment plans and the Proposition stated in the preceding section for the finite theory, it is both natural and important to determine whether the investment plan model of investor decision making is equivalent to the trading strategy model (10). To analyze this issue, it will first be shown that to each initial market value $v \ge 0$ and each investment plan θ there exists a corresponding predictable trading strategy ϕ satisfying the nonnegativity condition (10b), the self-financing condition (10c), and

$$V^*(\phi) = v\,\beta\,\mathcal{E}(\rho(\theta)), \tag{13}$$

where $\mathcal{E}(\rho(\theta))$ is the semimartingale exponential of $\rho(\theta)$, as in [2 , eqn. (4.3)].

To see this, simply take

$$\phi_t^k = v\,\theta_t^k \mathcal{E}_{t-}(\rho(\theta))/S_{t-}^k \tag{14}$$

for $0 \le t \le T$ and $k = 0,\ldots,K$. Clearly ϕ is predictable. Since

$$\phi_t^k S_t^k = \phi_t^k S_{t-}^k (1 + \Delta R_t^k) = v\, \theta_t^k \mathcal{E}_{t-}(\rho(\theta))(1 + \Delta R_t^k),$$

it follows that

$$V_t^*(\phi) = v\, \beta\, \mathcal{E}_{t-}(\rho(\theta))(1 + \Delta \rho_t(\theta)) = v\, \beta\, \mathcal{E}_t(\rho(\theta)),$$

which is (13). Since $\rho(\theta) \in \mathcal{R}$, one has (10b). To show (10c), by (13) and the meaning of semimartingale exponentials (see Section 4 of [2]), one can write

$$V_t^*(\phi) = V_0^*(\phi) + V_0^*(\phi) \int_0^t \beta_{s-}\, \mathcal{E}_{s-}(\rho(\theta)) d(\rho_s(\theta) - R_s^0),$$

so

$$\begin{aligned} V_t^*(\phi) &= V_0^*(\phi) + v \sum_{k=0}^K \int_0^t \beta_{s-} \mathcal{E}_{s-}(\rho(\theta))\, \theta_s^k\, d(R_s^k - R_s^0) \\ &= V_0^*(\phi) + \sum_{k=0}^K \int_0^t \phi_s^k \beta_{s-} S_{s-}^k d(R_s^k - R_s^0) \\ &= V_0^*(\phi) + \sum_{k=0}^K \int_0^t \phi_s^k\, dZ_s^k \\ &= V_0^*(\phi) + G_t^*(\phi), \end{aligned}$$

since $R^k - R^0$ is the return process for Z^k by [2, eqn. (4.11)]. Hence ϕ is self-financing, and (10c) holds.

Is the trading strategy ϕ given by (14) admissible? In other words, are (10a) and (10d) also satisfied? Looking at (13) one sees that (10d) is satisfied if and only if the investment plan θ is such that $\beta\, \mathcal{E}(\rho(\theta))$ is a martingale under the reference measure P^*. This martingale property is not automatically satisfied, so one needs to add this requirement about investment plans in order for (10d) to hold.

What about the integrability condition (10a)? This is a technical requirement, added to make sure the stochastic integral $\int \phi\, dZ$ is well-

defined. Actually, a corresponding requirement about θ has been ignored, for in making the definition (12) of $\rho(\theta)$ one is tacitly assuming that the investment plan θ is integrable with respect to the return process R. One could make a precise statement about the integrability of θ and then show this statement implies (10a), but it is better to table this matter. Getting bogged down with technical complications that only arise with the most unusual, esoteric examples would be a time-consuming digression from this discussion. It is preferable to skip ahead, on the grounds that in the vast majority of practical examples the integrability of θ with respect to R, defined in a natural way, implies (10a), the integrability of the corresponding trading strategy ϕ with respect to Z.

In summary, subject to a technicality that is rarely an issue anyway, each initial wealth v and predictable investment plan θ satisfying (11) with $\rho(\theta) \in \mathcal{R}$ and $\beta\mathcal{S}(\rho(\theta))$ being a martingale under P^* gives rise via (14) to an admissible trading strategy ϕ satisfying (13). Thus one-half of the continuous version of the Proposition is true. What about the converse? To each admissible trading strategy ϕ does there exist a corresponding investment plan θ? The following section is devoted to showing that the answer to this last question is, in general, no.

5. A Counterexample

Consider a continuous model with two securities. The zeroth security S^0 equals the constant one, while

$$S_t^1 = \exp(\sigma W_t - \tfrac{1}{2}\sigma^2 t), \qquad 0 \le t \le T,$$

where W is standard Brownian motion on some probability space $(\Omega,\mathcal{F},P)$ (this is the Black-Scholes model discussed in [2, sec. 1a] with $r = \mu = 0$). Then $S = Z$ is a martingale from the outset, and we can take P itself as the

reference measure P^*.

Consider the trading strategy ϕ defined by

$$\phi_t^k = \begin{cases} 2 & \text{if } k=0, \quad t \le \tau \wedge T, \\ -1 & \text{if } k=1, \quad t \le \tau \wedge T, \\ 0 & \text{otherwise,} \end{cases}$$

where $\tau \equiv \inf\{t: \ S_t^1 = 2\}$. Thus starting with one dollar, the investor buys two units of S^0 and sells short one unit of S^1, maintaining that position until time T or time τ when his wealth is reduced to zero, whichever happens first. It is easy to check that ϕ is an admissible trading strategy and that $V_T(\phi)$, his wealth at time T, is given by

$$V_T(\phi) = (2 - S_T^1)1_{\{\tau > T\}}.$$

Notice that $1 > P\{V_T(\phi) = 0\} > 0$.

Now S, and hence R, are continuous in this model, implying $\rho(\theta)$ is continuous for all investment plans θ. Thus $\rho(\theta) \in \mathcal{R}^+$ for all θ (recall $\mathcal{R}^+$ is the set of semimartingales X with $X_0 = 0$ and $1 + \Delta X > 0$), so by the theory of semimartingale exponentials, $\mathcal{E}(\rho(\theta))$ is strictly positive for all θ. In other words, the value process under any investment plan is strictly positive at all times, so there certainly can be no $v \ge 0$ and investment plan θ such that $v\,\mathcal{E}_T(\rho(\theta)) = V_T(\phi)$.

The moral of this story is that the set of investment plans, as defined in the preceding section, is not rich enough to adequately model all of the decisions the investor may wish to make, i.e., all of the admissible trading strategies ϕ. In particular, and using the terminology of [2], only strictly positive contingent claims are attainable with investment plans in this example. It is important to emphasize that this deficiency of the investment plan model of investor decision making is of a fundamental nature and not

simply due to some technicalities such as whether certain predictable processes are integrable. In order to remedy this situation, one must enlarge the set of legal investment plans θ to allow the possibility that $\rho_t(\theta) \to -\infty$ in a finite amount of time. This would obviously require some very delicate mathematics, and this subject will not be pursued any further here. It is worth pointing out, however, that the issue under discussion has no counterpart when trading takes place at discrete points in time.

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