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THE PERIOD OF COMMITMENT IN DYNAMIC GAMES

by

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ABSTRACT

When economic competition over time is modelled as a dynamic game, the appropriate formulation of the players' strategy spaces is an important issue. Two approaches have been adopted: the use of path strategies, which corresponds to the assumption that players make binding commitments that extend over the entire horizon of the game; and the use of decision rule strategies, which corresponds to the assumption that no commitments at all are possible. Here it is shown that intermediate choices are possible as well, and, by example, that the choice of the period of commitment can be crucial.

Oligopoly extraction of a nonrenewable, common property resource is analyzed as a noncooperative game in extensive form, by examining a family of discrete-period games in which the players are firms and their strategies are extraction plans. This family of games is parametrized by the length of the period over which firms can make commitments about their extraction rates. It is shown that the aggregate rate of resource depletion varies dramatically as the length of the period of commitment varies, approaching instantaneous depletion as the length of the period of commitment approaches zero.

I. Introduction

The theory of noncooperative dynamic games has provided an extremely powerful framework for studying many of the classic questions in industrial organization--for example, questions about advertising, research and development, investment in new capacity, and barriers to entry--where interactions over time among a small number of firms are involved. However, an important methodological issue arises when economic competition over time is modelled as a dynamic game. The issue is the appropriate formulation of the players' strategy spaces.

Two approaches have been adopted. One is to model players as choosing path strategies and to look at Nash equilibria; the other is to model them as choosing decision rule strategies and to look at subgame perfect Nash equilibria. (The resulting equilibria are also called "open loop" and "closed loop" equilibria.) However, it is not the equilibrium concept that distinguishes these two, since all Nash equilibria in path strategies are (trivially) subgame perfect. Rather, the difference is in the choice of the strategy space.

These two formulations correspond to extreme assumptions about players' ability to make commitments about their future actions. The use of path strategies corresponds to the assumption that commitments extend over the entire future horizon; the use of decision rule strategies corresponds to the assumption that no commitment at all is possible. Both approaches have been used in the industrial organization literature. For example, path strategies have been used to study investment in a new market (Spence [1979]), the learning curve (Spence [1981]); the extraction of nonrenewable resources (Crawford, Sobel and Takahashi [1980], Dasgupta and Heal [1979], Lewis and Schmalensee [1980], Loury [1980], and Salant [1979]); and cost-reducing investment (Flaherty [1979]); while decision rule strategies have been used to study the arms races (Simaan and Cruz [1975]); the extraction of renewable resources (Clemhout and Wan [1979], and Levhari and Mirman [1980]) and nonrenewable resources (Stokey [1981]); oligopoly theory (Clemhout, Leitmann and Wan [1973]); and research and development (Reinganum [1981a,b]).[1]

What is unfortunate is not that different formulations of players' strategy spaces are used. Since this work covers a wide variety of questions, this by itself would be neither surprising nor disturbing. Rather, what is disturbing is that the choice of one formulation over another is seldom defended or even discussed explicitly. As will be shown below, this choice can be crucial. Consequently, when formulating a model care should be taken to choose a strategy space that is appropriate for the situation under study. Path strategies may be appropriate in some situations, decision rule strategies in others, and intermediate formulations in still others. But the choice should not be made casually, since it can--as shown in the example below--dictate to a very large degree the nature of the conclusions.

In section II "Nash equilibrium" and "subgame perfectness" are defined, the distinction between perfect and imperfect equilibria is described, and the reason for confining attention to subgame perfect equilibria is discussed.

In section III the importance of the choice of strategy spaces is illustrated by example. Oligopoly extraction of a nonrenewable, common property resource is analyzed as a noncooperative game in extensive form, by examining a family of games in which the players are firms and their strategies are extraction plans. This family of games is parametrized by the length of the period over which firms can make commitments about their extraction rates. A stationary, isoelastic inverse demand function is used throughout, and it is assumed that there are no costs of extraction. We show that for any given length of period, planning horizon, and number of firms, there exists a symmetric, subgame perfect Nash equilibrium. We then compare aggregate extraction paths as the length of the period changes.

We use as a baseline for comparison the path of aggregate extraction that occurs if the period of commitment coincides with the planning horizon. (It happens that for the demand curve adopted here, this path maximizes total surplus and the price rises at the rate of interest over time. This is incidental for our purposes.) For any fixed number of firms greater than one and fixed planning horizon, shortening the period of commitment causes the resource to be depleted more rapidly. Compared with the baseline path, the price is initially lower but rises more rapidly. As the period of commitment becomes arbitrarily short, the resource stock is depleted arbitrarily quickly.

