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On the Informational Requirements for the Implementation of Social Choice Rules

by

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1. Introduction

A central issue in modern welfare economics is the investigation of alternative communication and coordination schemes that an organization might employ in order to achieve desirable outcomes.

Since the relevant knowledge about an organization is generally not in the hands of a central institution but dispersed among the participants, some form of communication has to take place, if certain performance criteria are to be met. The idea of a communication and coordination scheme was first formalized by Hurwicz (1959 and 1972) and Mount and Reiter (1974) through the notion of a resource allocation mechanism. The principle question then was to design informationally efficient mechanisms that minimize the size of the exchanged message units over all allocation mechanisms meeting the same performance criteria.

Numerous authors have shown that the competitive mechanism is informationally best in the class of all mechanisms that yield Pareto-optima for an exchange economy. One feature of this theory is that agents are assumed to be honest. This means that the participants follow the rules of the process rather than exploit the fact that strategic behavior might manipulate the mechanism to their advantage.

It is exactly this point that the incentive literature focuses on. The central question there is to examine what kind of social choice rules can be implemented by a game form. It turns out that for most behavioral equilibrium notions one can, without loss of generality, restrict attention to those game forms where the participants are asked to communicate their entire characteristics. The outcome then is determined on the basis of full information about the environment. For many authors this result, often referred to as the revelation principle, served as a justification for considering revelation processes, see for example Groves and Loeb (1975), Green and Laffont (1979), d'Aspremont and Gérard-Varet (1979), Myerson (1979), Satterthwaite and Sonnenschein (1980). Even though the revelation principle is helpful in determining what kind of social choice rules can be implemented, it completely disregards the informational aspect of the problem. In many important cases it might suffice to transmit a "sufficient statistic" rather than the full characteristics.

In my opinion a theory that attempts to provide criteria for choosing among alternative mechanisms has to take information and incentive aspects into consideration simultaneously. In other words a satisfactory theory should determine the informational requirements for the implementation of social choice rules.

Results in this direction are of twofold interest. Given a particular equilibrium notion, i.e. a notion of incentive compatibility, one might be in a position to analyze the informational "cost" of incentive compatible mechanisms as opposed to mechanisms that achieve the desired performance standards only if the participants are honest. This would illustrate a potential trade-off between self-enforcing mechanisms and manipulative schemes that might be enforced by some costly monitoring or auditing policy. Secondly such a theory might allow for at least partial comparison between the various notions of incentive compatibility. Every equilibrium concept requires some degree of sophistication and understanding on parts of the agents. Since the mechanism performs satisfactorily only if the behavioral equilibrium is reached, there is always a certain chance of failure. Again, one might find a trade-off between mechanisms that involve a "higher chance" of failure and mechanisms that reduce this chance, but necessitate the communication of larger message units.

This paper is structured as follows: In section two the essentials of the theory on resource allocation mechanisms are reproduced. The major results of the paper are contained in section three. As an application of the general theory we will consider a multidivisional firm that has to allocate a scarce resource among its divisions. In this context it is shown in section four that incentive compatibility requirements necessarily lead to an increase in the size of the message space.

2. Allocation Mechanisms

It is assumed that there is a set of agents $N = \{1...n\}$. All the relevant knowledge concerning agent i is specified by a point $e_i \varepsilon E_i$, where E_i denotes the space of possible characteristics. A social choice rule P associates with each environment $e = (e_1...e_n)$ a set of outcomes:

$$P : \overset{m}{\underset{i=1}{X}} E_i \twoheadrightarrow A$$

The following definitions are presented and discussed extensively in Mount and Reiter (1974), Walker (1978) and Osana (1977).

<u>Definition 2.1.</u>: Let $E \equiv \overset{m}{\underset{i=1}{X}} E_i$ and X be topological spaces; the message correspondence $\mu : E \twoheadrightarrow X$ is said to be privacy preserving, if there exist correspondences $\mu_i : E_i \twoheadrightarrow X$ such that

$$\mu(e) = \bigcap_{i=1}^{m} \mu_i(e_i) \quad \forall e \in E$$

The idea behind definition 2.1. is that, if the actual environment is $e \in E$, then agent i knows his own characteristics with certainty and therefore his messages can be based only on $e_i \in E_i$.

