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Option Pricing When the Underlying Asset is Non-Stored

by

Robert McDonald*

and

Daniel Siegel**

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* School of Management, Boston University

** Kellogg Graduate School of Management, Northwestern University; and Department of Economics, Massachusetts Institute of Technology

ABSTRACT

The Black-Scholes stock option formula can be derived using an "arbitrage" argument or a capital market "equilibrium" argument. However, the derivations need not yield the same option price. If the option is written on an asset which is "non-stored" (i.e., does not earn a rate of return sufficient to induce investors to store it) then the derivations yield different option prices. We show that in a competitive market, such an option will be priced according to the "equilibrium" formula, rather than the "arbitrage" formula. The apparent resulting arbitrage opportunity cannot be acted upon.

"Option Pricing When the Underlying Asset is Non-Stored"

In their classic paper, Black and Scholes [2] present two derivations of the european call option pricing formula. The first, which we will refer to as the "arbitrage derivation," has the implication that if an option is not priced according to the Black-Scholes formula, then there is a sure profit to be made by some combination of either short or long sales of the option and underlying asset. This argument is seemingly independent of considerations relating to capital market equilibrium. The second derivation, which we will call the "equilibrium derivation," evolves from a requirement that the option earn an expected rate of return commensurate with the risk involved in holding the option as an asset. This paper shows that if the underlying asset earns a rate of return which is too low for any investor to willingly store the asset, then the arbitrage derivation gives the wrong option price.

If the asset upon which the option is written earns an expected rate of return sufficient to induce investors to bear the risk in storing the asset, then the arbitrage and equilibrium derivations yield the same option price. If the asset upon which the option is written is "non-stored" (i.e., earns a rate of return lower than that which is necessary to induce investors to bear the risk in storing it), but is still tradeable[1], then the two derivations yield different option prices[2]. If the two derivations yield different option prices, one would (from the very nature of arbitrage) expect the arbitrage price to prevail. We show, to the contrary, that the equilibrium price[3] will prevail, despite the fact that this seemingly induces an arbitrage opportunity. Our argument is that, in cases where the asset underlying the option is non-stored, competition among suppliers of options will ensure that the option is priced according to the equilibrium formula. Furthermore, despite the fact that the option price is different from that implied by the arbitrage derivation, there will be no way for an arbitrageur to make money. The reason is that arbitrage in this case would require holding the option and short-selling the underlying asset. Because the underlying asset is non-stored, however, no one will lend the asset for the purposes of a short-sale, so the arbitrage cannot occur.

In Section I, we make the above argument explicitly. In Section II we use our analysis to discuss the problem of pricing a european call option on a commodity and compare our formula with that of Black [1]. Section III offers a summary and conclusions.

I. ARBITRAGE VS. EQUILIBRIUM PRICING

Consider a tradeable asset with price P, following the Ito process

$$\frac{dP}{P} = \alpha_p dt + \sigma_p dz_p \tag{1}$$

Suppose that the equilibrium expected rate of return necessary to compensate anyone for bearing the risk in holding this asset is $\alpha > \alpha_p$. Consequently, no one will store the asset solely to earn a return.

A. Arbitrage Pricing

Suppose now that an investor decides to write call options on the asset. If the investor stores the asset and sells call options, he can create a riskless position which must earn a risk-free rate of return. Following the Black-Scholes derivation, the call will then sell for

$$V(P_0, X, T) = P_0 N(d_1) - Xe^{-rT} N(d_2) \tag{2}$$

$$d_1 = [\ln(P_0/X) + (r+\sigma_p^2/2)T]/\sigma_p\sqrt{T}$$

$$d_2 = d_1 - \sigma_p\sqrt{T}$$

where X is the exercise price, r is the risk free rate of interest, T is the time to maturity, and N(•) is the standard normal cumulative density function.

