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DYNAMIC PRICING WITH STOCHASTIC ENTRY

by

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ABSTRACT

As new firms enter an industry, their interarrival times are assumed to be random variables whose distribution depends on the price policy selected by the firms already in the industry. The objective of an existing firm is to select an optimal sequence of prices so as to maximize its total expected discounted return over the infinite horizon. The problem is formulated as a semi-Markov decision process and its solution as more firms enter the industry is studied. It is shown that in the limit the firms' behavior tends to be monopolistic.

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I. Introduction:

It has been recognized that an intelligent oligopolist should take into consideration both potential as well as existing competitors while determining his price-output policies. Bhagwati [1] and Modigliani [6] present excellent surveys of various assumptions and strategies proposed by different authors in setting a "limit price" to prevent a new entry. The general conclusion is that an oligopolist can charge an entry-preventing price which is higher than the competitive price. Since the existing competition affects the firm's "short run" profit, while the "long run" profit is determined by potential competition, it seems reasonable that the firm should seek to maximize some weighted average of the two, as suggested by Hicks [4]. Consequently the optimal price should lie between the monopolistic price and the competitive one. These and similar other interesting conclusions about the optimal pricing strategies were also obtained by Gaskins [3] and Kamien and Schwartz [5] using the optimal control theory framework.

In this paper we propose a semi-Markov decision model for determining a sequence of optimal prices to be charged and the corresponding long run profit streams, along with their limiting behavior as the number of firms in the industry tends to infinity. The assumptions are stated and the model is formulated in the next section. The third section presents the main results about the transient and the limiting behavior of the model while the last section concludes with some discussion and possible extensions.

2. The Model:

Consider a firm producing a single commodity under constant returns to scale, c being its average cost of production. The industry is composed of n such identical firms and faces the rate of demand $D(p)$ as a negatively sloping, twice differentiable function of the product price p . It is assumed that the total industry demand is shared equally among its members so that an individual firm's rate of profit may be written as

$$\bar{\pi}(n,p) = (p-c) \frac{D(p)}{n} \quad (2.1)$$

to be denoted by $\frac{\pi(p)}{n}$, where $\pi(p)$, the total industry profit rate, is assumed to be a strictly concave function attaining its maximum at $p = \bar{p}$. The interest rate is denoted by r .

Depending on the price p charged new firms identical to the existing ones are assumed to enter the industry according to the Poisson process at rate $\lambda(p)$, i.e. the probability that k firms will enter the industry during an interval of length t is given by $e^{-\lambda(p)t} [\lambda(p)t]^k / k!$ and, given n and p , the distribution of time until a new arrival is exponential with mean $\frac{1}{\lambda(p)}$. Such an assumption requires that the number of entrants in the interval depends only on its duration t and not on when it began (stationarity), that the probability of an arrival in that interval is $\lambda(p)t + o(t)$ while the probability of more than one arrival is $o(t)$ and that the number of entrants in disjoint time intervals are independent random variables. Thus, the implicit assumption is that there are many potential entrants, they behave independently of one another and their choices of entry times are random in some sense. For $p > c$ the rate of entry $\lambda(p)$ is assumed to

be an increasing twice differentiable convex function, while for $p \leq c$, $\lambda(p)$ is taken to be zero; also assume $\lambda'(c) < \lambda'(\bar{p}) < \infty$. Thus the higher the price more attractive the entry is, while a firm making losses need not fear new competition. It is assumed that a firm is the sole determinant of the prices and since all the firms in the industry are identical it is sufficient to determine the price-output policy of any single firm. As a new firm enters the industry the existing n firms accept this fact, each sets a new price and the total industry demand is shared equally among the $(n+1)$ firms; thus the existing firms are assumed to enjoy no goodwill.

