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THE UTILITY HYPOTHESIS AND MARKET DEMAND THEORY

by

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Abstract

The utility hypothesis is the fundamental building block in economists' explanation of market demand. We add the strong hypothesis that each consumer receives a fixed proportion of a community income variable, and investigate the implications of these hypotheses taken together. A theorem is proved which shows that for an arbitrary price-income pair (p,I), there can exist no observations on demand and rates of change in demand, which a) satisfy homogeneity and the budget equation, and b) refute the utility and fixed proportions hypotheses.

Introduction

Market demand functions are defined as the sum of the demand functions of utility maximizing individuals. It is common to reduce the number of variables which explain market demand by assuming that a community income variable is shared among all consumers in fixed proportions. Under this assumption, market demand depends only on commodity prices and community income. As a consequence the necessity of continuously monitoring individual income is eliminated. Two restrictions which must be satisfied by a market demand function are obvious and well-known: the balance condition; i.e., the value of market demand must equal community income, and the homogeneity condition; i.e., market demand must be homogeneous of degree zero in the prices of all commodities and community income. Also, there are some global requirements which follow from the positivity of individuals' demands. Our main result provides a striking indication that the aforementioned restrictions largely exhaust the empirical implications of the utility hypothesis for market demand functions, even under the strong hypothesis that community income is shared equally.

Theorem: Consider any n-tuple of (positive) prices $\bar{p} = (\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_n)$, any (positive) value of community income $\bar{I}$, and any (positive) n-tuple $(D_1, D_2, \ldots, D_n)$ which satisfies the balance condition. Let (α_{ij}) be any $(n-1) \times (n-1)$ matrix and $(\beta_1, \beta_2, \ldots, \beta_{n-1})$ be any (n-1)-tuple. Then there exists a finite collection of consumers (a consumer is a utility function over positive n-tuples) who when they share equally in community income generate a market demand function $(h_1, h_2, \ldots, h_n)$ having the following properties.

$$(h_1(\bar{p}, \bar{I}), h_2(\bar{p}, \bar{I}), \ldots, h_n(\bar{p}, \bar{I})) = (D_1, D_2, \ldots, D_n),$$

$$\begin{pmatrix} \frac{\partial h_1(\bar{p},\bar{I})}{\partial p_1} & \frac{\partial h_1(\bar{p},\bar{I})}{\partial p_2} & \cdots & \frac{\partial h_1(\bar{p},\bar{I})}{\partial p_{n-1}} \\ \frac{\partial h_2(\bar{p},\bar{I})}{\partial p_1} & \vdots & & \vdots \\ \vdots & & & \\ \frac{\partial h_{n-1}(\bar{p},\bar{I})}{\partial p_1} & & \cdots & \frac{\partial h_{n-1}(\bar{p},\bar{I})}{\partial p_{n-1}} \end{pmatrix} = (\alpha_{ij}), \text{ and}$$

$$\left(\frac{\partial h_1(\bar{p},\bar{I})}{\partial I}, \frac{\partial h_2(\bar{p},\bar{I})}{\partial I}, \ldots, \frac{\partial h_{n-1}(\bar{p},\bar{I})}{\partial I} \right) = (\beta_1, \beta_2, \ldots, \beta_{n-1}).$$

(The rates of change of h_n and the rates of change with respect to p_n are determined by the balance and homogeneity conditions.)

The strength of this Theorem is that it precludes the discovery of any general relationship among a commodity price vector p, community income I, market demand at (p, I), and rates of change in community demand at (p, I), distinct from those which are implied by the balance and homogeneity conditions. Observe that every other potential relationship is contradicted by the conclusion to the Theorem![1]

The moral of the Theorem is simply this: if you put very little in, you get very little out. It is perhaps not so surprising that when individual demand is aggregated in the market place, and neither individual demand nor individual incomes are observed, there is little left of demand theory beyond homogeneity and balance. This result underlines the importance to demand theory of positing and then testing for restrictions based on something more than the utility hypothesis. It explains why one might group commodities in special ways (as is done by I.F. Pearce, [7]), or rule out inferiority of some goods, or assume that preferences are nicely distributed with a single peak, or derive demand from household production functions, or a "characteristics" approach, or identical homothetic preferences, etc. Despite the fact that the general form of community demand theory appears in every text on price theory, it does not have the same empirical force as the theory of gravitation! Unless it is combined with supplementary hypotheses it remains an "empty (empirical) box".

