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VECTOR MEASURES ARE OPEN MAPS

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ABSTRACT

Nonatomic vector measures are shown to be open maps from the σ-field on which they are defined to their range, where the σ-field is equipped with the pseudo-metric of the symmetric difference with respect to a given scalar measure.

We prove the following:

Main Theorem: Let $\lambda, \mu_1, \ldots, \mu_n$ be nonatomic, σ-additive, finite measures on a measurable space (I, Σ), and let λ be a nonnegative measure. Then the vector measure $\mu = (\mu_1, \ldots, \mu_n)$ is an open map from Σ to the range of μ, where Σ is equipped with the topology induced by the pseudo-metric d_λ defined by $d_\lambda(S,T) = \lambda[(S\setminus T) \cup (T\setminus S)]$, and the range of μ is equipped with its relative topology in R^n.

Let us introduce the following notations. For S in Σ we denote by $\bar{S}$ the complementary set I S. The symmetric difference of S and T, $(S\setminus T)\cup(T\setminus S)$ is denoted by $S\Delta T$. The Euclidian norm in R^n is denoted by $\|\ \|$, and the scalar product of ξ and x in R^n is denoted by $\langle\xi, x\rangle$. By the relative boundary of a closed set K in R^n we mean the set of all points in K which are not in the relative interior of K. The face of a convex set K in the direction ξ, is the set $F(\xi) = \{x | x \in K,\ \langle\xi, x\rangle = \max_{y\in K} \langle\xi, y\rangle\}$. We say that a set K in R^n is strictly convex if all the points on the relative boundary of K are extreme, or alternatively if for each $\xi \in R^n$, $F(\xi)$ is either K or a singleton. For a scalar measure λ, we denote by $|\lambda|$ the sum of the positive and the negative parts of λ. For a vector measure $\mu = (\mu_1, \ldots, \mu_n)$, $|\mu|$ is the sum $\sum_{i=1}^{n} |\mu_i|$. For each S we define $R(\mu,S) = \{\mu(T) | T \subseteq S\}$. Clearly $R(\mu,S) + R(\mu,\bar{S}) = R(\mu,I)$. By Lyapunov Theorem [1], $R(\mu,S)$ is a convex and compact set.

A convenient way to describe $R(\mu,I)$ is as follows. Let f_i be the Randon-Nikodym derivative of μ_i with respect to $|\mu|$. and let $f = (f_1, \ldots, f_n)$. Then $\mu(S) = \int_S f d|\mu|$ and for $\xi \in R^n$, $\langle\xi, \mu(S)\rangle = \int_S \langle\xi, f\rangle\, d|\mu|$. Obviously $\mu(S) \in F(\xi)$ if an only if $\{t|\ \langle\xi, f(t)\rangle > 0\} \subseteq S \subseteq \{t|\ \langle\xi, f(t)\rangle \geq 0\}$ almost everywhere with respect to $|\mu|$. It follows then, that $R(\mu,I)$ is strictly convex if and only if, the set $\{t|\langle\xi, f(t)\rangle = 0\}$ is of $\bar{\mu}$-measure zero for all supporting hyperplanes ξ which do not contain $R(\mu,I)$, or alternatively if for each subspace V of R^n of dimension smaller than that of $R(\mu,I)$, the set $\{t, f(t) \in V\}$ if of $\bar{\mu}$-measure zero.

We can prove now:

Lemma 1 There is a decomposition $R(\mu,I) = \sum_i R(\mu,S_i)$ such that $\bigcup_i S_i$ is a partition of I and $R(\mu,S_i)$ is strictly convex for each i.

Proof: The decomposition is built in n stages. In the stages $1,\ldots,k-1$ a family of disjoint sets S_i^j, $1 \leq j \leq k-1$, $1 \leq i < i_j$ if defined (i_j is possibly ∞ or 0) such that $R(\mu,S_i^j)$ is strictly convex and of dimension j. Moreover, for each k-1 dimensional subspace of R^n, V, the set $\{t | t \in I \setminus \bigcup_{i,j} S_i^j, f(t) \in V\}$ is of $\bar{\mu}$-measure zero. In the k-th stage we define the sets S_i^k, $1 \leq i < i_k$ which are all the subsets of $I \setminus \bigcup_{i,j} S_i^j$ of the form $\{t | f(t) \in V\}$ which have positive $\bar{\mu}$ measure, where V is a k-dimensional subspace of R^n. The disjointness of the sets S_i^k can be guaranteed since the intersection of such two sets is a set of t's for which f(t) belongs to a subspace of dimension less than k. The strict convexity of $R(\mu,S_i^k)$ follows similarly.

Q.E.D.