<u>Definition 2.2.</u>: A correspondence $\mu : X \twoheadrightarrow Y$, where X and Y are topological spaces, is locally threaded, if for every $x \in X$ there exists a neighborhood U(x) and a continuous function $t : U(x) \to Y$ such that $t(x') \in \mu(x')$ for all $x' \in U(x)$.

<u>Definition 2.3.</u>: The triple $\Lambda = \langle M, \mu, h \rangle$ is said to be a Mount-Reiter allocation mechanism, if $\mu : E \twoheadrightarrow M$ is a privacy preserving message correspondence from environments to the message space M and h is a function from M to A such that h is constant on $\mu(e)$ for all $e \in E$.

This specification of an allocation mechanism is the one step analog of an iterative adjustment process as first introduced by Hurwicz (1959). Λ is said to realize the social choice rule $P : E \twoheadrightarrow A$, if for all $e \in E$

$$(h \circ \mu)(e) \in P(e)$$

The set $\mu_i(e_i)$ should be interpreted as the collection of those messages, that, from i's point of view, are possible equilibrium messages. We call the mechanism Λ regular, if the message correspondence μ is locally threaded.[1]

Definition 2.4.:[2] Let M and M^* be topological spaces. M has as much information as M^*, denoted by $M \trianglerighteq M^*$, if there exists a subspace $M' \subset M$ such that $M' \cong M^*$, i.e., M' and M^* are homeomorphic.

Definition 2.4. provides a measure for the informational size of the message space.

3. Implementation by an Allocation Mechanism

Following Gibbard's (1973) approach the incentive literature examined whether particular social choice rules can be

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[1]Mount and Reiter (1974) discuss why the local threadedness condition is one way to rule out "smuggling" of information.

[2]This concept of Frechet size has been introduced by Walker (1978).

implemented in certain equilibrium strategies by a game form. In this paper two central equilibrium notions : dominant strategy equilibria and Bayesian equilibria, which are both applicable to the revelation principle, will be considered.

Denote by $\Pi_i(\cdot|e_i)$ an ordinal utility function that represents agent's i preferences over A given that his characteristics are $e_i \varepsilon E_i$.

a) Dominant Strategy Equilibria

Definition 3.1.:[1] The allocation mechanism $\Lambda = \langle M, \mu, h \rangle$ implements the social choice rule $P : E \twoheadrightarrow A$ in dominant strategies, if:

$$\forall i \varepsilon N \ \forall e_i \varepsilon E_i \ \exists \ e_i^* \varepsilon E_i \ \forall e_{-i} \varepsilon E_{-i}$$

a) $$\mu_i(e_i^*) \ \varepsilon \ \underset{m_i \varepsilon M_i}{\operatorname{argmax}} \ \Pi_i(h(m_i \cap \mu_{-i}(e_{-i}))|e_i)$$

$$\text{where } M_i = \{m_i \subset M | m_i = \mu_i(\bar{e}_i) \text{ for some } \bar{e}_i \ \varepsilon \ E_i\}$$

$$\mu_i(e_{-i}) = \bigcap_{\substack{j=1 \\ j \neq i}}^{n} \mu_i(e_i)$$

b) $h(\mu(e^*)) \ \varepsilon \ P(e)$ for every $e^* \ \varepsilon \ \rho(e)$

[1] Throughout this page $e_{-i} = (e_1 \cdots e_{i-1}, e_{i+1} \cdots e_n)$ and $(e_{-i}, e_i) = (e_1 \cdots e_{i-1}, e_i, e_{i+1} \cdots e_n)$

$$\text{where } \rho(e) = \{e^* \in E \mid e^* \text{ satisfies a) for all } i \in N \text{ given that the environment is } e \in E\}$$

Note that by definition $\rho(e)$ can be written as a cartesian product of individually dominating types, so that $\rho(e) = \overset{n}{\underset{i=1}{X}} \rho_i(e_i)$. The implementation is said to be proper, if ρ is globally threaded, i.e., ρ contains a continuous function.