The objective of an individual existing firm is to determine an optimal sequence of prices it should charge as a function of the number of firms in the industry so as to maximize its own infinite horizon expected discounted profit. Suppose the n^{th} firm enters the industry at time t and p_n is the price charged then on until the $(n+1)^{\text{st}}$ entry. Then the expected profit to an established firm during this interval discounted to time t is given by

$$\int_0^{\infty} \int_0^x \bar{\pi}(n, p_n) e^{-r\tau} \cdot \lambda(p_n) e^{-\lambda(p_n)x} d\tau dx \quad (2.2)$$

$$= \frac{\bar{\pi}(n, p_n)}{r + \lambda(p_n)}$$

which may be called the short run profit of the firm (i.e. until a new entry occurs). Denoting the probability that the n^{th} entry occurs in the interval $[t, t+dt]$ by $q(n, t)dt$ we may recursively define this probability for a given sequence of prices $\{p_1, \dots, p_{n-1}\}$ by

$$q(1,0) = 1$$

$$q(n,t) = \int_0^t q(n-1,t-y) \lambda(p_{n-1}) e^{-\lambda(p_{n-1})y} dy, \quad n \geq 2 \quad (2.3)$$

With this notation the original firm's problem is to determine an optimal sequence of prices to be denoted by $\{p_n^*\}_{n=1}^{\infty}$ so as to maximize

$$\int_0^{\infty} \sum_{n=1}^{\infty} \frac{\bar{\pi}(n,p_n)}{r + \lambda(p_n)} q(n,t) e^{-rt} dt \quad (2.4)$$

Denote by $V(1)$ the maximum of (2.4) over all sequences $\{p_n\}_{n=1}^{\infty}$ and in general let $V(n)$ be the optimal total expected discounted profit of the n^{th} entrant using his optimal policy $\{p_m^*\}_{m=n}^{\infty}$. Then it is well known (e.g. see Blackwell [2]) that the sequence $\{V(n)\}_{n=1}^{\infty}$ satisfies the following optimality equation

$$V(n) = \text{Max}_{p_n} \left[\frac{\bar{\pi}(n,p_n)}{r + \lambda(p_n)} + \int_0^{\infty} V(n+1) e^{-rx} \lambda(p_n) e^{-\lambda(p_n)x} dx \right]$$

i.e.
$$V(n) = \text{Max}_{p_n} \left[\frac{\bar{\pi}(n,p_n) + \lambda(p_n) V(n+1)}{r + \lambda(p_n)} \right] \quad (2.5)$$

If $V(\cdot)$ is the solution of this functional equation, the optimal price p_n^* is the one that maximizes the quantity in the squared brackets, which may be called the long run profit, denoted by

$$W(n,p_n) = \frac{r}{r + \lambda(p_n)} \cdot \frac{\pi(p_n)}{nr} + \frac{\lambda(p_n)}{r + \lambda(p_n)} \cdot V(n+1) \quad (2.6)$$

It is the sum of the short run return until a new entry occurs and the total return from then on following the optimal pricing

strategy. A better alternative interpretation is that it is the weighted average of the firm's optimistic infinite horizon return $\frac{\pi(p_n)}{nr}$ (i.e. if no entry occurs) and its pessimistic infinite horizon return $V(n+1)$ (i.e. if an entry occurs immediately.) As p_n increases the entry is hastened so that the pessimistic return becomes more probable and is given a greater weight. Similarly the higher the interest rate r less relevant is the future return $V(n+1)$ for the same price p_n .

It should be noted that in this model although all the firms in the industry pursue identical pricing policies this is simply due to the assumption of their identical structure and equal sharing of demand, and does not imply their complete mutual cooperation so as to maximize the total industry profit. In fact, the total industry profit $(p-c) D(p)$ is independent of the number of firms in the industry (due to our assumption of constant returns to scale) so that the infinite horizon industry profit is maximized by simply setting $p = \bar{p}$, because the industry as a whole has no motivation to retard an entry. However, as will be shown in the next section, such a monopolistic policy is not in the best interests of an individual firm, except in the limit when the number of firms in the industry tends to infinity.