Thus for a common property resource, the length of the commitment period is a crucial determinant of equilibrium extraction. The fact that the length of the period is critical in our model suggests that results from other dynamic game models of economic problems may also be sensitive to implicit assumptions made about the period of commitment.

II. Strategy Spaces and Equilibrium Concepts

Two alternative assumptions about the period of commitment correspond to the two strategy spaces commonly used in the literature on dynamic games. These two strategy spaces represent limiting assumptions about the length of the period over which players make commitments. The use of path strategies corresponds to the assumption that the period of commitment is the same as the planning horizon. That is, at the initial date each player must make a binding commitment about the actions it will take at all future dates. A set of path strategies (one for each player) constitutes a Nash equilibrium if for each player the following condition is satisfied: the path the player commits himself to must be an optimal response to the paths the other players have committed themselves to.

The use of decision rule strategies corresponds to the assumption that firms observe the values of relevant state variables and respond instantaneously by choosing their current actions. Therefore, the situation is equivalent to one in which at the initial date each player selects a decision rule that specifies which action is to be taken at each intermediate (date,state) pair. A set of decision rules (one for each player) constitutes a Nash equilibrium if the following condition is satisfied: the decision rule of each player, when viewed at the starting date of the game and the given initial conditions for the state variables, must be an optimal response to the decision rules chosen by the other players. In general there will be multiple equilibria. However, some of these equilibria can be ruled out.

The definition of a Nash equilibrium states only that each player's decision rule must be an optimal response when viewed from the initial (date,state) pair; the continuation of his decision rule is not required to be an optimal response when viewed from any intermediate (date,state) pairs. A Nash equilibrium is subgame perfect (Selten [1975]) if the continuation of the given decision rules constitutes a Nash equilibrium when viewed from any intermediate (date,state) pair, i.e., if they form a Nash equilibrium in every subgame of the original game. (From here on, "perfect" will always mean "subgame perfect.")

The distinction between perfect Nash equilibria and imperfect Nash equilibria consists of a difference in the points at which the Nash equilibrium conditions must hold. These conditions must hold at the initial node of every subgame in order for the equilibrium to be subgame perfect; they need only hold at the initial node of the game itself for a (possibly imperfect) Nash equilibrium. Thus perfect Nash equilibrium strategies satsify a sort of "principle of optimality"; imperfect Nash equilibrium strategies do not. Since imperfect Nash equilibria rely on threatened behavior at some decision points that players would not have an incentive to carry out if they arrived at those decision points, they are inappropriate for situations where commitment (to a particular decision rule) is impossible. In any discrete-period finite-horizon game, the subgame perfect Nash equilibria can be found by using backward induction.

In the next section we will examine a parametric family of dynamic games of resource extraction, where the games are indexed by the period of commitment. Because Nash equilibria will be computed recursively from the final contract to the initial one, the equilibria will be subgame perfect.

III. Oligopoly Exploitation of a Common Property Resource

Firms $j = 1,2,...J$ are engaged in the noncooperative extraction of a nonrenewable, common property resource. Exploitation begins at date $t = 0$ and ends no later than date $T \leq \infty$, and the stock of the resource at date $t = 0$, X_0, is known with certainty. Each firm can costlessly extract from the common pool. For any date $t \in [0,T]$, let $y_j(t) \geq 0$ denote the extraction rate of firm j, and $Y(t) \equiv \sum_{j=1}^{J} y_j(t)$ the aggregate extraction rate. We do not impose any upper bound on individual firms' extraction rates or on the aggregate extraction rate. Let $X(t)$ denote reserves at date $t \in [0,T]$. Hence $X(t)$ is described by:

$$X(0) = X_0 \; ; \qquad \dot{X}(t) = - Y(t), \qquad \text{for all } t.$$

Assume that storage after extraction is impossible, so that the rate of sales at any date is equal to the rate of extraction at that date. The market price is determined by the stationary inverse demand curve:

$$p(Y) = Y^{-\gamma}, \qquad\qquad 0 < \gamma < 1.$$

Since $\gamma < 1$, it follows that $|-1/\gamma| > 1$; the price elasticity of demand exceeds one in absolute value, and demand is elastic. The rate of interest, r, is constant over time, and interest is compounded continuously. Define $\rho \equiv r/\gamma$.