B. Equilibrium Pricing

To derive the equilibrium price of the call on our non-stored asset, we must first characterize capital market equilibrium explicitly. For ease of exposition, we will suppose that Merton's [4] Intertemporal Capital Asset Pricing Model (ICAPM) holds, but the argument should be able to encompass other plausible asset pricing models. We now consider an alternative strategy for an investor who decides to write calls on the asset, with the same exercise price and time to maturity as above. Instead of creating a risk-free net position, this investor creates a zero beta net position by holding a (non-dividend paying) security with equilibrium expected rate of return α. Therefore, the security pays the expected rate of return that the non-stored asset would have had to pay in order to induce investors to store it[4]. By planning to sell a fraction $(\alpha-\alpha_p)$ of the shares of the security that he holds at each instant, the investor generates an equity position which has the same expected rate of capital gains as does the non-stored asset, and at the same time pays a continuous proportional dividend at the rate $(\alpha-\alpha_p)$.

More formally, suppose that ρ_{pm} is the instantaneous correlation between the rate of return on the non-stored asset and that of the market portfolio.

The beta of the non-stored asset is then

$$\beta_p = \frac{\rho_{pm}\sigma_p}{\sigma_m} \tag{3}$$

Any security with price S, for which $\rho_{sm}\sigma_s = \rho_{pm}\sigma_p$ will have a total rate of return α, where $\alpha = r + \beta_p(r_m - r)$, r_m is the expected rate of return on the market portfolio, and σ_m is the variance of its return. If the investor holds this security and sells shares at the rate $(\alpha-\alpha_p)dt$, then the stochastic process governing the net equity position, $\hat{S}$, will be

$$\begin{aligned} \frac{d\hat{S}}{\hat{S}} &= (\alpha - (\alpha-\alpha_p))dt + \sigma_s dz_s \\ &= \alpha_p dt + \sigma_s dz_s \end{aligned} \tag{4}$$

If the investor sells one call and holds q shares of this security, the value of his portfolio is

$$H = q\hat{S} - W(P,T) \tag{5}$$

where $W(\cdot,\cdot)$ is the price of the call. Using Ito's lemma, the change in the value of the portfolio over time is

$$\begin{aligned} dH = {} & q d\hat{S} + q(\alpha-\alpha_p)\hat{S}dt - \frac{\partial W}{\partial P}dP \\ & - \frac{\partial W}{\partial t}dt - \frac{1}{2}\sigma_p^2 P^2 \frac{\partial^2 W}{\partial P^2}dt \end{aligned} \tag{6}$$

where we have used $d(q\hat{S}) = \hat{S}dq + qd\hat{S} = \hat{S}q(\alpha-\alpha_p)dt + qd\hat{S}$.

Expanding dS and dP, and rewriting yields

$$dH = (q\hat{S}-P\frac{\partial W}{\partial P})\alpha_p dt + q(\alpha-\alpha_p)\hat{S}dt + q\hat{S}\sigma_s dz_s \tag{7}$$

$$- P\frac{\partial W}{\partial P}\sigma_p dz_p - \frac{\partial W}{\partial t}dt - \frac{1}{2}\sigma_p^2 P^2 \frac{\partial^2 W}{\partial P^2}dt$$

Suppose we continuously adjust q, so that at each point in time $q\hat{S} = P\frac{\partial W}{\partial P}$. Then the only stochastic part of the portfolio's return is

$$P\frac{\partial W}{\partial P}[\sigma_s dz_s - \sigma_p dz_p] \tag{8}$$

Since by the construction of the net equity position with value $\hat{S}$, $\sigma_s\rho_{sm} = \sigma_p\rho_{pm}$. Therefore (8) is uncorrelated with the return on the market portfolio[5]. Consequently, the hedge portfolio has a zero beta. So if the hedge portfolio is a small part of any investor's portfolio, the investor will be satisfied to earn the risk free rate:

$$rHdt = E[dH] = -\frac{\partial W}{\partial t}dt - \frac{1}{2}\sigma_p^2 P^2 \frac{\partial^2 W}{\partial P^2}dt \tag{9}$$

$$+ (\alpha-\alpha_p)P\frac{\partial W}{\partial P}dt$$

and the partial differential equation for the option price becomes

$$\frac{\partial W}{\partial t} = rW - \frac{1}{2}\sigma_p^2 P^2 \frac{\partial^2 W}{\partial P^2} - (r - (\alpha-\alpha_p))P\frac{\partial W}{\partial P} \tag{10}$$

