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STOCHASTIC TECHNOLOGY, PRODUCTION PLANS AND THE THEORY OF THE FIRM*

by

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ABSTRACT

This paper is concerned with the formation of production plans by multi-product firms that face a stochastic technology. Two types of planning under uncertainty are considered: informed planning and dynamic informed planning. The first case represents static plans formation wherein inputs must be chosen before a random variable is revealed, but output decisions can be delayed until the state of nature is known. In the second case the firm progressively learns about the state of nature as it produces. Moreover, it can use resources to adjust the distribution on the state of nature. In both cases we show that a nondynamic expected technology exists that embodies all relevant information for plans formation. Implications for duality theory, econometric studies of technology and models of the firm with expected vs. actual supply are discussed.

1. Introduction

This paper is concerned with the formation of production plans by multi-product firms. Specifically, we will characterize plan formation for the case wherein the firm's technology is stochastic. This is in contrast to much of the literature, which has exhaustively examined problems of randomly varying output in terms of external factors: random demand or random input prices (see, for example, Sandmo [1971]). In the case at hand randomness in output will arise from purely internal reasons: the technological relationship between inputs and outputs (and amongst inputs and outputs) will be viewed as depending on an unknown state of nature.

Previous literature is sparse. Rothenberg and Smith [1971] provided a general equilibrium notion of stochastic prices arising from stochastic inputs faced by the firm. Feldstein [1971] and Olivera [1973] have considered the one-ouput stochastic production problem. Henn and Krug [1973] consider multiple output stochactic production correspondences, but do not couple the analysis with the firm as expected profit maximizer so as to derive a characterization of production plans. Mak [1981] has developed a notion of a "confidence indexed production correspondence" for a stochastic technology. He provides a model of production plans in the one output case. Finally econometricians have, for some time now, been attempting to measure various types of inefficiencies in a stochastic technology framework (see, e.g., Aigner and Schmidt [1980].

No one, however, has formalized these notions in a model of a multi-product firm that gives rise to a characterization of plans (i.e. planned levels of output and input purchases). Since firm supply (and general firm behavior) depends upon costs incurred, and these in turn depend upon commitments made to purchase inputs, a formal characterization of output/input plan formation is clearly called for. Moreover, none of the above papers has considered the firm is a general stochastic, dynamic environment where resources can be used to influence the distribution of the state of nature as well as produce output.

To say that a firm chooses levels of inputs and outputs (i.e. sets plans) so as to maximize expected profits presupposes a notion of what to take a mathematical expectation over. This, in turn, hinges on when and how randomness enters the planning process. In what follows we will consider two cases:

(1) Informed planning, wherein the random variable is observed after choosing inputs but before picking the output mix.

(2) Dynamic informed planning, wherein the random variable is sequentially revealed and wherein resources can be used to adjust the distribution on the state of nature as well as produce output.

In section two we consider the first case, developing a notion of static plans. We show how to construct the expected transformation function and we explore the duality between stochastic technology and revenue. We show that, in a certain sense, the introduction of randomness into the technology requires the employment of a nonsymmetric duality.

Section three considers firms in a dynamic, stochastic environment. Here we find that all relevant (i.e., to plans formation) information about the firms technology and optimal choices can be captured by a deterministic, static representation. Finally section four provides a summary and conclusions.

2. Stochastic Technology and Production Planning

2.1 Stochastic Technology

We consider a firm that uses inputs[1] $x \varepsilon R^n_+$ to make outputs $z \varepsilon R^m_+$ according to the state of nature $\omega \varepsilon \Omega$. Inputs are purchased at given prices $q \varepsilon \Gamma^n_{++}$ while outputs are sold at given prices $p \varepsilon \Gamma^m_{++}$. The firm's technology is summarized by the function $T: R^m_+ \times R^n_+ \times \Omega \Longrightarrow R$ and is written as $T(z,x,\omega)$.

Definition 1 Let $W(x,\omega) = \{z \varepsilon R^m_+ | \ T(z,x,\omega) \leq 0\}$ for $x \varepsilon R^n_+$, $\omega \varepsilon \Omega$. $W(x,\omega)$ is called the set of feasible outputs for the pair $(x,\omega) \varepsilon R^n_+ \times \Omega$. The firm's production possibilities set is $Y = \{(z,x,\omega) \varepsilon R^m_+ \times R^n_+ \times \Omega \mid z \varepsilon W(x,\omega)\}$.

We assume the following properties for $T(z,x,\omega)$ and $W(x,\omega)$:

A1. $W(x, \omega)$ is a non-empty, compact, convex subset of R^m_+ for each $(x,\omega) \varepsilon R^n_+ \times \Omega$. Moreover, $W(x,\omega) \subseteq A(x) \ \forall \omega \varepsilon \Omega$ where $A(x)$ is a compact subset of R^m_+.

A2. $W(0, \omega) = \{0\} \quad \forall \omega \varepsilon \Omega$.

A3. $T(\cdot,\cdot,\omega) \ \varepsilon \ C^2 \ \forall \omega \varepsilon \Omega$, $\nabla_z T(z,x,\omega) > 0$, $\nabla_x T(z,x,\omega) < 0$ $\forall (z,x,\omega) \varepsilon Y$.

A4. $\forall (x,\omega) \varepsilon R^n_+ \times \Omega$, $\forall z^1, z^2 \ \varepsilon W(x,\omega)$, $z^1 \neq z^2$ such that $T(z^1,x,\omega) = 0 = T(z^2,x,\omega)$, $\nabla_z T(z^1,x,\omega) \neq \nabla_z T(z^2,x,\omega)$.

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1 $R^k_+ = \{x \varepsilon R^k | x_i \geq 0, \ i=1,\ldots,k\}$, $R^k_{++} = \{x \varepsilon R^k | x_i > 0, \ i=1,\ldots,k\}$, $\Gamma^k_+ = \{x \varepsilon R^k_+ | \Sigma x_i = 1\}$, $\Gamma^k_{++} = R^k_{++} \cap \Gamma^k_+$

Condition (1) states that level sets in the output space are convex, non-empty and compact for each (x, ω) in $R_+^n \times \Omega$; this is a regularity condition. Furthermore, condition (1) eliminates certain pathological cases that might imply unbounded output for a finite input level. For example $W(x,\omega) = \{z \varepsilon R_+^m | \; z'z \leq x'x/\omega\}$, $\omega \varepsilon (0,1]$ is eliminated, since it would imply that there exists $\omega \varepsilon \Omega$ that allows arbitrarily high production levels for finite x. Condition (2) implies that positive output cannot be produced if all inputs are zero. Condition (3) is an efficiency of production condition; for efficient production $T(z,x,\omega) = 0$. Also, the transformation function is twice differentiable in x and z. Condition (4), along with (1) and (3), implies that the transformation surface in R_+^m is strictly concave and that each point on the surface has associated with it a unique gradient. Figure 1 represents the feasible output sets for ω_0, ω_1 $\varepsilon\Omega$, $m = 2$, and for a given level of inputs $\bar{x} \varepsilon R_+^n$.

Finally, we assume (A5) that the state of nature is an element of a probability space $\{\Omega,A,P\}$ where A is a σ- algebra on sets of Ω and P is a probability measure.

For convenience we shall refer to $\{T,P\}$ as the technology structure since the notation carries all the relevant information about technology, namely the transformation function $T(z,x,\omega)$ and the probability measure P.