A crucial feature of the model is the following assumption: firms can make binding commitments about their extraction rates over a limited horizon. Define a <u>period</u> to be the length of time over which firms can make such commitments, and let z be the length of a period. Thus, the extraction rates for each firm are chosen as follows. At date $t = 0$, firm j chooses $[y_j(t),\ 0 \leq t \leq z]$; at date $t = z$ it chooses $[y_j(t),\ z < t \leq 2z]$, and so on. The parameter z may be regarded as the duration of a futures contract. The firms' decisions consist of selecting a sequence of futures contracts, signed at the dates z, $2z$, $3z, \ldots$, which specify deliveries of resources during the intervals $[0,z]$, $[z,2z]$, $[2z,3z], \ldots$, respectively. As will be shown below, the duration of a contract, z, will be a critical determinant of the rate of depletion of the resource.

First consider a finite-horizon game with K periods, each of length $z \in (0,\infty)$; thus $T = Kz$ is the terminal sales date. Let k index the number of periods remaining, so that $k = 1$ indicates the last period, $k = 2$ the next-to-last, and so forth.

At the beginning of each period, each firm chooses its extraction path for the current period. A firm's choice can depend on the number of periods remaining, k, the beginning-of-period stock, x, and the length of period, z.

Let $\{u_j^k(s;x,z),\ s\varepsilon[0,z]\}$ denote the path of extraction planned by firm j for period k.

Definition 1: In a K-period game with length of period z, a <u>strategy</u> for firm j is a sequence of functions $U_j^K \equiv \{u_j^k: [0,z] \times R_+ \times z \to R_+\}_{k=1}^K$, where u_j^k is piecewise continuous in its first argument, for $k = 1,\ldots,K$.

Let $\mathbf{U}_j^K(z)$ denote the space of all such strategies, and let $\mathbf{U}^K(z) \equiv \mathbf{U}_1^K(z) \times \ldots \times \mathbf{U}_J^K(z)$. Define $u^k \equiv (u_1^k,\ldots,u_J^k)$ and $U^K \equiv (U_1^K,\ldots,U_J^K)$. We will refer to U^K as the strategies for a K-period game and to u^k as the strategies for period k.

Define $V_j^k(U^k,x,z)$ to be the payoff of firm j in a k-period game with length of period z if the strategies U^k are played and the initial stock is x. V_j^k is defined recursively by letting $V_j^0 \equiv 0$, all j, and then letting:

$$V_j^k(U^k,x,z) \equiv \begin{cases} \displaystyle\int_0^z e^{-rs}\Big[\sum_{i=1}^J u_i^k(s;x,z)\Big]^{-\gamma} u_j^k(s;x,z)\,ds \\ \displaystyle\qquad + e^{-rz} V_j^{k-1}\Big(U^{k-1},\Big(x - \int_0^z \sum_{i=1}^J u_i^k(s;x,z)\,ds\Big),z\Big), \\ \displaystyle\qquad\qquad \text{if } \int_0^z \Big[\sum_{i=1}^J u_i^k(s;x,z)\Big]ds \leqslant x, \\ 0, \qquad\qquad \text{otherwise} \end{cases}$$

for all $U^k = (U^{k-1},u^k)\ \varepsilon\ \mathbf{U}^k(z)$; $x \geqslant 0$; $j = 1,\ldots,J$; $z > 0$; $k = 1,2,\ldots$.

Note that if in period k of a k-period subgame the firms adopt extraction paths that are mutually inconsistent, i.e., if they collectively plan to extract more than the available stock, then every firm's payoff in that subgame is zero. Technically, this will ensure the existence of equilibria composed of mutually consistent plans. Economically, it might correspond to a situation in which the extraction of a common property resource is "regulated" to the extent that the extractors are required to agree on mutually consistent plans or else forfeit all claims to the resource. Alternatively, it might be interpreted as a situation in which firms establish temporary property rights to shares of the resource.

Definition 2: The strategies $U^{K*} \equiv (U_1^{K*},\ldots,U_J^{K*}) \in \mathbf{U}^K(z)$ are a perfect Nash equilibrium of the K-period game with period of length z if:

$$V_j^k(U^{k*},x,z) \geq V_j^k(U_1^{k*},\ldots,U_{j-1}^{k*},U_j^k,U_{j+1}^{k*},\ldots,U_J^{k*},x,z) ,$$

$$\text{for all } U_j^k \in \mathbf{U}_j^k(z);\ x \geq 0;\ j = 1,2,\ldots,J;\ k = 1,2,\ldots,K;$$

where U_j^{k*} is the k-period truncation of U_j^{K*}.