3. Results:

We shall need the following properties of the optimal value function $V(\cdot)$ stated as

Proposition 1: For all $n = 1, 2, \dots$

$$(a) \quad V(n) \geq 0 \quad (\text{non-negativity}) \quad (3.1)$$

$$(b) \quad V(n) \geq V(n+1) \quad (\text{monotonicity}) \quad (3.2)$$

$$(c) \quad V(n) - V(n+1) \geq V(n+1) - V(n+2) \quad (\text{"convexity"}) \quad (3.3)$$

Proof: To prove (a) note that in (2.5) by choosing $p_n = c$ we have $\pi(p_n) = \lambda(p_n) = 0$ yielding $W(n, p_n) = 0$, hence $V(n) = W(n, p_n^*) \geq 0$.

Next, define $V_M(n)$ as the optimal return to the n^{th} firm given that no more than M additional firms can enter in the long run. Then $V(n) = \lim_{M \rightarrow \infty} V_M(n)$, so that it suffices to prove (b) and (c) for $V_M(n)$ for all M and n by induction. Clearly $V_0(n) = \frac{\pi(\bar{p})}{nr}$ which satisfies (b) and (c). Suppose $V_{M-1}(n)$ satisfies (b) and (c) for all n , then note

$$V_M(n) = \text{Max}_p \left[\frac{\frac{\pi(p)}{n} + \lambda(p) V_{M-1}(n+1)}{r + \lambda(p)} \right] \quad (3.4)$$

which is a finite stage version of (2.5). Since $V_{M-1}(n+1)$ and $\frac{\pi(p)}{n}$ are decreasing in n and $\lambda(p) \geq 0$ we have $V_M(n)$ decreasing in n from (3.4), so that (b) holds for M . To show convexity let $n_1 \leq n_2 \leq n_3$ be three positive integers and $0 \leq \alpha \leq 1$ be such that $n_2 = \alpha n_1 + (1-\alpha)n_3$. Then from the induction hypothesis

$$V_M(\alpha n_1 + (1-\alpha)n_3) = \text{Max}_p \left[\frac{\frac{\pi(p)}{\alpha n_1 + (1-\alpha)n_3} + \lambda(p) V_{M-1}(\alpha n_1 + (1-\alpha)n_3 + 1)}{r + \lambda(p)} \right]$$

$$\leq \text{Max}_p \left[\frac{\alpha \frac{\pi(p)}{n_1} + (1-\alpha) \frac{\pi(p)}{n_3} + \alpha \lambda(p) V_{M-1}(n_1+1) + (1-\alpha) \lambda(p) V_{M-1}(n_3+1)}{r + \lambda(p)} \right]$$

$$\leq \alpha \operatorname{Max}_p \left[\frac{\frac{\pi(p)}{n_1} + \lambda(p)V_{M-1}(n_1+1)}{r + \lambda(p)} \right] + (1-\alpha) \operatorname{Max}_p \left[\frac{\frac{\pi(p)}{n_3} + \lambda(p)V_{M-1}(n_3+1)}{r + \lambda(p)} \right]$$

i.e. $V_M(n_2) \leq \alpha V_M(n_1) + (1-\alpha) V_M(n_3).$

In particular, letting $n_1 = n$, $n_2 = n+1$ and $n_3 = n+2$ yields (c) for M.

O.E.D.

Thus (a) says that an optimally acting individual firm never makes losses in the long run regardless of the size of the industry, while (b) implies that it is better off in the long run starting with fewer in the industry but, from (c), such an advantage wears off with a larger number in the industry. Therefore an individual firm will be less concerned with discouraging a new entrant in a larger industry.

The next proposition gives the relations between the optimal value function and the optimal pricing strategy.