2.2. Informed Plans

As stated above, this case corresponds to sequence: choose x, observe ω, choose z. Thus, maximization of expected profits for given $(p,q)\varepsilon\Gamma_{++}^m x \Gamma_{++}^n$ is expressed as

$$\max_x \; E[\max_z \; p'z - q'x].$$
$$\text{S.T. } \{T,P\}$$

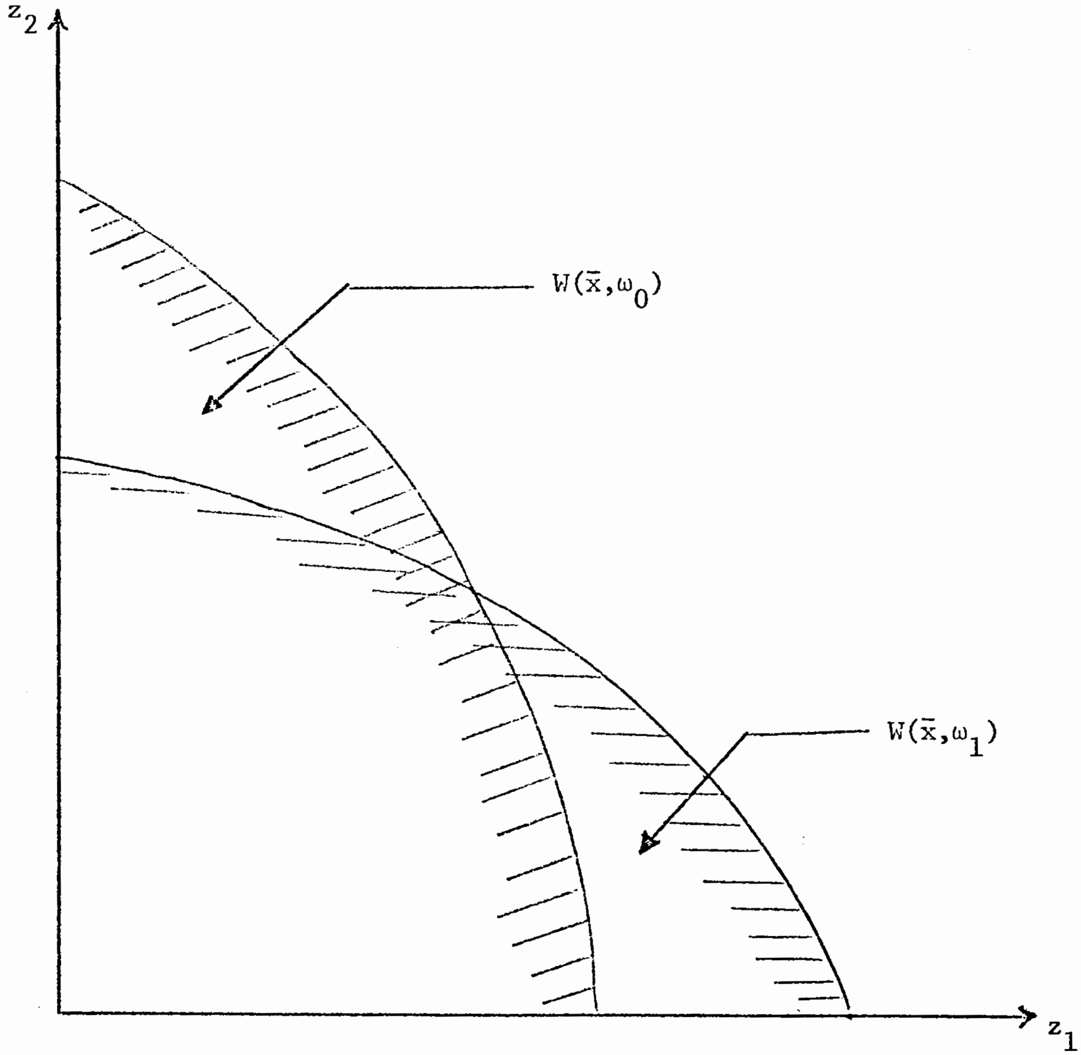

FIGURE 1

The inner-most problem is straightforward: for given $\bar{x}$ and $\bar{\omega}$ we wish to find a solution to the following problem:

$$\max_{z} p'z - q'\bar{x}$$
$$\text{S.T. } T(z,\bar{x}, \bar{\omega}) \leq 0.$$

This first order conditions imply that

$$p = \mu \nabla_z T(z,\bar{x}, \bar{\omega})$$

where the multiplier μ satisfies $\mu \varepsilon R_+$ and $\mu T(z,\bar{x},\bar{\omega}) = 0$ at the optimal z. Note that, in fact, $\mu = \mu(\bar{x},\bar{\omega})$ since changing $\bar{x}$ and $\bar{\omega}$ can result in a different multiplier. Since the expectation is taken over values of ω, we know that for a given $\bar{x}$, the first order conditions imply

$$p = \mu(\bar{x},\omega) \nabla_z T(z,\bar{x},\omega) \qquad \forall_{\omega \varepsilon \Omega} .$$

Hence, for each $\omega_o,\omega_1 \varepsilon \Omega$, optimality is characterized by equality of the marginal rate of transformation (MRT) over states of nature, i.e.

$$\frac{\partial T(z,\bar{x},\omega_o)/\partial z_i}{\partial T(z,\bar{x},\omega_o)/\partial z_j} = \frac{\partial T(z,\bar{x},\omega_1)/\partial z_i}{\partial T(z,\bar{x},\omega_1)/\partial z_j} \qquad \forall_{i,j} .$$

Thus the expectation can be viewed as being taken on a path segment in output space of constant MRT; this is illustrated in Figure 2. The figure shows two transformation surfaces for given input level $\bar{x}$ and states of nature ω_o and ω_1. The two points z^1 and z^2 have equal MRT. As a function of the probability measure P (and assuming $\Omega = \{\omega_o,\omega_1\}$), the expected output would be some particular convex combination of z^1 and z^2. By varying the MRT (or, equivalently, the price vector p), a surface in R^m_+ is traced out that relates input level x to expected (or <u>planned</u>) output ζ, which we will refer to as the planning surface $\tau(\zeta,x)$. The planning surface can then be

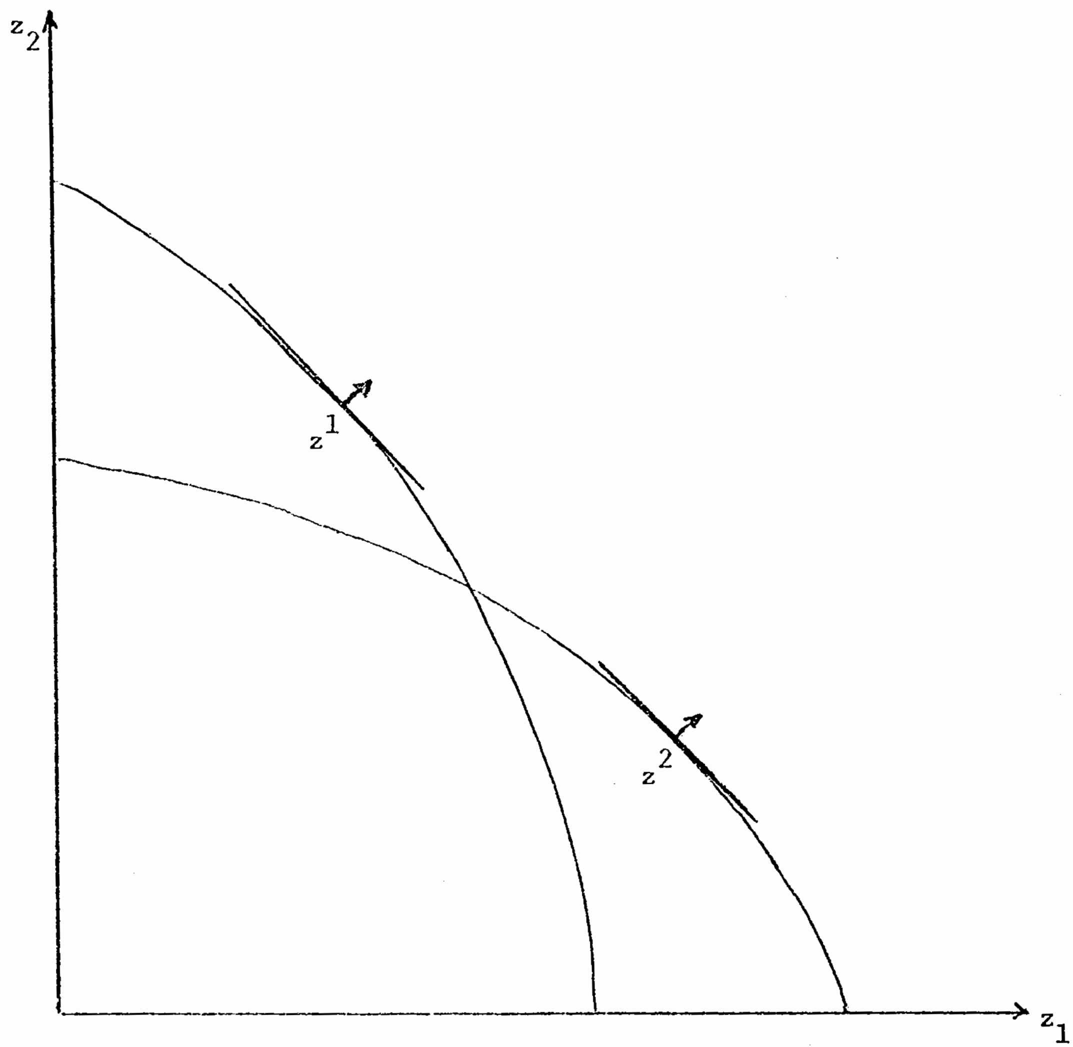

FIGURE 2

used in conjunction with a given set of prices p and q to provide an optimal informed plan (ζ^*, x^*). In this sense $\tau(\zeta, x)$ acts like a standard transformation surface in the deterministic case.