Perfect Nash equilibrium strategies for any K-period game, for $1 \leq K \leq \infty$, can be found by using backward induction. For an isoelastic demand curve these strategies can be calculated explicitly; this is done in the Appendix. As shown there, aggregate extraction at the symmetric Nash equilibrium can be described as follows. There is a constant $\Psi(k,z)$ describing the proportion of the beginning-of-period stock that will be extracted during period k, given that z is the length of the period of commitment.

For fixed z, $\Psi(k,z)$ is described by the first-order difference equation:

$$\Psi(k+1,z) = \Psi(k,z)/\left[\Psi(k,z) + e^{-\rho z}\left(1 - \frac{J-1}{J-\gamma}\,\Psi(k,z)\right)^{1/\gamma}\right], \text{ all } k \geq 1; \qquad (1)$$

with initial condition $\Psi(1,z) = 1$. Aggregate extraction $\bar{u}^k(s;x,z) \equiv \sum u_j^{k*}(s;x,z)$, at any date s in period k, given the beginning-of-period stock x and period length z, is:

$$\bar{u}^k(s;x,z) = e^{-\rho s}\rho x\Psi(k,z)/(1 - e^{-\rho z}), \qquad 0 \leq s \leq z. \qquad (2)$$

How important is the length of the period of commitment? Fix the number of firms, $J > 1$, and consider a situation where the stock at date $t = 0$ is X_0, and the selling horizon is $T > 0$. From (1) and (2) we see that for <u>any</u> period length the symmetric equilibrium always has the following properties:

i) Aggregate extraction falls at the rate ρ and the price rises at the rate r within each period.

ii) The resource is exhausted exactly at date T.

Now suppose that the length of the period is T (so that there is only one period)[3], and consider total extraction over the time interval [0, T/2]. Using (2) and the fact that $\Psi(1) = 1$, we see that this is given by:

$$\int_0^{T/2} \bar{u}^1(s;X_0,T)ds = \frac{\rho X_0}{1-e^{-\rho T}}\int_0^{T/2} e^{-\rho s}\,ds$$

$$= X_0/(1 + e^{-\rho T/2}).$$

Now suppose instead that the length of the period is T/2. Then total extraction over the same time interval is just $X_0\Psi(2,T/2) = X_0A/(A + e^{-\rho T/2})$, where $A \equiv ((J-\gamma)/(1-\gamma))^{1/\gamma}$. Since $J > 1$ implies $A > 1$, it follows that extraction over $[0,z]$ is greater (and over $[z, 2z]$ is smaller) for the shorter commitment period. Computations with (1) show that as the length of the period shrinks further, extraction is increasingly concentrated early in the interval $[0,T]$, as shown in Figure 1.

Next, consider what happens as the length of the period approaches zero. For a fixed horizon T, this implies that the number of periods approaches infinity. Since (1) is stable, for any fixed, positive value for z, $\lim_{k\to\infty} \Psi(k,z) = \Psi^*(z)$, where $\Psi^*(z)$ is the solution of:

$$\left(1 - \Psi^*(z)\right)^{\gamma} = e^{-\rho z}\left(1 - \frac{J-1}{J-\gamma}\,\Psi^*(z)\right).$$

Since $(1 - \Psi)^{\gamma}$ is concave in Ψ over the relevant range, a solution exists and it is unique, as shown in Figure 2. Thus, if the horizon is infinite, aggregate extraction within any period is a constant proportion $\Psi^*(z)$ of the beginning-of-period stock.

The effect of letting the period length approach zero can now be seen from the following experiment. Let $[t,t+\delta]$ be any fixed interval of time, let x be the stock at date t, and let h be the number of periods in the interval $[t,t+\delta]$. Thus $z = \delta/h$, and as the length of the period approaches zero the number of periods grows without bound. As the length of the period approaches zero, the proportion of the stock extracted over the length of time δ is given by:

$$\lim_{h\to\infty} \sum_{i=0}^{h-1} \Psi^*(\delta/h)\left(1 - \Psi^*(\delta/h)\right)^{i} = \lim_{h\to\infty} \left[1 - \left(\Psi^*(\delta/h)\right)^{h}\right] = 1.$$

This is true for any $\delta > 0$. Thus, as the length of the period approaches zero, virtually the entire stock is extracted within an arbitrarily short length of time. Note that the limiting equilibrium strategies as $z \to 0$ are a perfect equilibrium in decision rules.