This is the same partial differential equation as that describing the behavior of a call written on a stock which pays proportional dividends at the rate $(\alpha-\alpha_p)$. Since the boundary conditions, at expiration date t^*, for our option and the option on the dividend paying stock are both

$$W(P_t^*,0) = \max[0, P_t^*-X] \qquad (11)$$

the formula for our option is also the same as in the proportional dividend case. That formula is (Smith [5])

$$W(P_0,X,T;\delta) = P_0 e^{-\delta T} N(d_1^*) - Xe^{-rT} N(d_2^*) \qquad (12)$$
$$d_1^* = [\ln(P_0/X) + (r-\delta+\sigma_p^2/2)T]/\sigma_p\sqrt{T}$$
$$d_2^* = d_1^* - \sigma_p\sqrt{T}$$

where $\delta = \alpha - \alpha_p$.

C. Equilibrium in the Market for Options

It is clear from comparing (2) and (12) that $W(P_0,X,T;0) = V(P_0,X,T)$. Furthermore, it is easy to show that $\frac{\partial W}{\partial \delta} < 0$. Therefore $W(P_0,X,T;\delta) < V(P_0,X,T)$ if $\delta > 0$. Thus, the options offered by investors hedging with the security will sell for less than those offered by investors hedging directly with the non-stored asset. No one will buy the more expensive option, so in a competitive market with a large number of sellers, the market option price will be the equilibrium price W. The options written using the equilibrium "technology" will dominate those written using the arbitrage "technology".

D. Absence of Arbitrage Opportunities

We have argued above that the market price for a european call option on a non-stored asset will be less than the Black-Scholes price V. V was derived using an arbitrage argument, however, so it would appear that investors would make arbitrage profits if the call price is W instead of V. Because the market option price, W, is undervalued with respect to the Black-Scholes price, the arbitrage would involve short-selling the underlying asset and buying the option. Suppose investors buy calls at the price W and short-sell the underlying non-stored asset, setting $q = \frac{\partial W}{\partial P}$. We will verify that, if possible, this would be a scheme to make arbitrage profits; we then show that the scheme is impossible to effect.

The value of these investors' portfolio is:

$$\tilde{H} = - \frac{\partial W}{\partial P}P + W = - Xe^{-rT}N(d_2^*) < 0 \tag{13}$$

Because $\tilde{H} < 0$, the arbitrageurs are net borrowers as a result of these transactions. The net cost of borrowing $\tilde{H}$ is

$$\frac{d(\tilde{H})}{\tilde{H}} = \frac{-\frac{\partial W}{\partial t} - \frac{1}{2}\sigma_p^2 P^2 \frac{\partial^2 W}{\partial P^2}}{\frac{\partial W}{\partial P}P - W} < \tag{14}$$

$$\frac{-\frac{\partial W}{\partial t} - \frac{1}{2}\sigma_p^2 P^2 \frac{\partial^2 W}{\partial P^2} + (\alpha - \alpha_p)P\frac{\partial W}{\partial P}}{\frac{\partial W}{\partial P}P - W} = r$$

where the right hand side of the inequality comes from (9).

Thus the optimal strategy involves a negative net position, holding the option and short-selling the underlying asset. The arbitrageurs can in effect borrow risklessly at less than the risk free rate and reinvest the proceeds at the risk free rate.

This arbitrage scheme cannot, however, be undertaken, because no investor will lend the underlying asset for a short sale. By assumption the asset is non-stored, $\alpha > \alpha_p$, so that no investor would be willing to buy the asset, costlessly lend it to the short-seller, and then receive the asset back at some future time. Thus the short-sale cannot be undertaken, and W will stand unchallenged as the market option price.