To make the foregoing more precise, we define $\tilde{W}(x,\omega)$ as the <u>comprehensive set</u> associated with $W(x, \omega)$ $(\tilde{W}: R^n_+ \times \Omega \dashrightarrow R^m)$:

$$\tilde{W}(x,\omega) = \{z \varepsilon R^m | \exists_{z' \varepsilon W(x,\omega)},\ z' \geq z\}.$$

To construct $\tilde{W}(x,\omega)$, we simply pick points in $W(x,\omega)$ and then take all vectors z that are less then or equal to the picked point. This extends $W(x,\omega)$ to all of R^m. It is trivial to show that $\tilde{W}(x,\omega)$ is convex and that it contains the negative orthant R^m_-.

Let

$$\Psi = \{z: R^n_+ \times \Omega \dashrightarrow R^m_+ | \text{ z is P-integrable and } z(x,\omega) \varepsilon \tilde{W}(x,\omega)\}$$

i.e. Ψ is the set of selections from $\tilde{W}(x,\omega$ (i.e., random variables) that are P-integrable. The integral of the sets $\tilde{W}(x,\omega)$ is $\tilde{W}(x)$ (see, for example, Hildenbrand [1974]):

$$\tilde{W}(x) = \int_\Omega \tilde{W}(x,\omega)P(d\omega) = \{ \int_\Omega z(x,\omega)P(d\omega): z\varepsilon\Psi\};$$

Moreover $\tilde{W}(x)$ is convex. Furthermore for all $p\varepsilon\Gamma^k_+$:

$$\sup\{p'\zeta | \zeta \varepsilon \tilde{W}(x)\} = \int_\Omega \sup\ \{p'y | y\varepsilon \tilde{W}(x,\omega)\}P(d\omega),$$

i.e. the surface of $\tilde{W}(x)$ is a convex combination of points of equal MRT drawn from the surfaces of the sets $\tilde{W}(x,\omega)$. Since we are only interested in nonnegative plans, we define $W(x)$, the set of expected outputs as

$$W(x) = \{\zeta \varepsilon R^m_+ \mid \zeta \varepsilon\ \tilde{W}(x)\}.$$

Since $\widetilde{W}(x)$ is convex, so is $W(x)$. Figure 3 illustrates $W(x)$ for a two state example. The surface of $W(x)$ is the planning surface and is recovered as follows. Let, for $(\zeta,x) \varepsilon R_+^m \times R_+^n$:

$$D(\zeta,x) \equiv \min \{\lambda\varepsilon R_+ | \ \zeta/\lambda\varepsilon W(x)\},$$

and define the expected transformation function τ: $R_+^m \times R_+^n \rightarrow R$ as

$$\tau(\zeta,x) = D(\zeta,x) - 1$$

<u>Definition 2</u> $(\zeta,x) \varepsilon R_+^m \times R_+^n$ is a <u>feasible informed plan</u>

if $\tau(\zeta,x) \leq 0$.

As an example of finding $\tau(\zeta,x)$, let $W(x,\omega)$ be defined as follow:

$$W(x,\omega) = \{z\varepsilon R_+^m \ | \ \omega^2 z_1^2 + z_2^2 - x^2 \leq 0\}.$$

Moreover, assume P is the uniform density on the interval $[a,b]$, $b>a>0$. With some effort, it can be shown that

$$\tau(\zeta,x) = \zeta_1 - \frac{x}{b-a} \ln [b(u_1^{.5} + u_2)/a(u_1^{.5} + u_3)]$$

where

$$u_1 = \zeta_2^4(b-a)^2 + x^4(b+a)^2 - 2x^2\zeta_2^2(b^2 + a^2),$$

$$u_2 = x^2(b+a) - \zeta_2^2(b-a)$$

and

$$u_3 = x^2(b+a) + \zeta_2^2(b-a).$$

While not transparent, $\tau(\zeta,x)$ has output level sets as indicated in Figure 3.

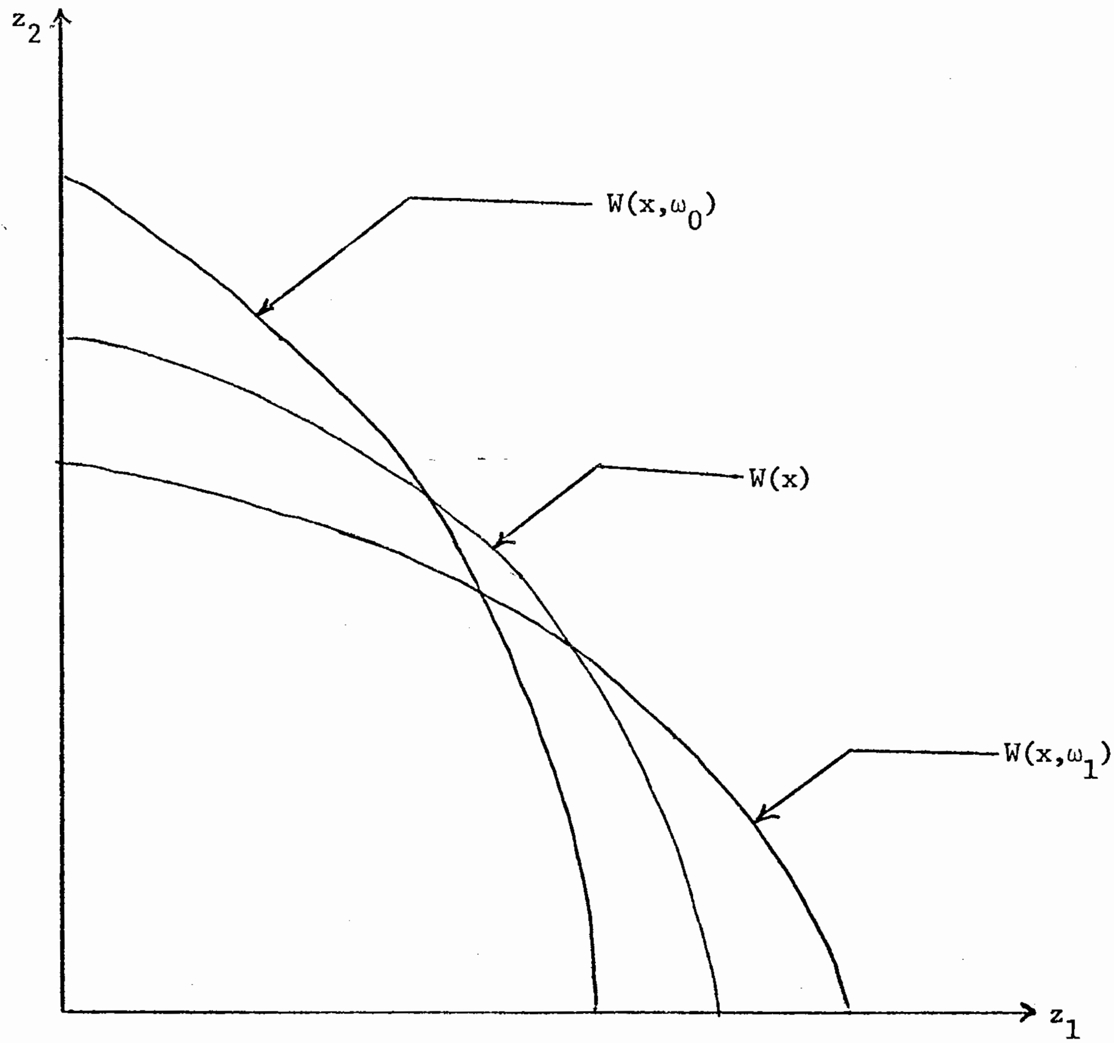

FIGURE 3

Finally, to construct optimal plans, we return to our original expected profit maximization problem which is now expressed as:

$$\pi(p,q) = \max_{x} \max_{\zeta} \{p'\zeta - q'x \mid \tau(\zeta,x) \leq 0\}$$

The function $\pi(p,q)$ represents expected profit and thus is non-stochastic. Following standard theory we have the following result via Hotelling's Lemma (see Varian [1978]):

<u>Theorem 1</u> The optimal informed plan is the m+n vector (ζ^*,x^*) where

$$\zeta^*(p,q) = \nabla_p \pi(p,q)$$
$$x^*(p,q) = -\nabla_q \pi(p,q).$$

2.3 Stochastic Duality for the Informed Planning Case

Duality relationships between production, cost and profit functions are well established for the deterministic case (see Fuss and McFadden [1978] or Shephard [1970]). When randomness enters the picture, a new level is added to the potential dual relationships. One might consider duality relationships between a stochastic technology description and a stochastic cost or profit function, and one might also consider duality relationships between the expected technology and various cost or profit functions.