The idea behind definition 3.1. is that an agent may pretend to be of any conceivable type, but in principle has to follow the rules of the process. We recall that a choice rule $P : E \twoheadrightarrow A$ can be implemented in dominant strategies by a game form, if there exists a pair $\langle S, g \rangle$ such that $S = \overset{n}{\underset{i=1}{x}} S_i$, $g : S \rightarrow A$ and

a) $\forall i \varepsilon N \; \forall e_i \varepsilon E_i \; \exists \; s_i^* \varepsilon S_i \; \forall s_{-i} \varepsilon S_{-i}$

$$s_i^* \; \varepsilon \; \underset{\bar{s}_i \, \varepsilon \, S_i}{\operatorname{argmax}} \; \Pi_i(g(\bar{s}_i, \; s_{-i}) \mid e_i)$$

b) $g(s^*) \; \varepsilon \; P(e)$ for all $s^* \; \varepsilon \; S$ that satisfy a) for all $i \; \varepsilon \; N$ at $e \; \varepsilon \; E$.

With an allocation process the outcome function can be based on the aggregated messages $\mu(e)$ whereas for a game from the domain of the outcome function is the cartesian product of the individual strategy spaces. Implementation by a game form is the more general concept because whenever $\Lambda = \langle M, \mu, h \rangle$ implements P in dominant strategies, one can construct the following game form:

$$S_i = M_i = \{m_i \subset M \mid m_i = \mu_i(\bar{e}_i) \text{ for same } \bar{e}_i \in E_i\}$$

and $g(s_1 \ldots s_n) = h(\bigcap_{i=1}^{n} s_i)$. $\langle S, g \rangle$ implements P in dominant strategies. Conversely, if $\langle S, g \rangle$ implements P, then there exists a regular allocation mechanism for which the "truth", i.e., sending messages according to one's true type, is a dominant strategy.

It is the purpose of this paper to outline how one can establish bounds on the size of the message space that is needed for the implementation of a social choice rule. Considering definition 3.1., it is not obvious whether implementing the rule P is informationally more "involved" than just realizing P. One even might imagine that the entire set of dominating types, i.e., $\rho(E)$, is contained in a "small" subenvironment so that the mechanism effectively only would have to work on this subenvironment.

Denote by

$$F = \{f : E \to A \mid f(e) \in P(e) : \forall i \in N\ \forall e_i \in E_i\ \forall e_i \in E_i\ \forall e_{-i} \in E_i$$
$$\Pi_i(f(e) \mid e_i) > \Pi_i(f(\bar{e}_i, e_{-i}) \mid e_i)\}$$

Green and Laffont (1979) call a choice function $f \in F$ s.i.i.c. (strongly individually incentive compatible).

<u>Lemma 3.2.</u>: Let $\mu : X \twoheadrightarrow Y$ be a locally threaded correspondence and $\rho : X \twoheadrightarrow X$ be a globally threaded correspondence. Then $\mu \circ \rho : X \twoheadrightarrow Y$ defined by: $\mu(\rho(x)) = \bigcup_{x' \varepsilon \rho(x)} \mu(x')$ is locally threaded.

<u>Proof</u>: Denote by $\omega : X \to X$ a continuous function such that $\omega(x) \varepsilon \rho(x)$ for all $x \varepsilon X$.
Let $\bar{x} \varepsilon X$ and $U(\omega(\bar{x}))$ be a neighborhood of $\omega(\bar{x})$ such that $t : U(\omega(\bar{x})) \to Y$ is a local thread of μ. Since ω is continuous, $\omega^{-1}(U(\omega(\bar{x})))$ is a neighborhood of $\bar{x}$ and $t o \omega$ is a local thread of $\mu o \rho$ on $\omega^{-1}(U(\omega(\bar{x})))$.

<u>Theorem 3.3.</u>: If $\Lambda = \langle M, \mu, h \rangle$ is a proper dominant strategy implementation of P on E, then there exists a regular realization of P using the same message space M.

<u>Proof</u>: Λ being proper yields that $\rho(e) = \mathop{X}_{i=1}^{n} \rho_i(e_i)$ is globally threaded. Define $\Lambda' = \langle M, \mu', h \rangle$ by choosing $\mu'_i = \mu_i o \rho_i$. Obviously μ' is privacy preserving and by lemma 3.2. also locally threaded. Part b) of definition 3.1. implies that Λ actually realizes P on E.