Proposition 2: For all $n = 1, 2, \dots$

$$(a) \quad V(n) = \frac{\pi(p_n^*)}{nr} - \frac{\pi'(p_n^*)}{nr \lambda'(p_n^*)} \lambda(p_n^*) \quad (3.5)$$

$$(b) \quad V(n+1) = \frac{\pi(p_n^*)}{nr} - \frac{\pi'(p_n^*)}{nr \lambda'(p_n^*)} [r + \lambda(p_n^*)] \quad \text{and hence} \quad (3.6)$$

$$(c) \quad V(n) - V(n+1) = \frac{\pi'(p_n^*)}{n \lambda'(p_n^*)} \quad (3.7)$$

Proof: Since p_n^* maximizes $W(n,p)$, we have $\frac{\partial W(n,p_n^*)}{\partial p} = 0$.

$$\begin{aligned} \text{i.e.} \quad \frac{\pi'(p_n^*)}{n} &= \frac{\lambda'(p_n^*)}{r+\lambda(p_n^*)} \left[\frac{\pi(p_n^*)}{n} - r V(n+1) \right] \\ &= \lambda'(p_n^*) [V(n) - V(n+1)] \end{aligned}$$

since

$$V(n) = \frac{\frac{\pi(p_n^*)}{n} + \lambda(p_n^*) V(n+1)}{r + \lambda(p_n^*)}, \text{ thus proving (c).} \quad (3.8)$$

Substituting $V(n+1) = V(n) - \frac{\pi'(p_n^*)}{n \lambda'(p_n^*)}$ into (3.8) we obtain (a)

and then (a) along with (c) yields (b). The second order conditions for a maximum at p_n^* require $\frac{\partial^2 W(n,p_n^*)}{\partial p^2} \leq 0$, which requires

$$\left[r + \lambda(p_n^*) \right] \frac{\pi''(p_n^*)}{n} - \lambda''(p_n^*) \left[\frac{\pi(p_n^*)}{n} - r V(n+1) \right] \leq 0$$

which is true since π is concave, λ convex and $V(n+1) \leq \frac{\pi(p_n^*)}{nr}$.

Q.E.D.

To interpret (a) we may write it as

$$V(n) = \frac{\pi(p_n^*)}{nr} - \{ \lambda(p_n^*) [V(n)-V(n+1)] + [1-\lambda(p_n^*)] \cdot 0 \} \quad (3.9)$$

where the first term is the expected long run optimistic profit if no more entry occurs, while the quantity within the braces is the expected decrease in the long run profit if an entry were to take place immediately (which happens with probability $\lambda(p_n^*)$). The relative individual advantage in starting with n rather than $(n+1)$

firms is given by (c), which may be written as

$$\frac{\pi'(p_n^*)}{n} = \lambda'(p_n^*) [V(n) - V(n+1)] \quad (3.10)$$

where $\lambda'(p)$ is the increase in the probability of an instantaneous entry due to setting price p . Then from (3.10) the optimal price balances the marginal increase in the short run profit with the marginal expected decrease in the long run profit due to a possible entry.

From (3.1) and (3.5) and the assumption that $\lambda(p) = 0$ for $p \leq c$, it follows that $p_n^* \geq c$; in fact $p_n^* = c$, the competitive price, if and only if $V(m) = 0$ for all $m \geq n$, so that from (3.7)

$\pi'(c)/n \lambda'(c) = 0$ which is impossible for a finite n (unless $\pi'(c) = 0$) and hence $p_n^* > c$ for all n , so that each firm acting optimally enjoys a positive premium over the competitive price. Also from (3.2) and (3.7) we have $\frac{\pi'(p_n^*)}{n \lambda'(p_n^*)} = V(n) - V(n+1) \geq 0$,

so that $\pi'(p_n^*) \geq 0$, thus implying $p_n^* \leq \bar{p}$ by concavity of π . The two cases to be distinguished here are $\lambda(\bar{p}) = 0$ and $\lambda(\bar{p}) > 0$. If $\lambda(\bar{p}) = 0$ then $p_1^* = \bar{p}$, $V(1) = \frac{\pi(\bar{p})}{r}$ and the original firm is a traditional monopolist who need not fear an entrant's appearance.