Depending upon the source of the randomness, complete duality relationships may, or may not, be describable. We will examine duality at both the stochastic and expected levels for a firm facing a stochastic technology $T(z, x, \omega)$, a probability measure P, known input prices q and known output prices p. In general we will find that duality is incomplete: stochastic duality does not give rise to an expectational duality, except in special cases.

2.3.1 Incomplete Duality

The firm faces a stochastic transformation function $T(z, x, \omega)$, probability measure P, output prices p and input prices q. Define the stochastic distance function $D(z, x, \omega)$ as

$$D(z, x, \omega) = \{\min_{\lambda>0} \lambda \mid T(\lambda^{-1}z, x, \omega) \leq 0\}.$$

$D(z, x, \omega)$ is non-negative, continuous, convex, positive linear homogeneous (PLH) function of $z \varepsilon R^m_+$ (Fuss and McFadden [1978]). $D(z, x, \omega) = 1$ describes points on the surface of $W(x, \omega)$ while $D(z, x, \omega) < 1$ describes points in the interior of $W(x, \omega)$:

$$W(x, \omega) = \{z \varepsilon R^m_+ \mid D(z, x, \omega) \leq 1\}.$$

Dual to $D(z, x, \omega)$, for given x and ω, is the stochastic revenue function $R(p, x, \omega)$:

$$R(p, x, \omega) = \{\max_z p'z \mid z \varepsilon W(x, \omega)\} = \{\max_z p'z \mid D(z, x, \omega) \leq 1\}$$

This function is a non-negative, continuous[2], convex, PLH function of $p \varepsilon \Gamma^m_+$. Since $W(x, \omega)$ is convex, there is a one-to-one correspondence between $W(x, \omega)$ and $R(p, x, \omega)$, i.e. given $R(p, x, \omega)$ we can recover $W(x, \omega)$.

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[2] To be more precise, continuity is on the interior of the domain, but extension to the boundary is often performed (see Blackorby, Primont and Russell [1978]). To simplify exposition we will simply use the word continuous, since the issues to be discussed do not concern jumps at the boundary.

The expected output set W(x) is derived as before:

$$W(x) = \int_{\Omega} W(x, \omega)\ P(d\omega)$$

and, the revenue function associated with W(x), namely R(p, x) is simply

$$R(p, x) = \{\max_{\zeta} p'\zeta \mid \zeta \in W(x)\}.$$

R(p,x) is the integral of the stochastic revenue function, i.e.

$$R(p, x) = \int_{\Omega} R(p, x, \omega)\ P(d\omega),$$

and is thus a non-negative, continuous, convex, PLH function of $p \in \Gamma_+^m$. From standard duality theory we know that dual to R(p, x) is a distance function on W(x) which we denote as $D(\zeta, x)$:

$$D(\zeta, x) = \{\min_{\lambda>0} \lambda \mid \zeta/\lambda \in W(x)\}.$$

Unfortunately, in general, $D(\zeta, x)$ is not the integral of $D(z, x, \omega)$, i.e.

$$D(\zeta, x) \neq \int_{\Omega} D(z, x, \omega)\ P(d\omega).$$

This is because the construction of W(x) involves picking the "right" random outputs $z(x, \omega)$ (those of equal MRT) and integrating them to provide expected output. To see that the integration does not hold, consider the following two-state stochastic technology:

$$T(z, x, \omega) = (1 + \omega)z_1^2 + (2 - \omega)z_2^2 - x^2,$$

$\omega = 0$ or 1 with equal probability. For fixed x, the output sets for the two states of the world are symmetric about the 45° line. It is easy to show that $\zeta_1 = \zeta_2 = \sqrt{3/8}\ x$ belongs to the surface of W(x) (by letting $p_1 = p_2 = 1/2$ and solving) and that $T(\sqrt{3/8}\ x, \sqrt{3/8}\ x, x, \omega) > 0$ for both states of the world, i.e. $D(z, x, \omega) > 1$ at $z = (\sqrt{3/8}\ x, \sqrt{3/8}\ x)$. Thus, integrating the stochastic distance function does <u>not</u> provide the expected distance function $D(\zeta, x)$ since $D(\zeta, x) = 1$ at $\zeta = (\sqrt{3/8}\ x, \sqrt{3/8}\ x)$.

Figure 4 illustrates the results described above. The horizontal lines represent duality relationships that come from standard (deterministic) analysis. The one vertical arrow reflects that only the revenue function integrates to a function of interest. The duality is <u>incomplete</u> in the sense that $D(z, x, \omega)$ does not, in general, integrate to $D(\zeta, x)$.

2.3.2 Completing the Duality: State-Homothetic Technology

The fact that the standard distance functions do not integrate in general does not mean that it is impossible to find special classes of technology with distance functions that do. Consider the following definition[3]:

<u>Definition</u>: The transformation $T(z, x, \omega)$ represents a <u>state-homothetic</u> technology if there exists a function $\Phi: \Omega \to R_{++}$ which attains a minimum at $\omega_0 \in \Omega$, with $\Phi(\omega_0) = 1$, such that

$$T(z, x, \omega) = T(z/\Phi(\omega), x, \omega_0).$$

Thus, a state-homothetic technology is one wherein, for fixed input level x, points on the transformation surface associated with state ω are simply

[3] Spady [1981] uses a similar definition (for the single product case) to consider econometric estimation of distance function for firms that imperfectly optimize.

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
D(z,\, x,\, \omega) \arrow[r, shift right, "\{\max\limits_{z} p'z \mid D(z,\, x,\, \omega) \leq 1\}"'] & R(p,\, x,\, \omega) \arrow[l, shift right, "\{\max\limits_{p} p'z \mid R(p,\, x,\, \omega) \leq 1\}"'] \arrow[d, "\int R(p,\, x,\, \omega)\, P(d\omega)"] \\
D(\zeta,\, x) \arrow[r, shift right, "\{\max\limits_{\zeta} p'\zeta \mid D(\zeta,\, x) \leq 1\}"'] & R(p,\, x) \arrow[l, shift right, "\{\max\limits_{p} p'\zeta \mid R(p,\, x) \leq 1\}"']
\end{tikzcd}
$$

FIGURE 4

Incomplete Duality for Stochastic Technology fixed ray expansions of points on a reference surface (state ω_0). In fact, it is easy to show that:

$$W(x, \omega) = \Phi(\omega)\, W(x, \omega_0).$$

Note that this implies that MRT is preserved along rays from the origin. Let

$$\tilde{D}(z, x, \omega) = \{\max_{\sigma>0} \sigma \mid T(\sigma z, x, \omega) \leq 0\}$$

and let

$$\tilde{D}(\zeta, x) = \{\max_{\sigma>0} \sigma \mid \sigma\zeta \in W(x)\}$$

For fixed x and ω, $\tilde{D}(\zeta, x)$ and $\tilde{D}(z, x, \omega)$ are non-negative continuous, concave positive homogeneous of degree minus one functions of $\zeta \in R^m_+$ and $z \in R^m_+$ respectively. Moreover:

$$\tilde{D}(z, x, \omega) = 1/D(z, x, \omega)$$

and

$$\tilde{D}(\zeta, x) = 1/D(\zeta, x).$$

The following theorem provides completeness for the duality.