Proof And A Conjecture

The Theorem will be proved as a consequence of the following Lemma, which is specific to economies with two commodities.

Lemma:[2] Consider _any_ ordered pair of positive prices $\bar{p} = (\bar{p}_1, \bar{p}_2)$, _any_ positive value of community income $\bar{I}$, and _any_ ordered pair of continuously differentiable functions $d = (d_1, d_2)$ which are both positive at $(\bar{p}, \bar{I})$ and satisfy the homogeneity and balance conditions. There exist two utility maximizing consumers, and a neighborhood $N(\bar{p}, \bar{I})$, such that when these consumers share equally in community income, the market demand function they generate is identically equal to d on $N(\bar{p}, \bar{I})$.

The Lemma tells us that locally (i.e., in small neighborhoods) any system of demand equations, such as (f_1, f_2), which satisfies the balance and homogeneity requirements can be generated by the utility maximizing choices of two individuals. Thus it is impossible for observations of demand to contradict (locally) the hypothesis that demand is generated by utility maximizing individuals who share a community income in fixed proportion. The Lemma can be viewed as an extension of a standard diagram, used by Hicks to illustrate the fact that the weak axiom of revealed preference need not hold for market demand ([2], p. 55). See also [10], p. 143.

We now proceed to a proof of the Lemma.

Proof of the Lemma: Because the function $d = (d_1, d_2)$ is positive homogeneous of degree zero and satisfies the balance condition, there exists a unique continuously differentiable function f with the property that

$$(f(p_1/p_2, I/p_2), I/p_2 - (p_1/p_2)f(p_1/p_2, I/p_2))$$

$$= (d_1(p_1,p_2,I), d_2(p_1,p_2,I)),$$

for all $p_1, p_2, I > 0$.

Furthermore, the association is one-to-one; i.e., no function f can come from distinct functions which satisfy homogeneity and balance. Since demand functions generated by utility maximizing behavior must satisfy homogeneity and balance, and since any demand function for two commodities which satisfies the weak axiom of revealed preference can be generated by a utility function, it is sufficient to prove the following stronger statement.

Consider any positive real number $\bar{p}$, any positive real number $\bar{I}$, and any positive continuous function f which satisfies the condition $\bar{I} - \bar{p}f(\bar{p},\bar{I}) > 0$, and has its difference quotients bounded (in both variables) by K. Then there exist two demand functions $h^1 = (h_1^1, h_2^1)$ and $h^2 = (h_1^2, h_2^2)$ (each of which satisfies the weak axiom of revealed preference), and a neighborhood $N(\bar{p},\bar{I})$, such that for all $(p,I) \in N(\bar{p},\bar{I})$,

$$f(p,I) = h_1^1(p,I/2) + h_1^2(p,I/2).$$

The functions h_1^1 and h_1^2 are constructed as follows. Since $\bar{I} - \bar{p}f(\bar{p},\bar{I}) > 0$, there exist positive numbers a and b such that $b > a$, $a + b = f(\bar{p},\bar{I})$, $\bar{I}/2 - \bar{p}a > 0$ and $\bar{I}/2 - \bar{p}b > 0$. Define $\alpha = (3/4)a + (1/4)b$ and $\beta = (1/4)a + (3/4)b$. Choose $r,s > \max\{1,2K\}$ and then $\eta > 0$, so that

$$a < \alpha < s/r < (s + K)/(r - 2K) < \beta - \eta < \beta < \beta + \eta < b,$$

and $s + K - \beta(r - sK) < -\eta(r + K)$. Next define

$$h_1^1(p,I/2) = rI/2 - sp - r\bar{I}/2 + s\bar{p} + \alpha, \text{ and}$$

$$h_1^2(p,I/2) = f(p,I) - f(\bar{p},\bar{I}) - rI/2 + sp + r\bar{I}/2 - s\bar{p} + \beta.$$