Let us call a vector measure $\mu = (\mu_1,\ldots,\mu_n)$ monotonic if each $\mu_i (1 \leq i \leq n)$ is either nonnegative or nonpositive. We will show now that it suffices to prove the Main Theorem for monotonic μ with strictly convex range $R(\mu,I)$. Indeed, there is a partition $I = \bigcup_{i=1}^{2^n} I_i$ such that the restriction of μ to each I_i is monotonic. We can decomposite, furthermore, each I_i

according to Lemma 1, to get eventually a partition $I = \bigcup_i S_i$ and a decomposition $R(\mu,I) = \sum_i R(\mu,S_i)$ such that for each i, μ is monotonic on S_i and $R(\mu,S_i)$ is strictly convex. For $\varepsilon > 0$ and $S \in \Sigma$ denote $\Omega_i(S,\varepsilon) = \{T | T \subsetneq S_i,\ d_\lambda(T, S \cap S_i) < \varepsilon\}$ and $\Omega(S,\varepsilon) = \{\bigcup_i T_i | T_i \in \Omega_i(S,\varepsilon)\}$. It is easy to verify that the family of sets $\Omega(S,\varepsilon)$ where S ranges over Σ and ε ranges over the positive reals, is a basis to the topology induced by d_λ on Σ. Moreover $\mu(\Omega(S,\varepsilon)) = \sum_i \mu(\Omega_i(S,\varepsilon))$. But $\mu(\Omega_i(S,\varepsilon)) \subsetneq R(\mu,S_i)$, $R(\mu,S_i)$ is strictly convex and the restriction of μ to S_i is monotonic. Therefore by proving the Main Theorem for monotonic μ with strictly convex range we prove that $\mu(\Omega_i(S,\varepsilon))$ is relatively open in $R(\mu,S_i)$ which says that $\mu(\Omega(S,\varepsilon))$ is relatively open in $R(\mu,I)$.

We assume now that μ is monotonic and that $R(\mu,I)$ is strictly convex. We start by proving the following lemma.

<u>**Lemma 2**</u> If $x_o = \mu(S_o)$ then for each $1 \leq i \leq n$ and $\varepsilon > 0$ the set $\mu\{S | d_{|\mu_i|}(S,S_o) < \varepsilon\}$ contains a set $\{x | x \in R(\mu,I),\ \|x - x_o\| < \delta\}$ for some $\delta > 0$.

We first prove the lemma in the case that x_0 is in the relative interior of $R(\mu,I)$, using lemma 3.

<u>**Lemma 3.**</u> If $x_0 = \mu(S_0)$ is in the relative interior of $R(\mu,I)$, then the intersection of the relative interiors of $R(\mu,S_0)$ and $R(\mu,\bar{S}_0)$ is not empty.

<u>**Proof of Lemma 3:**</u> Indeed, if this intersection is empty then there exists a hyperplane which separates the two sets and for at least one of them, say $R(\mu,S_o)$, contains only points from its relative boundary. Since $0 \in R(\mu,S_0) \cap R(\mu,\bar{S}_0)$ we conclude that there exists $\xi \in R^n$ such that

$\langle\xi,x\rangle \geq 0$ for $x \in R(\mu,S_0)$ and $\langle\xi,x\rangle \leq 0$ for $x \in R(\mu,\bar{S}_0)$ and moreover for some x in the relative interior of $R(\mu,S_0)$, $\langle\xi,x\rangle > 0$. Now let $S \in \Sigma$ and denote $S_1 = S \cap S_0$, $S_2 = S \cap \bar{S}_0$. We have: $\langle\xi,\mu(S_2)\rangle \leq 0 \leq \langle\xi,\mu(S_0 \;\; \bar{S}_1)\rangle = \langle\xi,\mu(S_0)\rangle - \langle\xi,\mu(S_1)\rangle$ and therefore, $\langle\xi,\mu(S)\rangle = \langle\xi,\mu(S_1) + \mu(S_2)\rangle \leq \langle\xi,\mu(S_0)\rangle$. This inequality holds for each S in Σ and moreover, for some S the inequality is strict which shows that $\mu(S_0)$ is in the relative boundary of $R(\mu,I)$, contrary to our assumption.

Q.E.D.