The significance of theorem 3.3. is that dominant strategy implementation requires a message space at least as large as realization of P does. The following result offers a way to establish lower bounds on the size of the message space for dominant strategy implementations.

Theorem 3.4.: Let $\bar{E}$ be a subspace of E and denote by Y a normed linear space. Suppose that $g : \bar{E} \to Y$ is a continuous, locally sectioned[1] function such that $g(\bar{E})$ contains a set G that is homeomorphic to a linear subspace of Y. If for any regular mechanism $\Lambda = \langle M, \mu, h \rangle$ that realizes a function $f \in F$:

$$g(e) \neq g(\bar{e}) \Rightarrow \mu(e) \cap \mu(\bar{e}) = \emptyset \quad \forall e, \bar{e} \in \bar{E}$$

then every proper dominant strategy implementation of P uses a message space that has as much information as G.

Proof: Choose $a \in G \subset g(\bar{E})$. Since g is locally sectioned, there exists a set $U(a) \subset G$ such that U(a) is open relative to G, and a function $\phi : U(a) \to \bar{E}$ satisfying : $(g \circ \phi)(a') = a'$ for all $a' \in U(a)$.

Consider any mechanism $\Lambda = \langle M, \mu, h \rangle$ that realizes a function $f \in F$. Denote by s a local thread of μ on the neighborhood $V(\phi(a))$.

Since Y is a normed linear space, there exists a compact

[1]A function is locally sectioned if its inverse is locally threaded.

neighborhood (relative to G) $U^* \subset U(a)$ such that $\phi(U^*) \subset V(\phi(a))$.

Consider the mapping $s \circ \phi : U^* \to M$. It is continuous and if it can be verified that the mapping is 1-1, then U^* and $(s\circ\phi)(U^*)$ are homeomorphic.[1]

By construction ϕ is 1-1 from $U(a)$ to $\phi(U(a))$ and $g(e) \neq g(\bar{e})$ for all $e, \bar{e} \in \phi(U(a))$. By hypothesis this implies $\theta(e) \cap \theta(\bar{e}) = \emptyset$ and therefore $s(e) \neq s(\bar{e})$, i.e., s is 1 - 1 on $U(\phi(a))$.
It will be shown in the appendix that U^* has as much information as G. Let $\Gamma = \langle Z, v, q \rangle$ be a proper dominant strategy implementation of P. Then we can construct a regular mechanism
$\Gamma' = \langle Z, v', q \rangle$ that realizes a function $f \in F$, as shown in theorem 3.3. Therefore it follows that $Z \trianglerighteq G$.

The above theorem says that, if one constructs a sufficiently regular function on a subset of E and verifies that every realization of a s.i.i.c. function in P "reveals"[1] as much as g on $\bar{E}$, then the range of g gives a lower bound for the size of the message space for any dominant strategy implementation. The crucial

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[1] This is Lemma 8, page 18, in Dunford and Schwartz (1957).

[1] Compare Reiter (1974) for the exact definition which corresponds to the condition in theorem 3.4.

aspect about 3.4. is that every dominant strategy implementation gives rise to a realization that uses the same message space.

The following definition is of particular relevance in economic models.

Definition 3.5.: A choice function $f \in F$ is said to be individually sensitive on E, if

$$\forall i \in N \; \forall e_i \in E_i \; \forall \bar{e}_i \in E_i \; \exists \breve{e}_{-i} \in E_{-i}$$

$$\Pi_i(f(e_i, \breve{e}_{-i})|e_i) \neq \Pi_i(f(\bar{e}_i, \breve{e}_{-i})|e_i)$$

Theorem 3.7.: Let g be as in theorem 3.4. If $\Lambda = \langle M, \mu, h \rangle$ is a regular realization of an individually sensitive choice rule $f \in F$ such that $\mu(E) \cong G$, then G is the informationally efficient message space size among all proper dominant strategy implementations of P.