Suppose $\lambda(\bar{p}) > 0$ ("ineffectively blockaded entry" according to Hicks [4]) but that for some n for the first time $p_n^* = \bar{p}$. Then, from (3.5) and (3.7), $V(n) = V(n+1) = \frac{\pi(\bar{p})}{nr}$ and, by (3.2) and (3.3), we get $V(m) = \frac{\pi(\bar{p})}{nr}$ for all $m \geq n$. However, this is impossible because, as will be shown in the next proposition, $V(m)$ decreases down to zero (unless of course $\pi(\bar{p}) = 0$ or $D(c) = 0$). Thus, the optimal price each firm charges is less than the myopic price it

would charge if it ignored the possibility of an entry. We summarize the above discussion, without the stated exceptions, in the following

Corollary: For all n , $c < p_n^* < \bar{p}$, so that the optimal price charged lies strictly between the monopolistic and the competitive ones regardless of the industry size.

The final result is concerned with the limiting behavior of the firms as the size of the industry tends to infinity.

Proposition 3:

$$(a) \quad \lim_{n \rightarrow \infty} V(n) = 0 \quad (3.11)$$

$$(b) \quad \lim_{n \rightarrow \infty} p_n^* = \bar{p} \quad (3.12)$$

$$(c) \quad \lim_{n \rightarrow \infty} nV(n) = \frac{\pi(\bar{p})}{r} \quad (3.13)$$

Proof: Since $V(n)$ is nonnegative and monotonically decreasing (from Proposition 1) $\lim_{n \rightarrow \infty} V(n) = V \geq 0$ exists. From (2.6) and (3.2) $W(n,p)$ is decreasing in n for all p and from (2.5) $V(n) = \text{Max}_p [W(n,p)]$ where the maximum is attained at $p_n^* \in [c, \bar{p}]$. Therefore the following computations are justified.

$$\begin{aligned} V &= \lim_{n \rightarrow \infty} V(n) = \lim_{n \rightarrow \infty} \text{Max}_{c \leq p \leq \bar{p}} [W(n,p)] \\ &= \text{Max}_{c \leq p \leq \bar{p}} [\lim_{n \rightarrow \infty} W(n,p)] \\ &= \text{Max}_{c \leq p \leq \bar{p}} \left[\frac{h(p)V}{r+h(p)} \right] \end{aligned}$$

and hence $V = 0$, proving (a). To prove (b) note that, with $n > m$,

$$\begin{aligned} V(m) &= \sum_{k=m}^{\infty} [V(k) - V(k+1)] \geq \sum_{k=m}^n [V(k) - V(k+1)] \\ &\geq (n-m) [V(n) - V(n+1)] \end{aligned}$$

using monotonicity and convexity of $V(\cdot)$. Thus for any $n > m$

$$0 \leq n [V(n) - V(n+1)] \leq V(m) + m[V(n) - V(n+1)]$$

so that

$$0 \leq \limsup_{n \rightarrow \infty} n[V(n) - V(n+1)] \leq V(m) \text{ for any } m.$$

Hence, from (a) by letting $m \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} n[V(n) - V(n+1)] = 0. \quad (3.14)$$

Thus from (3.7), $\lim_{n \rightarrow \infty} \frac{\pi'(p_n^*)}{\lambda'(p_n^*)} = 0$

so that strict concavity of π and the assumption $\lambda'(\bar{p}) < \infty$ implies $\lim_{n \rightarrow \infty} p_n^* = \bar{p}$ which is (b).

Finally, from (3.5)

$$nV(n) = \frac{\pi(p_n^*)}{r} - \frac{\pi'(p_n^*)}{r\lambda'(p_n^*)} \lambda(p_n^*)$$

so that, taking limits and using (b), (c) follows.

Q.E.D.

Here $nV(n)$ is the total long-run industry profit given that there are n firms in the industry and that each follows its own optimal policy. (Note from (3.5) that $nV(n) \leq \frac{\pi(p_n^*)}{r} < \frac{\pi(\bar{p})}{r}$, the optimal profit for the whole industry, so that the optimal behavior

of individual firms is not optimal from the industry's viewpoint.) Then as the number of firms in the industry tends to infinity (a) implies that a new entrant can not make positive long run profits, (b) implies that the optimal price each firm charges increases to the monopolistic price, while (c) implies that the total industry profits increase to the monopolistic profits.