II. COMMODITIES AS NON-STORED ASSETS

To see how a non-stored, but tradeable asset can arise, consider a commodity, which is currently not being stored, and which has supply and demand functions given by

$$Q_t^s = aP_t^\gamma \qquad \text{(supply)} \qquad (15)$$

$$Q_t^d = b\theta_t P_t^{-\varepsilon} \qquad \text{(demand)} \qquad (16)$$

where

$$\frac{d\theta}{\theta} = \alpha_\theta dt + \sigma_\theta dz_\theta \qquad (17)$$

is the process governing the evolution of the demand uncertainty parameter θ_t. Therefore, demand today is known, has a secular rate of change, but is increasingly uncertain over time.

Equilibrium occurs at each point in time, where $Q_t^s = Q_t^d$. This implies that

$$P_t = A\theta_t^{-\beta} \tag{18}$$

where

$$\beta = \gamma + \varepsilon ; \qquad\qquad A = \frac{b}{a}^{-\beta}$$

Equilibrium price dynamics are found by applying Ito's lemma:

$$\frac{dP}{P} = \alpha_p dt + \sigma_p dz_p \tag{19}$$

where

$$\alpha_p = -\beta\alpha_\theta + \frac{1}{2}\beta(\beta+1)\sigma_\theta^2$$

$$\sigma_p = -\beta\sigma_\theta \qquad\qquad dz_p = dz_\theta$$

Thus, the price dynamics are determined by the instantaneous equating of supply and demand. There will be no relationship in general between α and α_p, where α is the expected rate of price increase required to induce storage of the commodity. There is no storage, and no incentive for any to occur. For there to be no storage, it is only required that $\alpha_p < \alpha$.

From the analysis in Section I, it follows that formula (12) is the price of a european call option on this commodity. Because the formula includes $\delta = \alpha - \alpha_p$, this option price depends upon the beta of the commodity. This occurs because there need not be any particular relationship between the systematic risk of the commodity price and its expected rate of price increase.

Black [1] has also derived a formula for this kind of option, assuming that futures contracts exist for the commodity. Black's formula does not depend upon the systematic risk of the commodity; nevertheless, we will show that, in the presence of futures contracts, his formula is the same as ours. Black's formula may be derived from ours by noting that $P_0e^{-\delta t}$ is the expression for a futures price when cash in advance is required. When cash in advance is not required, then the futures price is $F = P_0e^{-(\delta-r)t}$. If this expression is used to eliminate P_0 in (12), Black's formula is obtained.

III. SUMMARY AND CONCLUSIONS

While the Black-Scholes european call option formula can be derived using an "arbitrage" method, we have shown that when the underlying asset is non-stored this is not consistent with capital market equilibrium. Another formula, with a lower value, is appropriate. Because short sales of the non-stored asset are not possible, a call price differing from the Black-Scholes price will not result in arbitrage profits. The proper formula, unlike other option pricing formulas, will depend upon the systematic risk of the non-stored asset.

FOOTNOTES

1. By tradeable, we mean that the asset can be bought, sold and held.

2. Constantinides (1978) discusses options on "mispriced" (and therefore non-stored) but non-tradeable assets and derives the same formula as (12) in the text below. The distinction between the arbitrage and equilibrium derivations, however, only occurs if the asset is tradeable. Competitively produced commodities, for which no stockpile exists, are examples of mispriced but tradeable assets. See Section II for an example.

3. For the purposes of this paper, the equilibrium price refers to the option price derived using the equilibrium derivation and the market price refers to the prevailing price.

4. Using the ICAPM it is easy to create the necessary security using a portfolio of any stock and a position in the riskless asset.

5. Consider just the term in brackets. The covariance of this term with the market return is

$$[\sigma_s dz_s - \sigma_p dz_p]dz_m = \sigma_s \rho_{sm} - \sigma_p \rho_{pm} = 0.$$

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