Proof: According to theorem 3.4. G gives a lower bound. Since Λ realizes an $f \in F$, the "truth" is a dominating type and by individual sensitivity the unique dominant strategy. Hence Λ is a proper dominant strategy implementation of P using a message space that is homeomorphic to $g(\bar{E})$.

b) Bayesian Equilibria

For the Bayesian equilibrium concept the line of arguments is practically identical. We will restate the results without proof.

Denote by $B(E_{-i})$ the Borel σ-algebra on E_{-i} and let τ_i be a probability measure on $B(E_{-i}) \times E_i$. τ_i represents player i's beliefs about the other participants given that player i is of type $e_i \in E_i$. The functional form of the τ_i is assumed to be known a priori.

Definition 3.8.: The allocation mechanism $\Lambda = \langle M, \mu, h \rangle$ implements $P : E \twoheadrightarrow A$ in Bayesian equilibrium strategies, if there exist functions $\gamma_i : E_i \to E_i$, $i \in N$, such that $\forall i \in N$ $\forall e_i \in E_i$

a) $\mu_i(\gamma_i(e_i)) \in \underset{m_i \in M_i}{\operatorname{argmax}} \int_{E_{-i}} \Pi\,(m_i \cap \mu_{-i}(\gamma_{-i}(e_{-i})) | e_i)\ \tau_i(de_{-i} | e_i)$

b) $h(\mu(\bar{\gamma}(e))) \in P(e)$ for all $e \in E$ whenever

$\bar{\gamma} \in \Gamma \equiv \{(\gamma_1 \ldots \gamma_n) | \gamma_i : E_i \to E_i \text{ satisfies a) for all } i \in N\}$

The Bayesian implementation is said to be proper, if there exists a $\gamma \in \Gamma$ that is continuous. In the above definition $\Pi_i(\cdot | e_i)$ should be interpreted as a v. Neumann-Morgenstern utility function. Define:

$$\hat{F} = \{\hat{f} : E \to A | \hat{f}(e) \in P(e) \text{ for all } e \in E \text{ and } \forall i \in N\ \forall e_i \in E_i\ \forall \bar{e}_i \in E_i$$

$$\int_{E_{-i}} \Pi_i(\hat{f}(e_i, e_{-i}) | e_i) \tau_i(de_{-i} | e_i) \geq \int_{E_{-i}} \Pi_i(\hat{f}(\bar{e}_i, e_{-i}) | e_i) \tau_i(de_{-i} | e_i)\}$$

d'Aspremont and Gerard-Varet (1980) call functions $\hat{f} \in \hat{F}$ B.i.c. (Bayesian incentive compatible).

Theorem 3.9.: If $\Lambda = \langle M, \mu, h \rangle$ is a proper Bayesian implementation of P on E, then there exists a regular allocation mechanism $\Lambda' = \langle M, \mu', h \rangle$ that realizes P on E.

The important aspect of theorem 3.9. is that Λ and Λ' use the same message space.

Theorem 3.10: Denote by Y a normed linear space and let $\bar{E}$ be a subset of E. Consider a function $g : \bar{E} \to Y$, continuous and locally sectioned such that $g(\bar{E})$ contains a subset G that is homeomorphic to a linear subspace of Y. If for any regular mechanism $\Lambda = \langle M, \mu, h \rangle$ that realizes a function $\hat{f} \in \hat{F}$:

$g(e) \neq g(\bar{e}) \Rightarrow \mu(e) \cap \mu(\bar{e}) = \emptyset \quad \forall e, \bar{e} \in \bar{E}$,

then every proper Bayesian implementation of P uses a message space that has as much information as G.

4. An economic application

To illustrate the relevance of our results we will consider a simple economic model of a multidivisional firm that has to allocate a scarce resource among n divisions. Let ω be a positive number representing the overall available quantity of the resource. We take:

$$E_i = \{e_i : [0,\omega] \to R_+ \mid e_i \in C^{(1)}([0, \omega]),\ e_i \text{ monotone increasing and strictly concave, } e_i(\omega) \leq K\}$$

By choosing $d(e, \bar{e}) = \sum_{i=1}^{n} \max_{x \varepsilon [0,\omega]} |e_i(x) - \bar{e}_i(x)|$

(E,d) becomes a compact metric space. The interpretation of this environment is that the divisions can use the resource productively and the e_i represent profit functions.