Before commenting further on the above results let us consider a simple example with linear demand and entry-rate functions. Let $\lambda(p) = a(p-c)$ and $D(p) = b - dp$ where a , b and d are positive constants. Then we have $\pi(p) = (p-c)(b-dp)$ leading to $V(n) = \frac{d(p_n^* - c)^2}{nr}$ which is quadratic in p_n^* and $p_n^* = c + \left[\frac{nrV(n)}{d} \right]^{\frac{1}{2}}$. Thus $\left[\frac{nrV(n)}{d} \right]^{\frac{1}{2}}$ is the premium the oligopolist can command over the competitive price, which is increasing in the rate of interest and decreasing in the elasticity of demand. Then $\lim_{n \rightarrow \infty} nV(n) = \frac{(b-dc)^2}{4dr}$, so that $\lim_{n \rightarrow \infty} p_n^* = \frac{b+dc}{2d} = \bar{p}$.

The next section concludes with further remarks and indicates some extensions.

4. Remarks:

In the preceding sections we have analyzed the price setting behavior of an intelligent firm, taking into account the possibility of new firms entering the industry. It was shown that, given a finite number of firms in the industry, the optimal price each should charge is strictly less than the (myopic) monopolistic price, because the threat of entry and the resulting damage to the future profits is appreciable with finite n . The difference between the

optimal price and the monopolistic price may be thought of as a measure of the opportunity cost for fearing a new entry. Also, given a finite number of firms in the industry, the optimal price each should charge to maximize its own long run profits is different from the price they would charge collectively (namely the monopolistic price, since the industry profits are unaffected by the number of firms) if their objective was to form a cartel and maximize the total industry profits; thus the individual and the industry interests are different.

As the number of firms in the industry increases to infinity, the marginal effect of one more entry on an existing firm becomes truly negligible in the mathematical sense, so that the existing firm loses any motivation for sacrificing immediate profits to retard an entry. Thus, with infinitely many firms, the interests of an individual firm coincide with those of the industry. Such an industry then acts like a "super firm" with no fear of entry, thus resembling a traditional monopolist charging the exploitative price \bar{p} , producing a small amount $D(\bar{p})$ and reaping the highest profits $\frac{\pi(\bar{p})}{r}$. (The myopic policy also turns out to be optimal.) This super firm is therefore socially less desirable than the industry with only a finite number of firms. Similarly, since $\lim_{n \rightarrow \infty} V(n) = 0 \leq V(n)$, each individual firm is also better off with a finitely many competitors than in the limit because its own market share declines to zero as the number of firms in the industry increases to infinity. In general, since $V(n)$ decreases to zero and p_n^* increases to \bar{p} , the industry with fewer firms is more desirable for both an individual firm and the society.

It should be noted that as $n \rightarrow \infty$, the convergence of the model to the traditional monopoly rather than the perfect competition is a result of the basic assumption that the firms are price setters rather than price takers. Rather than the number of firms in the industry their behavior pattern determines the market mechanism. It may be possible to develop a similar dynamic version of the Cournot model allowing stochastic entry, where each firm chooses its optimal output rate as a function of the number of firms in the industry and the prevailing price so as to maximize the long run return while assuming everybody else's output fixed. Such a model may then converge to the competitive solution. Of relevance here is a paper by Ruffin [7].

Another possible extension of our model may allow for exits from the industry by defining an exponentially distributed lifetime of each firm as an increasing function of the existing price, thus yielding a birth and death process controlled by the pricing policy. The entry (exit) rate may also be made a decreasing (increasing) function of the number of firms in the industry. Also we may allow different firms to have different structures by, say, assuming a probability distribution over their cost functions and the firms may share the total demand unequally; the total demand itself may be allowed to grow with time as in Kamien and Schwartz [5]. Finally, the assumption of constant returns to scale may be modified.

Some of these refinements and extensions do not change the model or the results significantly but the others, while yielding a more realistic and sophisticated version, make it formidable in terms of obtaining results.