IV. Conclusion

It is evident that the length of the period of commitment can be a crucial determinant of perfect Nash equilibrium behavior. The uncritical use of path strategies, which corresponds to a single period of length $z = T$, would lead in this case to implausible conclusions regarding the efficiency of oligopolistic extraction of a common property resource: that it is perfectly efficient and is independent of the number of firms. On the other hand, the uncritical use of decision rule strategies, which corresponds to the assumption that no commitment is possible ($z = 0$), leads to results that are equally unrealistic: instantaneous extraction of the entire stock. The use of a commitment period that can be varied parametrically permits intermediate--and more plausible--outcomes. It also permits one to study how limitations on firms' abilities to commit themselves affects industry behavior at equilibrium.

The sharp results of the simple model above suggest that caution should be exercised when modeling economic problems as dynamic games. In particular, care should be taken to investigate the institutional context of the problem, to determine the extent to which opportunities for commitment are available to the relevant agents.

FOOTNOTES

[1]The last four papers describe situations in which the path and decision rule equilibria coincide. The two types of strategies are compared and contrasted in Kydland [1975].

[2]A related issue is that of dynamic inconsistency. In our context, dynamic consistency would require that along the equilibrium path through the game tree, the continuation of the Nash equilibrium strategies remains a Nash equilibrium. Subgame perfection is much stronger, requiring that this property hold at every subgame, not just those along the equilibrium path. For example, the open-loop Nash equilibrium is dynamically consistent, but not subgame perfect (for $K \geq 2$). Dynamic inconsistency is frequently a feature of games with a leader/follower structure. There is a sizable macro literature on this topic; see Newbery (1980) for an analysis of dynamic inconsistency in a resource extraction context.

[3]For an isoelastic demand curve and no costs, it happens that this path is independent of the number of firms, and also that it maximizes total surplus (see Weinstein and Zeckhauser [1975]). Thus oligopolistic extraction from a common property resource is socially efficient (in the sense of surplus-maximizing) if resource demand curves are isoelastic and futures contracts are of the same duration as the planning horizon.