Let $A \equiv \Delta \times \mathbb{R}^n$ where $\Delta = \{(x_1 \ldots x_n) | x_i \geq 0, \sum_{i=1}^{n} x_i \leq \omega\}$
The i-th division's utility functions is of the form
$\Pi_i((x,t)|e_i) = e_i(x_i) + t_i$

An outcome consists of a feasible allocation and monetary transfers for the divisions. Utility is fully transferable in this model so that efficient and incentive compatible mechanisms may exist.

If the social choice rule would only require that the resource is allocated in a jointly efficient way, then the competitive mechanism, using a message space that is homeomorphic to $\mathbb{R}^n$, would be the informationally efficient realization.

$$\text{Define } P(e) = \{(x,t) \ \varepsilon \ A | \sum_{i=1}^{n} t_i = 0, \ x \ \varepsilon \ \underset{\bar{x} \varepsilon \Delta}{\text{argmax}} \sum_{i=1}^{n} e_i(\bar{x}_i)\}$$

P is exactly the Pareto correspondence. It is well known, see for example Walker (1980), that there exists no mechanism that implements P in dominant strategies.[1)] But as d'Aspremont and Gerard-Varet (1979) show, P can be implemented in Bayesian strategies, if the players' beliefs can be made part of the mechanism. In particular, we

[1]If one drops the requirement of balanced transfers, then dominant strategy mechanisms will exist. A related model has been studied in Reichelstein (1980).

will adopt the well-known independence condition:

$$\tau_i(\cdot|e_i) = \tau_i(\cdot|\bar{e}_i) \text{ for all } e_i, \bar{e}_i \in E_i$$

A possible interpretation of the independence condition is that the characteristics are drawn independently from an underlying probability distribution that all players agree upon.

In order to analyze the impact of the Bayesian incentive compatibility requirement on the size of the message space, we will first need a characterization of $\hat{F}$.

Laffont and Maskin (1979) and d'Aspremont and Gerard-Varet (1979) have established that

$$\hat{f} \in \hat{F} \text{ if and only if } \hat{t}_i(e) = u_i(e_i) - \sum_{j \neq i} \frac{1}{n-1} u_j(e_j) + h_i(e)$$

where $\hat{f}(e) \equiv (\hat{x}(e), \hat{t}(e))$, $\hat{x}(e)$ denotes the unique optimal allocation,

$$(1) \quad u_i(e_i) \equiv \int_{E_{-i}} \sum_{j \neq i} T_j(e_i, \tilde{e}_{-i}) \tau_i(d\tilde{e}_{-i}) \text{ with}$$

$T_j(e_i, e_{-i}) = e_j(\hat{x}_j(e_i, e_{-i}))$ and finally the $\{h_i\}_{i=1}^n$ are arbitrary functions that satisfy:

$$\sum_{i=1}^n h_i \equiv 0$$

and $\int_{E_{-i}} h_i(e_i, \tilde{e}_{-i}) \, \tau_i(d\tilde{e}_{-i})$ is constant for all $e_i \in E_i$.

It turns out to be difficult to invoke theorem 3.10. at this point because of the undeterminate structure of the $\{h_i\}$ functions.

We call an allocation mechanism $\Lambda = \langle M,\mu,h \rangle$ discretionary with respect to the transfers (or just discretionary), if

$$\forall e \in E \ \forall_i \in N \ \exists \ \hat{f}^1_{t_i} : E_i \to \mathbb{R}, \ \hat{f}^2_{t_i} : E_{-i} \to \mathbb{R} \text{ such that}$$

$$\hat{f}_{t_i}(e) = \hat{f}^1_{t_i}(e_i) + \hat{f}^2_{t_i}(e_{-i})$$

Restricting attention to discretionary mechanisms means that for the Bayesian implementations of the Pareto rule the rest terms $\{h_i\}$ are not only independent in expectation but actually independent of $e_i \in E_i$.