Observe that $h_1^1(\bar{p},\bar{I}/2) = \alpha$, $h_1^2(\bar{p},\bar{I}/2) = \beta$, and $h_1^1(p,I/2) + h_1^2(p,I/2) = f(p,I)$ for all (p,I) in the positive orthant of R^2. Let $N(\bar{p},\bar{I})$ be chosen so that $h_1^1(p,I/2) \in]a,s/r[$, and $h_1^2(p,I/2) \in]\beta - \eta, \beta + \eta[$. It is left as an exercise to verify that $h^1 = (h_1^1,h_2^1)$ and $h^2 = (h_1^2,h_2^2)$ both are positive and satisfy the weak axiom of revealed preference on $N(\bar{p},\bar{I}/2) = \{(p,I/2) : (p,I) \in N(\bar{p},\bar{I})\}$. [In order to verify the weak axiom, check that for all $(\tilde{p},\tilde{I})$, $(\hat{p},\hat{I}) \in N(\bar{p},\bar{I})$ and $\hat{p}h_1^1(\tilde{p},\tilde{I}) + h_2^1(\tilde{p},\tilde{I}) = \hat{I}$, imply $(\hat{p}-\tilde{p})\ \ (h_1^i(\hat{p},\hat{I}) - h_1^i(\tilde{p},\tilde{I})) < 0$, $i = 1,2$.]

Sketch of a Proof of the Theorem:[3] It is sufficient to prove the theorem for $n = 3$, since the result follows for arbitrary n by forming the union of three commodity economies. Assume (without loss of generality) that $p_3 = 1$, and consider the pair of equations

$$d_1(p_1, p_2, 6I/M) = a_1p_1 + b_1I + c_1$$
$$+ 2I/(Mp_1) - a_2p_2^2/p_1^2 - b_2p_2I/p_1^2 - c_2p_2/p_1, \quad \text{and}$$

$$d_2(p_1, p_2, 6I/M) = a_2p_2/p_1 + b_2I/p_1 + c_2$$
$$+ 2I/(Mp_2) - a_3/p_2^2 - b_3I/p_2^2 - c_3/p_2.$$

By choosing M sufficiently large and $V = (a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3)$ appropriately, it is possible to find six agents who generate the above pair of equations as their demand for the first two commodities when they each receive a 1/M share of community income. At the same time it is possible to obtain $\partial d_i(\bar{p}_1, \bar{p}_2, 6\bar{I}/M)/\partial p_j = \alpha_{ij}$, $\partial d_i(\bar{p}_1, \bar{p}_2, 6\bar{I}/M)/\partial I = \beta_i$, $D_i - d_i(\bar{p}_1, \bar{p}_2, 6\bar{I}/M) > 0$, and $(M-6)\bar{I}/M - p_1(D_1 - d_1) - p_2(D_2 - d_2) > 0$, $i, j = 1, 2$. To achieve this construction one applies the lemma three times, each time with a different temporary numeraire and a different commodity not consumed. The proof of the theorem is completed by adding M - 6 identical consumers to the above six consumers. Each of these individuals receives 1/M of the community income variable I and has the constant positive demand $(D_1 - d_1(\bar{p}_1, \bar{p}_2, 6\bar{I}/M))/(M - 6)$ and $(D_2 - d_2(\bar{p}_1, \bar{p}_2, 6\bar{I}/M))/(M - 6)$ for commodities one and two in a neighborhood of $(\bar{p}_1, \bar{p}_2, \bar{I})$.

The preceding proof is neither elegant nor instructive. A superior proof may be obtained as a corollary of the following conjecture, which is the n commodity analog of the previous lemma. It is easily seen to imply all of the results of the paper.