<u>Theorem 2</u>: If $T(z, x, \omega)$ is state-homothetic then

$$\tilde{D}(\zeta, x) = \int_\Omega \tilde{D}(\zeta, x, \omega)\, P(d\omega).$$

<u>Proof</u>: It is easy to show that

$$\tilde{D}(z, x, \omega) = \Phi(\omega)\, \tilde{D}(z, x, \omega_0)$$

and therefore

$$\int_\Omega \tilde{D}(z, x, \omega) P(d\omega) = \tilde{D}(z, x, \omega_0) \int_\Omega \Phi(\omega) P(d\omega)$$

Note that:

$$\begin{aligned}
R(p, x, \omega) &= \{\max_z p'z \mid D(z, x, \omega) \leq 1\} \\
&= \{\max_z p'z \mid \tilde{D}(z, x, \omega) \geq 1\} \\
&= \{\max_z p'z \mid \Phi(\omega) \tilde{D}(z, x, \omega_0) \geq 1\} \\
&= \Phi(\omega) \{\max_z p'(z/\Phi(\omega)) \mid \tilde{D}(z/\Phi(\omega), x, \omega_0) \geq 1\} \\
&= \Phi(\omega) R(p, x, \omega_0).
\end{aligned}$$

Hence,

$$\begin{aligned}
R(p, x) &= \int_\Omega R(p, x, \omega) P(d\omega) \\
&= R(p, x, \omega_0) \int_\Omega \Phi(\omega) P(d\omega).
\end{aligned}$$

To retrieve $D(z, x, \omega)$ from $R(p, x, \omega)$ we construct it as follows:

$$\begin{aligned}
D(z, x, \omega) &= \{\max_p p'z \mid R(p, x, \omega) \leq 1\} \\
&= D(z, x, \omega_0)/\Phi(\omega)
\end{aligned}$$

which corresponds to $\tilde{D}(z, x, \omega) = \Phi(\omega) \tilde{D}(z, x, \omega_0)$ since $\tilde{D}(z, x, \omega) = 1/D(z, x, \omega)$. Similarly

$$\begin{aligned}
D(\zeta, x) &= \{\max_p p'\zeta \mid R(p, x) \leq 1\} \\
&= D(\zeta, x, \omega_0) / \int_\Omega \Phi(\omega) P(d\omega)
\end{aligned}$$

which corresponds to $\widetilde{D}(\zeta, x) = \widetilde{D}(\zeta, x, \omega_0) \int_\Omega \Phi(\omega)\, P(d\omega)$ since

$\widetilde{D}(\zeta, x) = 1/D(\zeta, x)$.

Figure 5 illustrates the extension of the relationships in Figure 4 and the resulting completion of the duality relationships.

$$
\begin{tikzcd}[column sep=large, row sep=huge]
\tilde{D}(z, x, \omega) \arrow[r, shift right, "1/\tilde{D}(z, x, \omega)"'] \arrow[d, "\int \tilde{D}(\zeta, x, \omega)\, P(d\omega)"'] & D(z, x, \omega) \arrow[l, shift right, "1/D(z, x, \omega)"'] \arrow[r, leftrightarrow] & R(p, x, \omega) \arrow[d, "\int R(p, x, \omega)\, P(d\omega)"] \\
\tilde{D}(\zeta, x) \arrow[r, shift right, "1/\tilde{D}(\zeta, x)"'] & D(\zeta, x) \arrow[l, shift right, "1/D(\zeta, x)"'] \arrow[r, leftrightarrow] & R(p, x)
\end{tikzcd}
$$

FIGURE 5

Completed Duality for State-Homothetic Stochastic Technology

3. Dynamic Informed Plans

We now consider a very important generalization of the previous case. In the preceeding analysis the firm knew the realized state of nature before it had to choose the mix of outputs to produce. Once the state was known there was no uncertainty as to what the firm could produce for the period even though production had not begun. Casual observation would suggest that this is the exception and not the rule. A more realistic representation would treat the problem of incomplete information concerning the state of nature, both at the begnning of the production process and throughout the production period. Intuitively, we think of the firm as planning weekly production while producing on a daily basis. The state of nature for the week is a vector of the states of nature that occur each day of the week. On Monday, the firm sees the realized state for that day and produces accordingly; but the firm does not know what states of nature will obtain through the rest of the week.

An obvious response of a firm is to apply resources to control the production process; i.e. alter the distribution of later states of nature. The use of resources both to produce daily output and control future production clearly separates this case from the previous one; in section 2, no control was needed.

3.1 Dynamic Production and Control

We will formalize this scenario in the following manner. We assume that at the beginning of the production period, the firm signs contracts for a fixed level of daily services and a fixed stock of resources. Furthermore we explicitly rule out the possibility of purchasing (or selling) resources on a daily basis. Thus, in the middle of the production week the firm cannot buy or sell resources on a spot market.

Let x_0 be the quantity of "flow" resources (i.e., services) contracted for at the beginning of the week and let y_0 be the stock of materials purchased. On day t, the firm must choose a level of services x_p and quantity of resources y_p to be used for daily production. We assume that given the allocation x_p and y_p, today's production possibility set is independent of past production levels and is specified by the daily production relationship $T(z,x_p,y_p,\omega_t) \leq 0$ where ω_t is the state of nature[4] on day t. The remainder of the flow services $x_c = x_0 - x_p$ is used for control along with the allocation of stock inputs y_c. The control technology and the evolution of the daily states of nature are specified by a transition probability function $P(\omega,\omega';x_0 -x_p,y_c)$ which gives the probability that tomorrow's state will be ω' given that todays state is ω and control resources $x_0 - x_p$ and y_c are used. We assume that $(x, y) \rightarrow P(\omega, \omega'; x, y)$ is continuous.[5]

Let $r(p,x_p,y_p,\omega)$ be the value of maximized revenue given the vector of prices p, the state of nature ω and the use of productive resources x_p and y_p:

$$r(p,x_p,y_p,\omega) = \max_{z \varepsilon W(x_p,y_p,\omega)} pz \tag{1}$$

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[4] To avoid measurability problems, which are tangential to the issues we want to address, we assume that Ω is countable. Extension to Ω uncountable merely complicates the exposition without contributing to the analysis.

[5] In general let f: $R^n \times R^m \rightarrow R^k$. The notation "$x \rightarrow f(x,y)$ is continuous" is a shorthand for "$f(x,y)$ is continuous in x for each y."

The assumed properties of ω imply that $(p,x_p,y_p) \to r(p,x_p,y_p,\omega)$ is continuous, and $p \to r(p,x_p,y_p,\omega)$ is convex. Let ν be the initial distribution on the states of nature. The planning problem that the firm faces given x_0 and y_0 is to pick sequences of allocations $\{x_p(t)\}$, $\{y_p(t)\}$ and $\{y_c(t)\}$ that maximize expected profit over the T period planning horizon, subject to resource restrictions. This is expressed as follows:

$$\max_{\omega_1} \Sigma\nu(\omega_1)\ E\left[\left\{\sum_{t=1}^{T} r(p,x_p(t),y_p(t),\omega_t)\right\} | \omega_1,\ x_p(1),\ y_c(1)\right]$$

$$\text{S.T.} \qquad \sum_{t=1}^{T} y_p(t) + y_c(t) \leq y_0$$

$$\begin{array}{cc} x_p(t) \leq x_0 & \\ x_p(t),\ y_p(t),\ y_c(t) \geq 0 & t = 1,\ldots,T \end{array}$$

where

$$W(x_p(t),y_p(t),\omega_t) = \{z\varepsilon R_+^m | T(z,x_p(t),y_p(t),\omega_t) \leq 0\}.$$

As before, we assume that $(z,x_p,y_p) \to T(z,x_p,y_p,\omega)$ is continuous and that $W(x_p,y_p,\omega)$ is convex for all $(x,y,\omega)\varepsilon R_+^n \times R_+^m \times \Omega$. Notice that a zero salvage value is attributed to unused resources.

