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NATURAL ENERGY RESOURCE DECISIONS AND PRICES INVOLVING UNCERTAINTY*

by

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Abstract

This paper presents a general model for determining the optimal decisions and the corresponding shadow prices associated with a nonrenewable natural energy resource. The model takes into account the uncertain events of (a) exhaustion of the resource stock, (b) discovery of an additional stock, and (c) development of a producible substitute. One decision variable is the resource consumption rate; this yields utility but hastens exhaustion. Another, the exploration or research effort rate, incurs costs but expedites the search or developmental activity. With the objective of maximizing the expected discounted utility net of costs, dynamic programming methods are used to characterize the optimal consumption and exploration policies. Martingale methods are used to derive a fundamental new characterization of the stochastic process that represents the price of the natural resource. Several special cases are examined, and the related literature is reviewed.

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1. Introduction

With the increasing scarcity of natural resources in recent years there has been a growing interest in problems of optimal management of these resource stocks. Starting with the classic paper by Hotelling (1931), the more recent extensive literature on the economics of natural exhaustible resources is represented by Solow's (1974) exposition of the basic theory, essays in the 1974 Symposium of the Review of Economic Studies, and the monograph by Dasgupta and Heal (1978).

In general, however, only a few of the studies have explicitly incorporated the crucial element of uncertainty in their analyses. These studies involving uncertainty may be broadly classified into three categories. Models in the first category are concerned with optimal resource extraction decisions when the total supply of the resource stock is unknown and may be suddenly exhausted, as in Kemp (1976,77), Cropper (1976), Loury (1978), and Gilbert (1978) (or it may be expropriated as in Long (1975)). In the second category of models the uncertainty is regarding the availability of new supplies due to discovery of additional resource stocks through search and exploration. Arrow and Chang (1980) and Deshmukh and Pliska (1980) have studied optimal consumption and exploration decisions that affect the uncertain timings and magnitudes of discoveries. (See also MacQueen (1961,64), Heal (1978) and Pindyck (1980) for related models involving stochastic discoveries and Pindyck (1978) for the certainty case.) Finally, the third category of models involves uncertainty on the demand side, namely about the time at which a perfect producible substitute becomes available so as to eliminate the dependence of the economy on the natural resource. Dasgupta and Heal (1974) and Dasgupta and Stiglitz (1981) analyze optimal extraction decisions when the probability distribution of the uncertain timing of innovation of a substitute is specified exogenously, while Dasgupta, Heal and Majumdar (1977) and Kamien and Schwartz (1978) also permit the innovation process to be controlled through R & D expenditures. (In a related model, Hoel (1978) assumes that the time of innovation is known but the cost of producing the substitute is uncertain.)

In all three cases, therefore, the uncertainty is about the time of occurrence of a particular event: exhaustion, discovery of an additional stock, or development of a substitute product. Adopting this point of view, we present in this paper a single model that can capture each kind of uncertainty. In particular, we provide a stochastic dynamic programming formulation to analyze optimal decisions regarding extraction (for consumption) and exploration (to discover new stocks) or research (to develop a substitute) activities that control these uncertainties over time. We also study the corresponding resource price process resulting from these decisions. Finally, within the unifying framework of our model, we present as special cases a systematic derivation of the results available in the literature.

The general model is presented in section 2. The optimal decisions and the price process are studied in sections 3 and 4, respectively. Several special cases are studied in the four subsequent sections in light of the existing literature, while the final section concludes with some remarks and possible extensions.

2. The Model

The distinguishing characteristic of a natural energy resource (such as oil or natural gas) is that it is nonproducible and nonrenewable. Consequently, the future supply of the resource cannot be determined or controlled with certainty. In an extreme event, the resource may be exhausted, thereby imposing a severe hardship on the economy. At the other extreme, a perfect producible substitute may become available, rendering the natural resource inessential. Between these possibilities of extremely unfavorable and favorable events, several interesting random events may occur. One is the discovery of an additional stock of the same resource and another (apparently not studied in the literature) is an invention, such as electric car, that results in a major change in the demand for the resource.

In this section, we assume, for simplicity and consistency, that only one type of random event may occur and that it can occur only once. The time at which the event takes place is a random variable, and its probability distribution can be controlled through the extraction and exploration rate decisions. Resource extraction (and consumption) yields social utility but depletes the stock on hand and hastens exhaustion. On the other hand, exploration (i.e., searching for more resource or a substitute) involves search or R & D expenditures but also expedites the occurrence of the desirable event (i.e., discovery of an additional stock or development of a producible substitute). The problem then is to determine optimal extraction and exploration policies in face of the uncertain timing of occurrence of the event of interest.