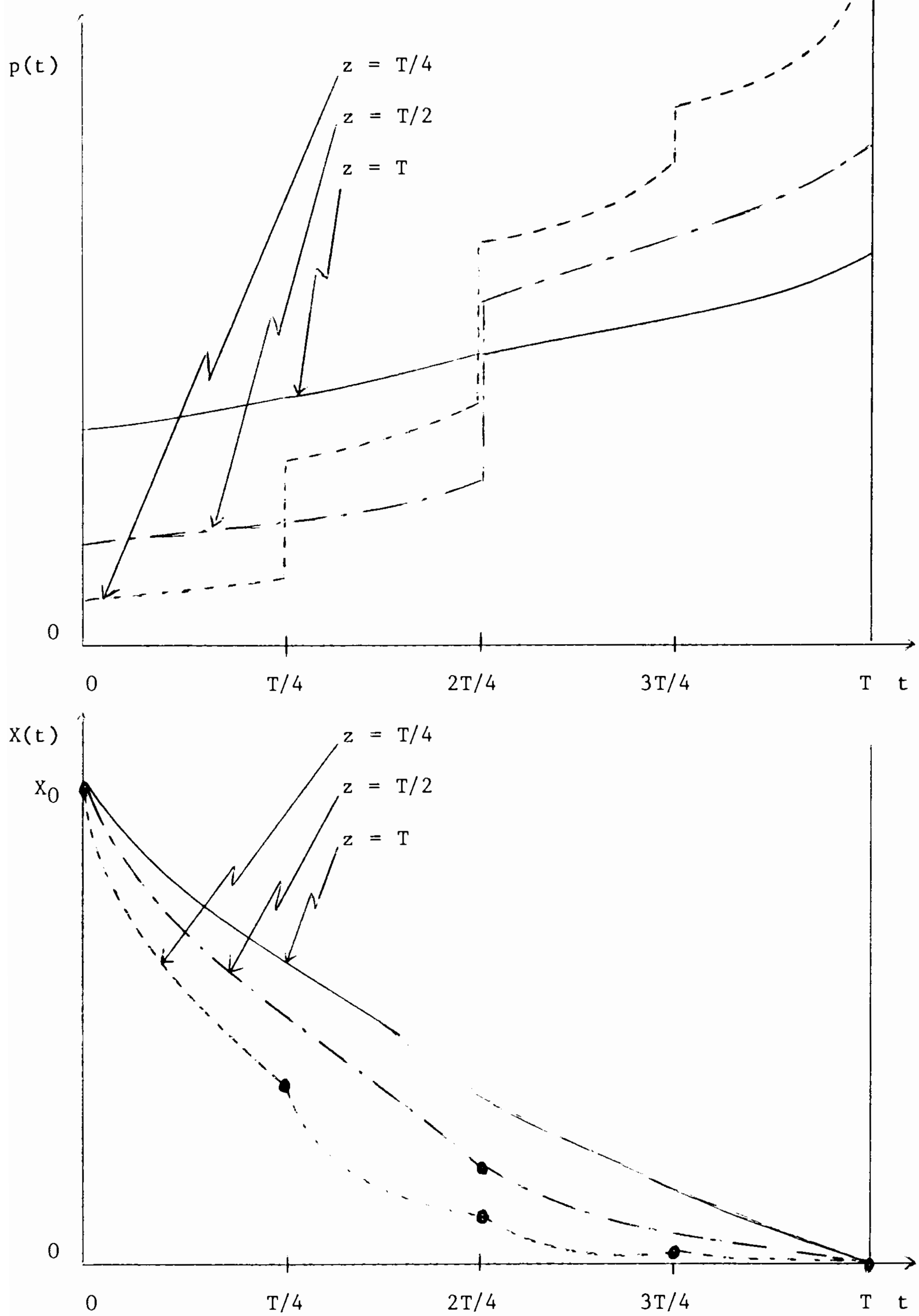
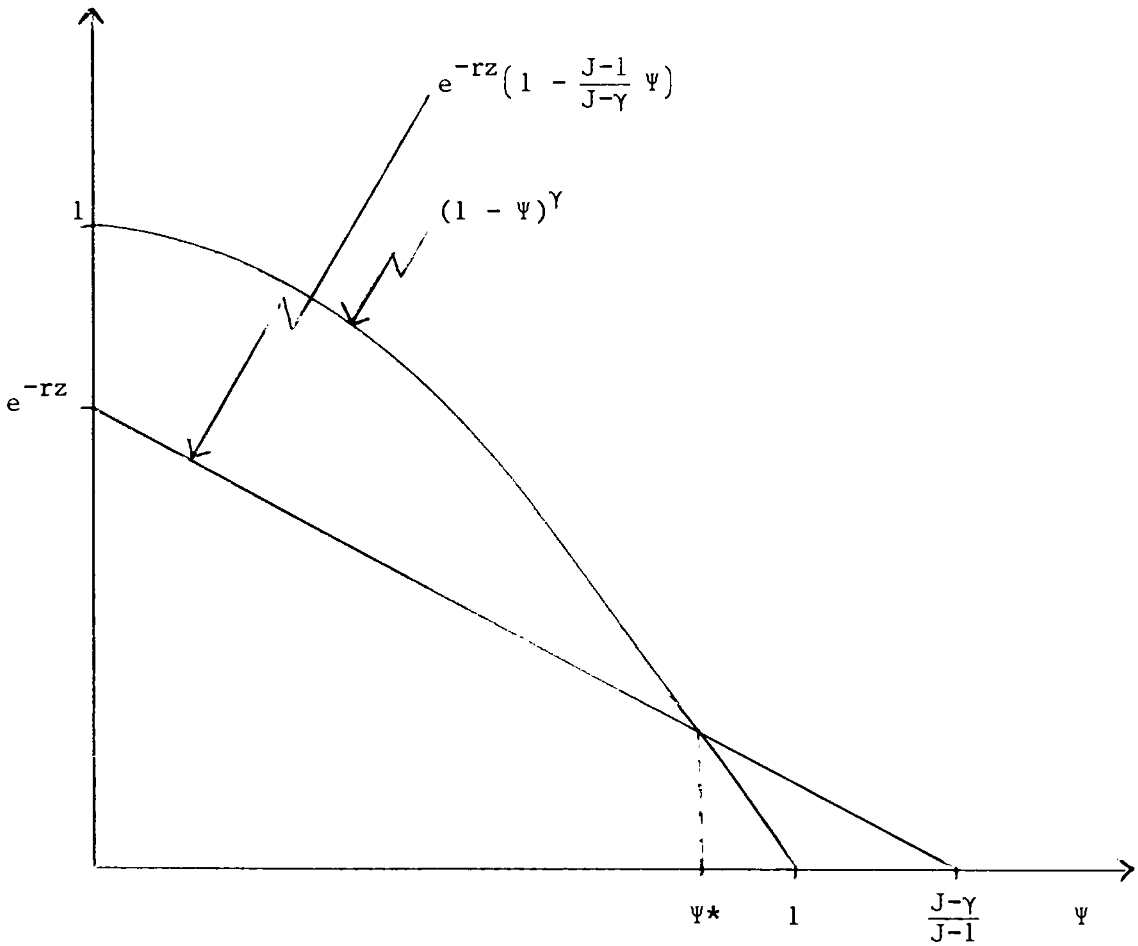