The problem for the firm represented by (2) can be solved via dynamic programming. The optimal use of resources (productive flow resources, productive stock inputs and control stock inputs) is then specified for each date t, each state of nature ω and each level of unused stock resources u. Thus, for $(x_0,\ y_0)\varepsilon R_+^n \times R_+^m$ given, the firm searches for a strategy $\mathcal{S}$: $\{1,\ldots,\ T\} \times \Omega \times R_+^n \to R_+^{3n}$ via the following recursion formulation which is equivalent to (2):

$$V_t(u,\omega;x_0,y_0,p) = \max \{r(p,x_p,y_p,\omega) + \sum_{\omega'} V_{t+1}(u-y_p-y_c,\omega';x_0,y_0,p)\ P(\omega,\omega';x_0-x_p,y_c)\}$$

$$t = 1,\ldots, T - 1$$

$$\text{S.T.} \tag{3}$$

$$(x_p,y_p,y_c)\varepsilon\ \beta(x_0,u)$$

$$V_T(u,\omega';x_0,y_0,p) = r(p,x_0,u,\omega)$$

where

$$\beta(x_0,u) = \{(x_p,y_p,y_c)\varepsilon R_+^{3n} \mid x_p \leq x_0,\ y_p + y_c \leq u\}$$

Thus, $V_t(u,\omega;x_0,y_0,p)$ is the expected optimal revenue from date t to date T, given that date t's state is ω and there is u unused stock resources. The strategy $\mathcal{S}$ is a function mapping period, state of nature and level of unused stock resources into a trajectory of allocations of productive (x_p and y_p) and control (y_c, and implicitly, x_0-x_p) inputs. We have assumed that $(x,y) \to r(p,x,y,\omega)$ and $(x,y) \to P(\omega,\omega';x,y)$ are continuous. Thus $u \to V_T(u,\omega;x,y,p)$ is continuous and if we assume that $u \to V_{t+1}(u,\omega;x,y,p)$ is also then

$$r(p,x_p,y_p,\omega) + \sum_{\omega'} V_{t+1}(u-y_p-y_c,\omega';x_0,y_0,p)P(\omega,\omega';x_0-x_p,y_c) \tag{4}$$

is continuous in u. Furthermore since $u \leq y_0$, $u \to \beta(x_0,u)$ is compact valued. Thus a maximum exists and, by the Berge Maximum Theorem (Berge [1963]) the maximized value of (4) is continuous in u. The induction argument thus shows that: 1) the value functions V_t exist and 2) the optimal strategy exists. We further make the assumption that the optimal strategy function $\mathcal{S}^*(t,\omega,u;x_0,y_0,p)$ is single valued.

Notice that the vector of unused stock resources at time s, u_s, is a function of the state-of-nature vector $\omega^{s-1} = (\omega_1,\ldots,\omega_{s-1})$ given the feasible strategy $\$$ (of course feasibility of $\$$ makes it a function of x_0 and y_0):

$$u_1 = y_0 \quad u_2 = y_0 - \$_y(1,\omega_1,y_0)$$

$$u_3 = y_0 - \$_y(1,\omega_1,y_0) - \$_y(2,\omega_2,y_0-\$_y(1,\omega_1,y_0))$$

etc., where

$$\$(t,\omega,u) \equiv (\$_{x_p}(t,\omega,u),\ \$_{y_p}(t,\omega,u),\ \$_{y_c}(t,\omega,u))$$

and[6]

$$\$_y(t,\omega,u) \equiv \$_{y_p}(t,\omega,u) + \$_{y_c}(t,\omega,u).$$

Thus, for a feasible, single-valued strategy $\$$, the beginning-of-the-week revenue $R(x_0,y_0,p,\$)$ is given by the following:

$$R(x_0,y_0,p,\$) = \Sigma\, \pi(\omega_1)E_{\$}\left[\left\{\sum_{t=1} r(p,\$_p(t,\omega_t,u_t),\omega_t)\right\} \Big| \omega_1, u_1 = y_0\right] \quad (5)$$

where $E_{\$}$ indicates expectation given that the control strategy is defined by $\$$.

The set of feasible outputs at date t is determined by the strategy function $\$$, the vector of states of nature $\omega^t = (\omega_1,\ldots,\omega_t)$ and the state of

---

[6] Also, for convenience, let $\$_p(t,\omega,u) \equiv (\$_{x_p}(t,\omega,u),\ \$_{y_p}(t,\omega,u))$ and $\$_c(t,\omega,u) \equiv (x_0 - \$_{x_p}(t,\omega,u),\ \$_{y_c}(t,\omega,u))$

nature for date t, ω_t. The strategy function $\$$ relates todays uncertainty and available stock resources (u) to an allocation of resources to production and control. The availability of the resources themselves are a function of the vector of states of nature ω^{t-1} through the strategy function $\$$. Finally, output is influenced by todays uncertainty ω_t. This will be capsulized by denoting the set of feasible outputs at date to by $W_t(\$,\omega^t)$. where

$$W_t(\$,\omega^t) = W(\$_{x_p}(t,\omega_t,u_t),\ \$_{y_p}(t,\omega_t,u_t),\ \omega_t).$$

Of course since u_t is a function U of t, the vector of states up to t (ω^{t-1}) and the strategy $\$$, we have

$$W_t(\$,\omega^t) = W(\$_{x_p}(t,\omega_t,U(t,\omega^{t-1},\$)),\ S_{y_p}(t,\omega_t,U(t,\omega^{t-1},\$)),\ \omega_t),$$

random output is

$$z_t(\$,\omega^t,p) = \underset{z\varepsilon W_t(\$,\omega^t)}{\arg\max}\ pz\ ,$$

and aggregate output for the period is

$$z(\$,\omega^T,p) = \sum_t z_t(\$,\omega^t,p)$$

Given the strategy $\$$, the probability that ω^t will occur is

$$P^t(\omega^t) = \pi(\omega_1)P(\omega_1,\omega_2;\$_c(1,\omega_1,y_0))\cdots P(\omega_{t-1},\omega_t;\$_c(t-1,\omega_{t-1},u_{t-1}) \qquad (6)$$

and thus expected revenue can be written as:

$$R(x_0,y_0,p,\$) = \sum_{t=1}^{T} \sum_{\omega_1} \sum_{\omega_2} \cdots \sum_{\omega_t} P^t(\omega^t) \max_{z \varepsilon W_t(\$,\omega^t)} pz \tag{7}$$

In a manner similar to the discussion in section two, we can integrate over the sets $W_t(\$,\omega^t)$ to produce $W_t(\$)$:

$$W_t(\$) = \sum_{\omega_1} \cdots \sum_{\omega_t} P^t(\omega^t)\ W_t(\$,\omega^t)$$

and thus (7) becomes:

$$R(x_0,y_0,p,\$) = \sum_{t=1}^{T} \max_{\zeta \varepsilon W_t(\$)} p\zeta \tag{8}$$

$$= \max_{\zeta\ \varepsilon\ W(\$)} p\zeta \tag{9}$$

where

$$W(\$) = \sum_{t} W_t(\$).$$

Evaluating W at the solution, $\$^*(x_0,y_0,p)$, of the dynamic program, define

$$W^*(x_0,y_0,p) = W(\$^*(x_0,y_0,p)) \tag{10}$$

and

$$R^*(x_0,y_0,p) = \max_{\zeta \varepsilon W^*(x_0,y_0,p)} p\zeta \geq R(x_0,y_0,p,\$) \tag{11}$$

for any other feasible strategy $\$$.

In general, $\mathcal{S}^*$ is a function of p and hence $W^*(x_0,y_0,p)$ will depend nontrivially on p. Thus, W^* reflects more than simply the technology of the firm. Due to the multi-period nature of the analysis, W^* also reflects the prices that output can be sold for. Except for this dependence, the expression in (11) looks remarkably like the static case discussed earlier in section two. It is this dependence on the output prices that we will now examine. We shall find that (11) can be rewritten into a form wherein prices only appear in the objective function.

Before reexpressing (11) we first examine the function $p \to R^*(x_0,y_0,p)$ and the correspondence $p \to W^*(x_0,y_0,p)$. Properties of these correspondences are provided by the following lemma

Lemma 1: $W^*(x_0,y_0,p)$ is a continuous, compact, convex valued correspondence in p and $R^*(x_0,y_0,p)$ is a continuous function of p.

Proof: Recall from above that

$$W^*(x_0,y_0,p) = \sum_{t=1}^{T} \sum_{\omega_1} \cdots \sum_{\omega_n} P^t(\omega^t) W_t(\mathcal{S}^*(x_0,y_0,p),\omega^t).$$

Recalling that, for a feasible strategy $\mathcal{S}$,

$$W_t(\mathcal{S},\omega^t) = W(\mathcal{S}_{x_p}(t,\omega_t,U(t,\omega^{t-1},\mathcal{S})),\ \mathcal{S}_{y_p}(t,\omega_t,U(t,\omega^{t-1},\mathcal{S})),\ \omega_t)$$

and

$$\mathcal{S}_{x_p}(t,\omega_t,u_t) \leq x_0\ , \qquad \mathcal{S}_{y_p}(t,{}_t,u_t) \leq y_0\ .$$

Then

$$W_t(\mathcal{S},\omega^t) \subseteq W(x_0,y_0,\omega_t).$$

Since $W(x,y,\omega)$ is compact and is contained in a compact subset of R^m_+ for all ω, we know that $W^*(x_0,y_0,p) = W(\mathcal{S}^*(x_0,y_0,p))$ is compact-valued.