Let the nonnegative random variable X_t denote the state of the natural resource in the economy at time $t \geq 0$. For instance, X_t may be the size of proven reserves on hand at time t or it may represent the cumulative amount extracted and consumed by t. Suppose the central planner's decision variable $c_t \varepsilon [0, \bar{c}]$ denotes the consumption (extraction) rate at which the resource stock is depleted at time t. This yields a social utility (net of extraction costs) to the economy at rate $U(c_t)$, where the utility function $U(\cdot)$ is usually assumed to be increasing and concave with $U(0) = 0$ or $-\infty$. Denote by $\alpha > 0$ the rate at which future utilities and costs are discounted.

While c_t advances the date of exhaustion, the exploration expenditure rate $e_t \ \varepsilon \ [0, \bar{e}]$ expedites the discovery of an additional stock or of a substitute through search or R & D activities. Let the random variable T represent the time when the event of interest occurs, and (borrowing terminology from reliability theory) let $\lambda(x,c,e)$ denote the hazard rate (success or failure rate) associated with the event time T, i.e., $\lambda(x,c,e)$ is the probabilistic rate of occurrence of the event at t, given that $T > t$, $X_t = x$, $c_t = c$ and $e_t = e$. Roughly, $\lambda(x, c, e)\, dt$ is the conditional probability that the event will occur during (t, t+dt), given that it has not occurred by time t, the resource state is $X_t = x$ and the consumption and exploration rate decisions are $c_t = c$ and $e_t = e$. One may assume that λ is nondecreasing in (c,e) in order to reflect the advancing of exhaustion through c or of discovery through e.

Once the uncertain event occurs at time T, the planner's problem becomes the relatively easy one of determining the optimal consumption pattern under certainty. Let W(x) denote the maximum attainable total discounted utility over $[T, \infty]$, given $X_T = x$. For instance, in the event of exhaustion and with X_t being the cumulative amount extracted, $W \equiv 0$. Similarly, with X_t as the resource stock on hand, in the case of discovery of size z, W(x+z) is the total utility from consuming the stock $(x + z)$ optimally, as in Hotelling (1931). In the case of a substitute discovery, $W(x)$ is the value of the optimal program of the substitute production and resource consumption, as in Dasgupta and Heal (1974). In general, we have $W(x) > V(x)$ if the random event is a favorable one and the reverse inequality holds in the unfavorable case.

In any event, we shall treat W as a specified terminal reward at time T. The planner's problem is then to determine $\{(c_t, e_t);\ 0 \leq t < T\}$ so as to maximize

$$E\{\int_0^T \exp(-\alpha t)\ [U(c_t) - e_t]dt + \exp(-\alpha T)\ W(X_T)|X_0=x\}.$$

Let $V(x)$ denote the <u>optimal value</u> of this program starting in the resource state $X_0 = x$. We now present a formal (as opposed to rigorous) derivation of the functional equation which V must satisfy. We simultaneously address two cases; X may be either the stock on hand or the cumulative consumption.

Selection of the constant decisions (c,e) during the interval $[0,t]$ with $t < T$, yields net utility $[U(c) - e]t$, and the resource state changes to $X_t = x - ct$ (if X is the stock on hand) or $X_t = x + ct$ (if X is the cumulative consumption). Also, the uncertain event occurs in $(0,t)$ with probability $\lambda(x,c,e)t + o(t)$, (in which case the optimal value from then on is determined by $W(X_t)$), and with probability $[1 - \lambda(x,c,e)t] + o(t)$ the event does not occur (in which case the optimal value from then on is $V(X_t)$). The dynamic programming argument then yields

$$V(x) = \underset{c,e}{\text{Max}} \{[U(c)-e]t + \exp\ (-\alpha t)[\lambda(x,c,e)t\ W(X_t) + (1-\lambda(x,c,e)t)V(X_t) + o(t)]\}$$

Using $\exp(-\alpha t) = 1-\alpha t + o(t)$, $X_t = x - ct$ (or $X_t = x + ct$) and the Taylor's expansion of $V(\cdot)$ and $W(\cdot)$ around x, dividing by t, and letting $t \downarrow 0$ yields the optimality equation

$$(1) \qquad \alpha V(x) = \underset{c,e}{\text{Max}} \{U(c) - e - cV'(x) + \lambda(x,c,e)[W(x)-V(x)]\}, \; x > 0.$$

Equation (1) is for the case where X represents the stock on hand, the case we shall primarily use for expositional continuity. If X is the cumulative consumption, then $V'(x)$ is replaced by $-V'(x)$.