<u>Proposition 4.1.</u>: If $\Lambda = \langle M, \mu,h \rangle$ is a proper and discretionary Bayesian implementation of P on E, then $M \trianglerighteq \mathbb{R}^{2n}$

To demonstrate proposition 4.1., we construct the following subenvironment

$$\bar{E}_i = \{e_i \in C^1([0,\omega]) | e_i(x) = K_i[a_i x - x^2/2], \ a_i > \omega, \ K_i > 0, \ (a_i,K_i) \in B\}$$

where B is an open ball in $\mathbb{R}^2$ such that for every $e \in \bar{E}$: $\hat{x}(e) \gg 0$

Let $g : \bar{E} \to \mathbb{R}^{2n+1}$ be given by $g(e) = (p(e),\hat{x}(e), u_1(e_1)\ldots u_n(e_n))$. $p(e)$ denotes the Lagrange multiplier of the resource constraint, $\hat{x}(e)$ the optimal allocation and the $u_i(e_i)$ are as defined in (1).

One can verify that $g(\overline{E})$ is homeomorphic to $\mathbb{R}^{2n}$ (since $\sum \hat{x}_i(e) = \omega$). Furthermore g is continuous on $\overline{E}$, because the T_j are uniformly continuous on $\overline{E}$ and therefore the u_i are continuous.

In view of theorem 3.10. it needs to be verified that every regular and discretionary mechanism $\Lambda = \langle M, \mu, h \rangle$ that realizes a function $\hat{f} \in \hat{F}$ is such that

$$g(e) \neq g(\overline{e}) \Rightarrow \mu(e) \cap \mu(\overline{e}) = \emptyset \ \forall e, \overline{e} \in \overline{E}$$

Suppose that $g_x(e) \neq g_x(\overline{e})$, i.e. $\hat{x}(e) \neq \hat{x}(\overline{e})$. Then it must be that $\mu(e) \cap \mu(\overline{e}) = \emptyset$, otherwise $h(\mu(e)) = h(\mu(\overline{e}))$ and Λ does not realize a function $\hat{f} \in \hat{F}$.

If $p(e) \neq p(\overline{e})$ the assumption $\mu(e) \cap \mu(\overline{e}) \neq \emptyset$ leads to the following contradiction:

$\mu(e) \cap \mu(\overline{e}) \neq \emptyset \Rightarrow \mu(e) \cap \mu(\overline{e}_i, e_{-i}) \neq \emptyset$ for all $i \in N$, since μ is privacy preserving. Moreover $\hat{x}(e) = \hat{x}(\overline{e}) = \hat{x}(\overline{e}_i, e_{-i})$ and therefore:

$$p(\overline{e}_i, e_{-i}) = e_j'(\hat{x}_j(\overline{e}_i, e_{-i})) = p(e) \neq p(\overline{e}) = \overline{e}_i'(\hat{x}_i(e))$$

$$= \overline{e}_i'(\hat{x}_i(\overline{e}_i, e_{-i})) = p(\overline{e}_i, e_{-i})$$

Finally assume $u_i(e_i) \neq u_i(\overline{e}_i)$. Since Λ is discretionary and realizes a function $\hat{f} \in \hat{F}$ we know from the above characterization that

$$(2) \qquad (h \circ \mu)_{t_i}(e) = u_i(e_i) - \frac{1}{n-1} \sum_{j \neq i} u_j(e_j) + \overline{h}_i(e_{-i})$$

Then $\mu(e) \cap \mu(\overline{e}) \neq \emptyset$ would yield $(h \circ \mu)_{t_i}(e) = (h \circ \mu)_{t_i}(\overline{e}_i, e_{-i})$ by the fact that μ is privacy preserving. But this contradicts the structure of (2).

It needs to be verified that g is locally sectioned.

Let $m \in g(\bar{E})$ and consider a "small" neighborhood of m, say U(m). We want to find a function $\phi : (g(\bar{E}) \cap U(m)) \to \bar{E}$ that is continuous and $(\mu \circ \phi) = id|_{g(\bar{E}) \cap U(m)}$.