Since $(x,y) \to T(z,x,y,\omega)$ is continuous then $(x,y) \to W(x,y,\omega)$ is a continuous correspondence. Moreover, since $p \to r(p,x_p,y_p,\omega)$ is continuous, so is $p \to V_T(u,\omega;x_0,y_0,p)$. Again, using and induction argument as before:

$$p \to r(p,x_p,y_p,\omega) + \sum_{\omega'} V_{t+1}(u-y_p-y_c,\omega';x_0,y_0,p)P(\omega,\omega';x_0-x_p,y_c)$$

is continuous for all t. Thus, since $\mathcal{S}^*$ is the maximizer it is upper hemi-continuous due to the Maximum Theorem. Since $\mathcal{S}^*$ is assumed to be single valued, $p \to \mathcal{S}^*(t,\omega,u;x_0,y_0,p)$ is a continuous function. Note that date t's production possibility set at the optimum allocation is a function of p through the continuous function S^*, and hence date t's production possibility set is a continuous correspondence of a continuous function and hence a continuous correspondence in p. Finally, $W^*(x_0,y_0,p)$ is a weighted sum of continuous correspondences where the weights are continuous functions of p. Thus $p \to W^*(x_0,y_0,p)$ is a continuous correspondence in p that is compact and convex valued (due to the assumptions on $W(x,y,\omega)$). Applying the Maximum Theorem to $R^*(x_0,y_0,p)$ we see that $p \to R^*(x_0,y_0,p)$ is a continuous function, and thus the lemma is proved.

At this point, we have established that the optimal expected revenue can be expressed as a maximization over a nonstochastic "feasible production set" that is a function of p. We next establish the analogy with section two: optimal expected revenue can be expressed as a maximization over a nonstochastic feasible production set which is purely a function of x_0 and y_0. Moreover, we will then show that this set is (in an appropriate sense) an envelope set of the $W^*(x_0,y_0,p)$ sets.

First, define $\bar{W}(x_0,y_0)$ as follows[7]:

$$\bar{W}(x_0,y_0) = \{\zeta\varepsilon R^m_+ | p\zeta \leq R^*(x_0,y_0,p),\ p\varepsilon\Gamma^m_+\}.$$

We now prove Lemma 2.

Lemma 2

$$R^*(x_0,y_0,p) = \max_{\zeta\varepsilon\bar{W}(x_0,y_0)} p\zeta$$

Proof

Clearly $\bar{W}(x_0,y_0)$ is compact and convex, and thus

$$\bar{R}(x_0,y_0,p) = \max_{\zeta\varepsilon\bar{W}(x_0,y_0)} p\zeta$$

is well defined with $p \to \bar{R}(x_0,y_0,p)$ continuous, homogeneous of degree one and convex. If $\zeta\varepsilon\bar{W}(x_0,y_0)$ then for all p

$$p\zeta \leq R^*(x_0,y_0,p)$$

and in particular

$$\bar{R}(x_0,y_0,p) = \max_{\zeta\varepsilon\bar{W}(x_0,y_0)} p\zeta \leq R^*(x_0,y_0,p)$$

Now consider an arbitrary $p_0\varepsilon\Gamma^m_+$ and $\zeta_0\varepsilon W^*(x_0,y_0,p_0)$. By the definition of $R^*(x_0,y_0,p_0)$, $p_0\zeta_0 \leq R^*(x_0,y_0,p_0)$. Let $p_1\varepsilon\Gamma^m_+$ be another price vector. Then

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[7] As was mentioned in section 2.3 above, this is more rigorously done by defining the set for $p\varepsilon\Gamma^m_{++}$ and then extending the definition by continuity.

$$
\begin{aligned}
p_1\zeta_0 &\leq \max_{\zeta\varepsilon W^*(x_0,y_0,p_0)} p_1\zeta \\
&= \max_{\zeta\varepsilon W(\mathcal{S}^*(x_0,y_0,p_0))} p_1\zeta \\
&\leq \max_{\zeta\varepsilon W(\mathcal{S}^*(x_0,y_0,p_1))} p_1\zeta = R^*(x_0,y_0,p_1)
\end{aligned}
$$

Thus $p_1\zeta_0 \leq R^*(x_0,y_0,p_1)$ and since p_1 was arbitrary $p\zeta_0 \leq R^*(x_0,y_0,p)$ for all $p\varepsilon\Gamma_+^m$. Since ζ_0 was picked arbitrarily, we see that

$$W^*(x_0,y_0,p_0) \subseteq \bar{W}(x_0,y_0)$$

and, since p_0 was arbitrary:

$$W^*(x_0,y_0,p) \subseteq \bar{W}(x_0,y_0) \qquad \forall_{p\varepsilon\Gamma_+^m}.$$

Therefore

$$
\begin{aligned}
R^*(x_0,y_0,p) &= \max_{\zeta\varepsilon W^*(x_0,y_0,p)} p\zeta \\
&\leq \max_{\zeta\varepsilon \bar{W}(x_0,y_0)} p\zeta \\
&= \bar{R}(x_0,y_0,p) \leq R^*(x_0,y_0,p)
\end{aligned}
$$

from above. Therefore, the lemma is proved:

$$R^*(x_0,y_0,p) = \max_{\zeta\varepsilon\bar{W}(x_0,y_0)} p\zeta.$$

Lemma 2 provides an indirect method of construction for the nondynamic, nonstochastic feasible production set. It is indirect in the sense that the sets $W^*(x_0,y_0,p)$ are used to construct $R^*(x_0,y_0,p)$ which, via standard duality theory, should be dual to a distance function representing a production set (namely $\bar{W}(x_0,y_0)$).

What is the direct relationship between the $W^*(x_0,y_0,p)$ sets and $\bar{W}(x_0,y_0)$? We shall find that $\bar{W}(x_0,y_0)$ is essentially an envelope of the $W^*(x_0,y_0,p)$ set, as p is varied and that

$$\bar{W}(x_0,y_0) = \bigcup_p W^*(x_0,y_0,p)$$

where the union is over $p\varepsilon\Gamma_+^m$. To prove this we need the following lemma to show that $\bigcup_p \bar{W}(x_0,y_0)$ is closed (this is not trivial since Γ_+^m is not finite).

Lemma 3

$p \to W^*(x_0,y_0,p)$ continuous implies that $\bigcup_p W^*(x_0,y_0,p)$ is closed.

Proof:

Let $\{\zeta^q\}$ be an arbitrary sequence in $\bigcup_p W^*(x_0,y_0,p)$ with $\zeta^q \to \zeta_0 \;\varepsilon\; R_+^m$. We need to show that $\zeta_0 \;\varepsilon \bigcup_p W^*(x_0,y_0,p)$. Since $\zeta^q \;\varepsilon \bigcup_p W^*(x_0,y_0,p)$, $\zeta^q \;\varepsilon\; W^*(x_0,y_0,p^q)$ for some p^q. The sequence p^q is contained in Γ_+^m, a compact set, and hence has a convergent subsequence $p^{qn} \to p_0 \;\varepsilon\; \Gamma_+^m$. Then we have $\zeta^{qn} \to \zeta_0$, $p^{qn} \to p_0$, $\zeta^{qn} \;\varepsilon\; W^*(x_0,y_0,p^{qn})$, and by continuity, ζ_0 is an element of $W^*(x_0,y_0,p_0)$, a subset of $\bigcup_p W^*(x_0,y_0,p)$.

We now employ the fact that $\bigcup_p W^*(x_0,y,p)$ is closed to provide the following theorem.

Theorem 3

If $\mathcal{S}^*$ is single valued then

$$\bar{W}(x_0,y_0) = \bigcup_p W^*(x_0,y_0,p).$$

Proof:

As was shown in Lemma 2,

$$W^*(x_0,y_0,p) \subseteq \bar{W}(x_0,y_0) \qquad \forall p\varepsilon\Gamma_+^m$$

and thus

$$\bigcup_p W^*(x_0,y_0,p) \subseteq \bar{W}(x_0,y_0).$$

We wish to show the reverse so as to prove the theorem. Thus, pick a point $\zeta_0 \geq 0$ not in $\bigcup_p W^*(x_0,y_0,p)$; we will show that it is also not in $\bar{W}(x_0,y_0)$. For a given $\varepsilon > 0$ define the price correspondence π by

$$\pi(p) = \{\pi\varepsilon\Gamma_+^m | \ \pi\zeta_0 \geq \pi\zeta + \varepsilon \text{ for all } \zeta\varepsilon W^*(x_0,y_0,p)\}.$$

If $\pi(p)$ has a fixed point (i.e., $p^*\varepsilon\Gamma_+^m$ such that $p^*\varepsilon\pi(p^*)$) then we know that $p^*\zeta_0 > R^*(x_0,y_0,p)$ and thus $\zeta_0 \not\in \bar{W}(x_0,y_0)$.