Proof of Lemma 2: Assume first that x_0 is in the relative interior of $R(\mu,I)$. Let E_0, E_1 and E_2 be the linear spaces spanned by $R(\mu,I)$ $R(\mu,S_0)$ and $R(\mu,\bar{S}_0)$ respectively, and denote by B_0, B_1 and B_2 the interesection of the unit ball in R^n with E_0, E_1 and E_2 respectively. Since $0 \in R(\mu,S_0) \cap R(\mu,\bar{S}_0)$, we find, using Lemma 3, a point w which belongs to the relative interiors of both $R(\mu,\bar{S}_0)$ and $R(\mu,S_o)$ and for which $\|w\| < \frac{\varepsilon}{4}$. Choose now $0 < \eta < \frac{\varepsilon}{4}$ such that $w + \eta B_1 \subseteq R(\mu,S_0)$ and $w + \eta B_2 \subseteq R(\mu,\bar{S}_0)$. Clearly $E_0 = E_1 + E_2$ and therefore we can choose $0 < \delta < \frac{\varepsilon}{4}$ such that $\delta B_0 \subseteq \eta(B_2 + B_1) = \eta(B_2 - B_1)$. Now let $x \in R(\mu,I)$ with $\|x - x_o\| < \delta$ and denote $z = x - x_0$. Since $z \in \delta B_0$ there exist $z_1 \in \eta B_1$ and $z_2 \in \eta B_2$ such that $z = z_2 - z_1$. There exist also $S_1 \subseteq S_0$ and $S_2 \subseteq \bar{S}_0$ such that $\mu(S_1) = w + z_1$ and $\mu(S_2) = w + z_2$. Define $S = (S_0 \setminus S_1) \cup S_2$. We have

$$\mu(S) = \mu(S_0) - \mu(S_1) + \mu(S_2) = x_0 - z_1 + z_2 = x_0 + z = x,$$

and using the monotonicty of μ,

$$d_{|\mu_i|}(S,S_0) \leq \|\mu(S \Delta S_0)\| = \|\mu(S_1) + \mu(S_2)\| = \|2w + z_1 + z_2\| < 2\,\frac{\varepsilon}{4} + 2\eta < \varepsilon.$$

We continue now to prove Lemma 2 for x_0 on the relative boundary of $R(\mu,I)$. Consider a sequence $x_n = \mu(S_n)$ such that $x_n \to x_0$. We will show that $\mu(S_n \Delta S_0) \to 0$ which is more than we need to complete the proof of Lemma 3.

let $T'_n = S_n \cap S_0$ and $T''_n = S_n \cap \bar{S}_0$. Since the sequences $\mu(T'_n)$ and $\mu(T''_n)$ belong to the compact sets $R(\mu, S_0)$ and $R(\mu, \bar{S}_0)$ we can assume without loss of generality that $\mu(T'_n) \to \mu(T')$ and $\mu(T''_n) \to \mu(T'')$ where $T' \subseteq S_0$ and $T'' \subseteq \bar{S}_0$. It follows that $\mu(T' \cup T'') = \mu(S_0)$ and since $R(\mu, I)$ is strictly convex $T' = \bar{S}_0$ and $T'' = \phi$ almost everywhere with respect to μ, which shows that

$\mu(S_n \Delta S_0) = \mu(S_0) - \mu(T'_n) + \mu(T''_n) \to 0.$

Q.E.D.

To complete the proof of the Main Theorem we have to show that d_λ can replace $d_{|\mu_i|}$ in Lemma 2. There is a partition $I = S_1 \cup S_2$ of I such that the restriction of λ to S_0 is continuous with respect to $|\mu|$ and $|\mu|(S_2) = 0$. Define $\Omega_i(S, \varepsilon) = \{T | T \subseteq S_i,\ d_\lambda(T, S) < \varepsilon\}$ $i = 1,2$, and

$\Omega(S, \varepsilon) = \{T_1 \cup T_2 |\ T_i \in \Omega_i(S, \varepsilon),\ i = 1,2\}$. Clearly $\mu(\Omega_2(S, \varepsilon)) = 0$. But $\Omega_1(S, \varepsilon)$ is open in the topology induced by $d_{|\mu|}$ on the σ-field $\{T | T \in \Sigma,\ T \subseteq S_i\}$ and therefore by Lemma 2 $\mu(\Omega(S, \varepsilon)) = \mu(\Omega_1(S, \varepsilon))$ is relatively open in $R(\mu, S_1) = R(\mu, I)$.

Corollary: The projection $\pi : R^{n+1} \to R^n$ on the first n coordinates is an open map from $R((\mu_1, \ldots, \mu_{n+1}), I)$ onto $R((\mu_1, \ldots, \mu_n), I)$.

Proof: Denote $\mu = (\mu_1, \ldots, \mu_{n+1})$. Then $\pi = (\pi\mu)\mu^{-1}$. The result follows from the continuity of $\pi\mu$ and the fact that μ^{-1} is an open map.

Finally let us remark that the Main Theorem in an extension of Lemma 2 in [2]. This lemma states for a nonnegative vector measure μ, that for each x in $R(\mu, I)$ there exists S with $\mu(S) = x$ such that any neighborhood of S (with respect to d_μ), is mapped by μ to a neighborhood of x. The Main Theorem is used in [3] where the weaker result of [2] in not enough.

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