Figure 1



$e^{-rz}\left(1 - \frac{J-1}{J-\gamma}\,\Psi\right)$
$(1 - \Psi)^{\gamma}$
1
e^{-rz}
Ψ^*
1
$\frac{J-\gamma}{J-1}$
Ψ

Figure 2

APPENDIX

Fix the period of commitment z and temporarily suppress it as an argument of all functions. Given Nash equilibrium strategies $U^{(K-1)*}$ for the (K-1)-period subgame, we can find Nash equilibrium strategies for period K by using the theory of optimal control, with the discounted value function $e^{-rz}V_j^{K-1}(U^{(K-1)*},x')$ describing the value to firm j of having terminal resource stock x' at the end of period K. Equilibria will be described parametrically in x, the beginning-of-period stock.

For each firm j, define the Hamiltonian:

$$H_j(y,X,\lambda,s) = e^{-rs}\left(\sum_i y_i\right)^{-\gamma} y_j - \lambda_j \sum_i y_i ,$$

where $u_j^1(s;x)$, $X(s;x)$, and $\lambda_j(s;x)$ are the control, state and costate variables respectively, if the initial stock is x. Nash equilibrium strategies for period K, u^{K*}, must satisfy:

$$\partial H_j/\partial y_j \Big|_{y=u^{K*}} = e^{-rs}\left(\sum_i u_i^{K*}\right)^{-\gamma}\left(1 - \gamma\, u_j^{K*} / \sum_i u_i^{K*}\right) - \lambda_j = 0 , \tag{1a}$$

$$\dot{\lambda}_j = - \partial H_j/\partial X = 0 , \tag{1b}$$

$$\dot{X} = - \sum_i u_i^{K*}, \quad X(0;x) = x, \tag{1c}$$

$$\lambda_j(z;x) = \partial\left(e^{-rz}V_j^{K-1}\left(U^{(K-1)*},x'\right)\right)/\partial x' \Big|_{x'=X(z;x)} , \tag{1d}$$

$$\lambda_j(z;x) \geq 0, \qquad X(z;x) \geq 0, \qquad \lambda_j(z;x)X(z;x) = 0, \tag{1e}$$

where (1e) holds if K = 1, and (1d) otherwise.

First consider the case K = 1. Then (1e) states that either the resource stock is exhausted by the end of the period, $X(z;x) = 0$, or else the remaining stock is valueless, $\lambda_j(z;x) = 0$. Since (1a) cannot hold for $\lambda_j = 0$, it follows that $X(z;x) = 0$, for all $x \geq 0$; the stock is always exhausted at the end of the last (K = 1) period.

Since (1a) holds for all firms j, summing over j, using (1b), and using the boundary condition:

$$X(z;x) = x - \int_0^z \sum_j u_j^{1*}(s;x)ds = 0,$$

one finds that:

$$\bar{u}^1(s;x) \equiv \sum_{j=1}^{J} u_j^{1*}(s;x) = e^{-\rho s}\rho x/\left(1-e^{-\rho z}\right), \qquad \text{all } s,x. \tag{2}$$

Over the period aggregate extraction, $\bar{u}^1(s;x)$, falls at the rate ρ, so that the price rises at the rate of interest. Note that the aggregate extraction rate is homogeneous of degree one in x.

Although (2) is consistent with many sets of extraction paths for the individual firms, for simplicity we will focus on the symmetric equilibrium. The value function for each firm then is:

$$V^1(x) = cx^{1-\gamma}, \qquad \text{all } x;$$

where the subscript j and the equilibrium strategies have been suppressed, and $c \equiv \left((1-e^{-\rho z})/\rho\right)^{\gamma}/J$. Note that the value function V^1 is homogeneous of degree $(1-\gamma)$ in x.