Recall that Γ_+^m is a nonempty, compact, convex set in R_+^m and that π is a correspondence from Γ_+^m to Γ_+^m. From its definition it is clear that π is convex for all $p\varepsilon\Gamma_+^m$. Next we show that it is nonempty. From Lemma 3 $\bigcup_p W^*(x_0,y_0,p)$ is closed and thus its complement in R_+^m is open (in R_+^m).

Thus there is an open ball around ζ_0 which is contained in the complement of $\bigcup_p W^*(x_0,y_0,p)$, i.e., there exists $\varepsilon > 0$ with $0 < \varepsilon < \min \{\zeta_{oi}:\zeta_{oi} > 0\}$ such that we construct the vector $\hat{\zeta}$:

$$\hat{\zeta}_i = \begin{cases} \zeta_{oi} - \varepsilon & \text{if } \zeta_{oi} > 0, \\ 0 & \text{otherwise,} \end{cases}$$

which is nonnegative and in the complement of $\bigcup_p W^*(x_0,y_0,p)$, and thus (for arbitrary $p\varepsilon\Gamma^m_+$) outside of $W^*(x_0,y_0y,p)$. Since $W^*(x_0,y_0,p)$ is convex, there exists $\pi\varepsilon\Gamma^m_+$ such that

$$\pi\hat{\zeta} > \pi\zeta \qquad \forall \zeta\varepsilon W^*(x_0,y_0,p).$$

Let $A = \{i|\hat{\zeta}_i > 0\}$. By construction A is nonempty and, moreover, $\sum_{i\varepsilon A} \pi_i = \gamma > 0$, for the aforementioned π-vector, since otherwise

$$0 = \pi\hat{\zeta} > \pi\zeta \geq 0 \qquad \forall \zeta\varepsilon W^*(x_0,y_0,p).$$

We now use π to construct a member of $\pi(p)$, namely $\hat{\pi}\varepsilon\Gamma^m_+$:

$$\hat{\pi}_i = \begin{cases} \pi_i/\gamma & \hat{\zeta}_i > 0, \\ 0 & \text{otherwise.} \end{cases}$$

By construction $\hat{\pi}\hat{\zeta} = \pi\hat{\zeta}/\gamma$ and, for arbitrary $\zeta\varepsilon W(x_0,y_0,p)$ we have that

$$\hat{\pi}\zeta = \sum_A \pi_i\zeta_i/\gamma \leq \pi\zeta/\gamma$$

and therefore $\hat{\pi}\hat{\zeta} - \hat{\pi}\zeta \geq \pi\hat{\zeta}/\gamma - \pi\zeta/\gamma > 0$, since $\pi\hat{\zeta} > \pi\zeta$ for all $\zeta\varepsilon W^*(x_0,y_0,p)$. Again, by construction

$$\begin{aligned} \hat{\pi}\hat{\zeta} &= \hat{\pi}\zeta_0 - \varepsilon\sum_A \pi_i/\gamma \\ &= \hat{\pi}\zeta_0 - \varepsilon, \end{aligned}$$

which means that $\hat{\pi}\zeta_0$ exceeds $\hat{\pi}\zeta + \varepsilon$ for all $\zeta\varepsilon W^*(x_0,y_0,p)$ i.e., $\pi(p)$ is nonempty since p was arbitrary and ε was chosen independently of p.

Moreover, $\pi(p)$ is upper hemi-continuous. Suppose $\{p^q\}$ is a sequence in Γ^m_+ converging to p_0 and let $\{\pi^q\}$ be a sequence converging to π_0 with $\pi^q\varepsilon\pi(p^q)$.

We want to prove that $\pi_0 \varepsilon \pi(p_0)$. To see this, observe that since $\pi^q \varepsilon \pi(p^q)$ then

$$\pi^q \zeta_0 \geq \pi^q \zeta + \varepsilon \qquad \forall \zeta \varepsilon W^*(x_0, y_0, p^q)$$

Pick an arbitrary point ζ_1 in $W^*(x_0, y_0, p_0)$. By the continuity of $p \to W^*(x_0, y_0, p)$ there exists a sequence $\zeta^q \to \zeta_1$ with

$$\zeta^q \varepsilon W^*(x_0, y_0, p^q).$$

Therefore $\pi^q \zeta_0 \geq \pi^q \zeta^q + \varepsilon$, and thus $\pi_0 \zeta_0 \geq \pi_0 \zeta_1 + \varepsilon$. Since ζ_1 was arbitrarily chosen, $\pi_0 \zeta_0 \geq \pi_0 \zeta + \varepsilon$ for all $\zeta \varepsilon W^*(x_0, y_0, p_0)$ which implies that $\pi_0 \varepsilon \pi(p_0)$ and therefore π is upper hemi-continuous.

Thus, π is an upper hemi-continuous, convex valued, nonempty correspondence from a convex, compact set Γ_+^m to Γ_+^m. By Kakutani's fixed point theorem, there exists a p^* such that $p^* \varepsilon \pi(p^*)$ i.e., there exists p^* such that

$$p^* \zeta_0 \geq p^* \zeta + \varepsilon \qquad \forall \zeta \varepsilon W^*(x_0, y_0, p^*),$$

and in particular

$$\begin{aligned} p^* \zeta_0 &\geq \max_{\zeta \varepsilon W^*(x_0, y_0, p^*)} p^* \zeta + \varepsilon \\ &= R^*(x_0, y_0, p^*) + \varepsilon. \end{aligned}$$

and hence $\zeta_0 \notin \bar{W}(x_0, y_0)$, completing the proof.

Finally, let expected profit for the period (when it exists) be

$$\pi(p, q) = \max_{x_0, y_0} \max_{\zeta \varepsilon \bar{W}(x_0, y_0)} p\zeta - q_x x_0 - q_y y_0$$

with $q_x \varepsilon \Gamma_+^m$ and $q_y \varepsilon \Gamma_+^m$ the vectors of x and y purchase prices. The associated expected supply is $\zeta^*(p,q)$ and the random supply for the period is $z(\$^*(x_0^*,y_0^*,p),\omega^T,p)$, where x_0^* and y_0^* are the optimal input choices.

3.2 Implications of the Dynamic Models

The main implication of the foregoing analysis is embodied in Theorem 3: the expected profit maximizing firm operates in such a manner as to be (technologically) completely representable by a nonstochastic, nondynamic "feasible production set". The appearance of firm purchases of inputs (for the purpose of controlling output as opposed to producing it) such as product inspectors, quality control equipment, etc. is our only clue that the firm actually exists in a stochastic, dynamic environment.

The implication for econometric and policy analysis is also clear: an analyst may treat such a firm with standard, deterministic cost/production techniques, as long as output is expressed in terms of planned (or expected) output and not taken as observed output. This is an important caveat, since cost is a function of planned and not actual output; simulation experiments show that violating this provisio can result in serious inferential error (see Daughety [1979]).

An implication for the theory of the firm is that the above analysis provides (for the perfectly competitive firm) a complete foundation for models involving expected supply and and actual (random) supply. Such models, in a general equilibrium setting, give rise to randomly fluctuating prices (see Rothenberg and Smith [1971]). To the degree that the resulting random prices are independent of (or uncorrelated with) individual firm uncertainties (i.e., ω), the above analysis is valid and provides the micro-foundation for such models.

4. Conclusions

This paper has three basic results which correspond to the three theorems proved above. First, in the static setting, the expected profit maximizing firm behaves as if it were employing a nonstochastic technology which is the expected technology found by integrating over the stochastic production possibilities sets. Second, only in special cases does the stochastic distance function integrate to the expected distance function. The practical implication of this result for analysis of such firms is to suggest that studies using indirect techniques (i.e., cost or revenue functions) to study technology are more likely to properly represent expected technology then use of a direct approach (such as estimating the stochastic distance function). This holds as long as the investigator can provide a mechanism for constructing expected output.

Third, the most natural, but complex, environment for the firm involves both stochastic and dynamic aspects. Here we show the extension of theorem one, namely that even if the firm can use inputs both to produce output and to influence the state of nature, a well defined nondynamic expected technology can be constructed, which represents all the necessary information to link cost, revenue, factor demands and planned outputs.

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