Conjecture: Consider *any* ordered n-tuple of positive prices $\bar{p} = \bar{p}_1, \bar{p}_2, \ldots, p_n)$, *any* positive value of community income $\bar{I}$, and *any* ordered n-tuple of continuously differentiable functions $d = (d_1, d_2, \ldots d_n)$ which are both positive at $(\bar{p}, \bar{I})$ and satisfy the homogeneity and balance conditions. There exist n utility maximizing consumers, and a nieghborhood $N(\bar{p}, \bar{I})$, such that when these consumers share equally in community income, the market demand function they generate is identically equal to d on $N(\bar{p}, \bar{I})$.

That it is necessary to confine the realization of d to a neighborhood of $(\bar{p}, \bar{I})$ is readily demonstrated by observing that there does not exist a collection of utility maximizing individuals who generate the following two demand observations when they each receive a fixed percentage of community income. At prices (2,1) and community income 1, demand is (11/24, 2/24),[4] and at prices (1,2) and community income 1, demand is (2/24,11/24). Proof of this assertion is left to the reader. It relies on the requirement that consumers must choose non-negative commodity amounts.

The above conjecture requires as many consumers as commodities. That such a number may be necessary in order to realize an arbitrary function $(d_1, d_2, \ldots, d_n)$ is verified by observing that for each consumer's demand function h^k, the (n-1) × (n-1) matrix $(\partial h_i^k / \partial p_j)$ is the difference of the (n-1) × (n-1) Slutsky matrix and the matrix $(h_j^k \partial h_i^k / \partial I)$, which is of rank at most one. Since the Slutsky matrix is negative definite (almost everywhere), and since the sum of negative definite matrices is negative definite and thus non-singular, the (n-1) x (n-1) matrix of rates of change in market demand with respect to price changes has rank no less than (n-1) - (the number of commodities) (almost everywhere). Furthermore, if the vector of changes in demand for the first n-1 commodities with respect to income is identically zero, and if the number of consumers is n-1, the rank of this matrix must be at least one (almost everywhere).

Observe finally that a proof of the conjecture, because of the local premises of the conjecture, would not completely resolve the problem of characterizing the class of demand functions which are generated by summing utility maximizing behavior. The problem of obtaining a global characterization is completely open.

NOTES

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[1] Contrast this situation with results from the theory of the individual utility maximizing consumer. A function $d = (d_1, d_2, \ldots, d_n)$, which satisfies the balance and homogeneity conditions, belongs to the class of demand functions generated by a single utility maximizing individual if and only if the matrix $(\partial d_i/\partial p_j + d_j \partial d_i/\partial I)$ is symmetric (S) and negative semi-definite (N) evaluated at each point in the domain of d, [3]. The conditions (S) and (N) place severe restrictions on the class of demand functions which are generated by an individual utility maximizing consumer, and the fact that these conditions are met "exhausts the empirical implications of the utility analysis" (Samuelson's Foundations, p. 116). The Theorem I have stated denies the existence of analogous restrictions for the case of market demand functions, even under the strong hypothesis that community income is shared equally.

[2] Compare with Theorem 1, [9]. The proof of the theorem here is a good deal more tedious since commodity choices must be non-negative.

[3] In response to this theorem Rolf R. Mantel of the Instituto Torcuato Di Tella, Buenos Aires has shown that the required representation can be accomplished with as few consumers as commodities.

[4] Contrast this situation of "local realization" with the global theorems proved for excess demand functions in [1],[4],[5], and [9]. The excess demand theory differs in two important ways from the demand theory presented here. First, individual income is determined as the value of an initial endowment, and so community income is not an argument of the excess demand function. Also, implications of the requirement that consumers must pick non-negative bundles can be finessed in the excess demand theory by allowing consumers to have sufficiently large initial endowments. As demonstrated by the previous example, the requirement of non-negative choice has teeth in the demand theory.