Under suitable monotonicity and concavity assumptions on U, λ, and W, one may conclude the existence of a unique solution V of (1) with V satisfying certain monotonicity and concavity conditions. We shall not present a rigorous statement and derivation of these results here, since this would be a lengthy deviation from the objectives of this paper. Also, the methods involved are similar to those in the literature; see, for example, Deshmukh and Pliska (1980). Some additional remarks will be made about this later in this paper.

3. Optimal Decisions

Optimal decision policies $c^*(\cdot)$ and $e^*(\cdot)$ specify, as functions of the resource state $X_t = x$ at any time $t < T$, those consumption and exploration rates $c^*(x)$ and $e^*(x)$ that attain the maximum in (1). This leads to characterizations of these optimal policies. Throughout this section, we shall be dealing with the case where X represents the stock on hand; the case where X represents the cumulative amount consumed is similar and left to the reader. Note that under the optimal policies the stochastic process $X = \{X_t;\ 0 \leq t \leq T\}$ is a Markov process that terminates at T.

By equation (1), the optimal policies satisfy

$$(2) \quad \alpha V(x) = U(c^*(x)) - e^*(x) - c^*(x)V'(x) + \lambda^*(x)[W(x) - V(x)],$$

where we have written, for notational simplicity,

$$(3) \quad \lambda^*(x) = \lambda(x, c^*(x), e^*(x)).$$

Also, with the constraint that $X_t \geq 0$, we require $c^*(0) = 0$. Equation (2) may also be written in terms of the infinitesimal generator G of the Markov process X as follows. Define the expected rate of change in the optimal value $V(x)$ at time $t < T$ when $X_t = x$ as

$$GV(x) = \lim_{h \downarrow 0} \{E[V(X_{t+h}) | X_t = x,\ c_t = c^*(x),\ e_t = e^*(x)] - V(x)\}/h,$$

that is,

$$(4) \quad GV(x) = \begin{cases} -c^*(x)V'(x) + \lambda^*(x)[W(x) - V(x)], & x > 0 \\ \lambda^*(0)[W(0) - V(0)], & x = 0, \end{cases}$$

as may be seen by an analysis similar to the one leading up to equation (1). Thus, we may write equation (2) as

$$(5) \quad \alpha V(x) = r(x) + GV(x),\ x \geq 0,$$

where

$$(6) \quad r(x) = U(c^*(x)) - e^*(x)$$

is the net utility rate function under the optimal policies. By the theory of Markov processes, it follows (see Dynkin (1965) or Breiman (1968)) that total expected discounted utility $V(\cdot)$ following the policy $(c^*(\cdot), e^*(\cdot))$ is the unique solution of (5). Conversely, if $V(\cdot)$ satisfies (5) for some function r, then $V(x)$ is the total expected discounted return starting in state x if $r(X_t)$ is the running reward rate function and $W(X_T)$ is the terminal return function.

With these general remarks about the optimal value function $V(\cdot)$ we may now study the nature of optimal consumption and exploration policies. Assuming differentiability of the functions involved, interior optima $c^*(x)$ and $e^*(x)$ satisfy the first order conditions

$$(7) \quad V'(x) = U'(c^*(x)) + \frac{\partial \lambda^*(x)}{\partial c} [W(x) - V(x)]$$

and

$$(8) \quad \frac{\partial \lambda^*(x)}{\partial e} [W(x) - V(x)] = 1.$$

To interpret these conditions, recall that, if $t < T$ and $X_t = x$, then $\lambda^*(x)$ is the optimal probabilistic rate of occurrence of the event and $V(x)$ is the optimal expected long run net return over $[t,\infty]$. Thus, according to (7), optimal consumption rate balances the marginal reduction in the long-run return against the marginal instantaneous utility of consumption plus the marginal expected rate of change in the long-run return due to possible occurrence of the event. Similarly, according to (8), optimal exploration rate balances the latter against the cost of exploration. These conditions (7) and (8) together with the relevant properties of functions U, λ, W and V then enable us to characterize the structure of the optimal policies $c^*(\cdot)$ and $e^*(\cdot)$ as functions of the resource state x. For example, under appropriate conditions, $c^*(\cdot)$ is nondecreasing and $e^*(\cdot)$ is nonincreasing, implying greater consumption and less exploration in better resource states. Finally, since the resource level X_t is a (monotone nonincreasing) deterministic function of time t as long as $t < T$, we may also study the trajectory of the optimal consumption c_t^* and exploration rates e_t^* through time. We shall indicate these results in various special cases that are analyzed in the subsequent sections.