For a point $\bar{m} = (\bar{p}, \bar{x}, \bar{y}) \in g(\bar{E}) \cap U(m)$ there is an n-dimensional manifold in B^n that gives rise to $(\bar{p}, \bar{x})$. These remaining n variables are determined by $(\bar{u}_1, \ldots \bar{u}_n)$ and the outlined function will be continuous and invertible.

There exists a mechanism $\bar{\Lambda} = \langle \bar{M}, \bar{\mu}, \bar{h} \rangle$ realizing P on E for which the identity is a Bayesian equilibrium and $\bar{M} \cong \mathbb{R}^{2n}$.

This mechanism is given by:

$$\bar{\mu}_i(e_i) = \{(p, x_1, \ldots x_n, u_1, \ldots u_n) | \Sigma x_j = \omega, x_j \geq 0, \; x_i \in \psi_i(p|e_i),$$

$$u_i = \int_{E_{-i}} \sum_{j \neq i} T_j(e_i, \tilde{e}_{-i}) \; \tau_i(d\tilde{e}_{-i})$$

where $x_i \in \psi_i(p|e_i)$ if $x_i \in \underset{\bar{x}_i \geq 0}{\text{argmax}} \{e_i(\bar{x}_i) - p\bar{x}_i\}$

Clearly $\bar{\mu}$ is a privacy preserving, continuous function. By choosing

$$\bar{h}(\bar{\mu}(e)) = (x_1, \ldots x_n, t_1, \ldots t_n) \text{ with } t_i = u_i - \frac{1}{n-1} \sum_{j \neq i} u_j$$

we obtain a mechanism that leads to Pareto optimal outcomes.

5. Summary and Discussion

For a given choice rule $P : E \twoheadrightarrow A$ we have been concerned with the informational requirements for incentive compatible implementations. If the solution concept is such that the revelation principle holds, i.e., we consider direct mechanisms, then the following line of arguments applies:

i) Every imlementation gives rise to a "comparable" realization, therefore the informationally efficient realization provides a lower bound for all implementations.

ii) One constructs a sufficiently regular function g on a subset of E and verifies that every realization of a function $f \in F$ (the set of choice functions in P for which the "truth" always is an equilibrium strategy) distinguishes between any two points that g distinguishes between.

iii) The image of g under $\overline{E}$ gives a lower bound for the informational size of the message space of any implementation of P.

The essential fact about direct mechanisms is that the equilibrium correspondence "factors", which permits the construction of the "new" privacy preserving correspondence. If the solution concept had been Nash equilibria for example, our approach would not succeed any more. Up to this point there does not exist a theory for the analysis of non-direct mechanisms.

Appendix

We will show that $U^* \triangleright G$ as claimed in the proof of theorem 3.3.

Let Y' be a linear subspace of Y and denote by $H : G \to Y'$ a homeomorphism. Take $W_\varepsilon(a)$ to be an ε ball around a in Y such that $V_\varepsilon(a) \equiv (W_\varepsilon(a) \cap G) \subset U^*$. The claim is shown, if it can be verified $V\varepsilon(a) \cong G$.

Define the mapping $\overline{H} : V_\varepsilon(a) \to Y'$ by $\overline{H}(y) = H(y) - H(a)$. Clearly, $\overline{H}$ yields a homeomorphism between $V_\varepsilon(a)$ and $\overline{H}(V_\varepsilon(a))$. Since $\overline{H}^{-1}$ is continuous there exists a $\delta > 0$ such that $U_\delta(0) = \{y \in Y' \mid \|y\| < \delta\} \subset \overline{H}(V_\varepsilon(a))$. It remains to establish that $U_\delta(0) \cong Y'$.

Consider the continuous function $\phi : U_\delta(0) \to Y'$ defined by

$$\phi(y) = \frac{1}{\delta - \|y\|}\, y$$

Clearly ϕ is 1-1. For $\bar{z} \in Y'$ choose $\bar{y} = \delta \cdot \bar{z}\left(1 - \frac{\|\bar{z}\|}{1+\|\bar{z}\|}\right)$ to see that ϕ is onto.

The mapping $\phi^{-1}(\bar{z}) = \delta \cdot \bar{z}\left(1 - \frac{\|\bar{z}\|}{1+\|\bar{z}\|}\right)$ is the continuous inverse of ϕ.

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