Symmetric equilibria for longer games can be found recursively. Let $U^{(K-1)*}$ be a symmetric equilibrium for the (K-1)-period game, where $K > 1$, and assume that the value function for each firm associated with that equilibrium, $V^{K-1}(x)$, is homogeneous of degree $(1-\gamma)$ in x. We will show that there exists a unique set of strategies for period K, u^{K*}, such that $U^{K*} = (U^{(K-1)*}, u^{K*})$ is a Nash equilibrium for the K-period game; that symmetry of the equilibrium strategies persists; and that homogeneity of the (common) value function persists. Thus the assumption needed to begin the next stage of the induction will be satisfied.

Suppose that symmetric Nash equilibrium strategies for the (K-1)-period subgame are given, and that the associated (common) value function, $V^{K-1}(x)$, is homogeneous of degree $(1-\gamma)$. Then we can define v(K-1) by:

$$v(K-1) \equiv V^{K-1}(x)/cx^{1-\gamma}, \tag{3}$$

Equilibrium strategies for period K must satisfy (1a)-(1d). Since the value functions are identical for all firms, it follows from (1d) that $\lambda_j = \lambda$, for all j; from (1a) it then follows that the equilibrium strategies for period K will be symmetric. Moreover, since (3) holds, (1b) and (1d) together imply that:

$$\lambda(s;x) = (1-\gamma)e^{-rz}cv(K-1)X^{-\gamma}(z;x), \qquad \text{all } s,x,j. \tag{4}$$

Substituting (4) into (1a) and summing over j we find that:

$$\left(\bar{u}^{K}\right)^{-\gamma} = J\left(\frac{1-\gamma}{J-\gamma}\right)e^{r(s-z)}cv(K-1)X^{-\gamma}(z;x), \tag{5}$$

where $\bar{u}^{K}(s;x) \equiv \sum_j u_j^{K*}(s;x)$ is aggregate exteraction at any date s in period K. Raising each side of (5) to the power $-1/\gamma$, integrating over s, and substituting $X(z;x) = x - \int_0^z \bar{u}^{K}ds$, we find that:

$$\int_0^z \bar{u}^{K}ds = Ae^{\rho z}v^{-1/\gamma}(K-1)\left(x - \int_0^z \bar{u}^{K}ds\right), \tag{6}$$

where $A \equiv \left((J-\gamma)/(1-\gamma)\right)^{1/\gamma} > 1$. From (6) we see that $\int \bar{u}^{K}ds$, aggregate extraction over period K, is homogeneous of degree one in x, the beginning-of-period stock. Thus we can define $\Psi(K)$ to be the proportion of the beginning-of-period stock that is extracted within the period, $\Psi(K) \equiv \int \bar{u}^{K}ds/x$. Solving (6) we find that:

$$\Psi(K) = \frac{Ae^{\rho z}}{Ae^{\rho z} + v^{1/\gamma}(K-1)}. \tag{7}$$

Then, using (5) and (7), we find that:

$$\bar{u}^{K} = \rho e^{-\rho s}\Psi(K)x/(1 - e^{-\rho z}), \qquad \text{all s.} \tag{8}$$

The aggregate rate of extraction at any date during period K is homogeneous of degree one in x, and within the period aggregate extraction falls at the rate ρ, so that the price rises at the rate of interest. Since symmetry of the equilibrium strategies persists so does symmetry of the value functions, and since u^{K*} is homogeneous of degree one in x, it follows immediately that V^{K} is homogeneous of degree $(1-\gamma)$. From (7),(8) and the definitions of V^{K} and v(K) we find that:

$$v(K) = \Psi^{1-\gamma}(K) + e^{-rz}\left(1 - \Psi(K)\right)^{1-\gamma} v(K-1). \tag{9}$$

Solving (7) for v(K) and substituting into (9) to eliminate v(K) and v(K-1), we find that $\Psi(K)$ is completely characterized by the first-order difference equation:

$$\Psi(K+1) = \Psi(K)/[\Psi(K) + e^{-\rho z}\left(1 - \frac{J-1}{J-\gamma}\,\Psi(K)\right)^{1/\gamma}], \qquad \text{all } K \geq 1, \tag{10}$$

with initial condition $\Psi(1) = 1$.

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