4. Shadow Prices

As above, let X be the Markov process representing the stock on hand under the optimal policies. Recalling again that, given $t < T$ and $X_t = x$, V(x) is the maximum expected discounted net utility to the society over $[t,\infty)$, it follows that $V'(x)$, the marginal contribution of an incremental unit of the resource stock, corresponds to the rent or the imputed (shadow) price (net of extraction costs) of the resource stock at that time. Similarly, $W'(X_T)$ is the resource price at time T. Thus, upon defining a process $P = \{P_t;\ t \geq 0\}$ by letting

$$P_t = \begin{cases} \exp(-\alpha t)V'(X_t), & t < T \\ \exp(-\alpha T)W'(X_T), & t \geq T, \end{cases}$$

it is clear that P should be interpreted as the <u>discounted shadow price process</u>. (For convenience, we have taken P constant on $[T,\infty)$). The purpose of this section is to show that P can be characterized in a certain way in terms of martingales. Indeed, under stated conditions, it will be seen that P actually is a martingale and this result will be shown to have a meaningful interpretation.

Let G denote the operator defined in terms of differentiable $f:[0,\infty) \to \mathbb{R}$ by

$$Gf(x) = \begin{cases} -c^*(x)f'(x) + \lambda^*(x)[W'(x) - f(x)], & x > 0 \\ \lambda^*(0)[W'(0) - f(0)], & x = 0. \end{cases}$$

Thus, G is the same as G in equation (4), only that W' is now the terminal reward instead of W. Moreover, just as in equation (5), f satisfies $\alpha f = Gf + g$ if and only if f is the expected discounted reward when g is the reward rate function and W' is the terminal reward. Therefore, by Dynkin's identity (see Dynkin (1965), Theorem 5.1 and its corollary) or by working out the conditional expectations, it is easy to verify that the stochastic process $M = \{M_t;\ t \geq 0\}$ defined by

$$M_t = \begin{cases} \int_0^t \exp(-\alpha s)[\alpha f(X_s) - Gf(X_s)]ds + \exp(-\alpha t)f(X_t), & t < T \\ \int_0^T \exp(-\alpha s)[\alpha f(X_s) - Gf(X_s)]ds + \exp(-\alpha T)W'(X_T), & t \geq T \end{cases}$$

is a martingale. In particular, taking $f = V'$ gives the following result.

(9) Proposition. The process

$$P_t + \int_0^{t\wedge T} \exp(-\alpha s)[\alpha V'(X_s) - GV'(X_s)]ds$$

is a martingale.

Recalling that a martingale plus a constant (respectively, increasing, decreasing) process is a martingale (respectively, submartingale, supermartingale), one immediately obtains the next result.

(10) Corollary. If $\alpha V'-GV' = 0$ (respectively, > 0, < 0), then P is a martingale (respectively, supermartingale, submartingale).

Thus, the nature of the process P hinges on the sign of $\alpha V'-GV'$. To examine this function, we first look at the equation (2) which V satisfies. Differentiating and collecting terms yields, for $x > 0$,

$$\begin{aligned}\alpha V'(x) = {} & \frac{dc^*(x)}{dx}\,[U'(c^*(x)) - V'(x) + \frac{\partial\lambda^*(x)}{\partial c}\,(W(x) - V(x))] \\ & + \frac{de^*(x)}{dx}\,[-1+ \frac{\partial\lambda^*(x)}{\partial e}\,(W(x) - V(x))] \\ & + \frac{\partial\lambda^*(x)}{\partial x}\,[W(x) - V(x)] + GV'(x).\end{aligned}$$

But the first two terms on the right hand side equal zero, because c^* and e^* satisfy the first order optimality conditions (7) and (8), so for $x > 0$, it must be that

$$\alpha V'(x) - GV'(x) = \frac{\partial\lambda^*(x)}{\partial x}\,[W(x) - V(x)].$$

In a similar way, one can show this same equation holds for $x = 0$. Consequently, we have derived the following main result of this section.

(11) Theorem. If $\frac{\partial\lambda^*(x)}{\partial x}\,[W(x) - V(x)] = 0$ (respectively, > 0, < 0), then the discounted shadow price process P is a martingale (respectively, supermartingale, submartingale).

We should remark that this derivation was made for the case where the process X represents the stock on hand, but everything remains true with only two modifications for the case where X represents the cumulative amount consumed. The first modification is to change the sign of the term c^*f' in the expression for the infinitesimal generator G. The second is to call -P the discounted shadow price process rather than P. This is because V and W will now be decreasing functions. Thus, Theorem (11) now becomes: if $\frac{\partial\lambda^*}{\partial x}$ $[W(x) - V(x)] = 0$ (respectively, > 0, < 0), then the discounted shadow price process is a martingale (respectively, submartingale, supermartingale).

In the case where P is a martingale, we have $E[V'(X_t)|X_0=x] = e^{\alpha t} V'(x)$, which is the stochastic analog of the well-known deterministic result (Hotelling (1931)): the shadow price rises at the rate of discount. The following sections will shed additional light and interpretation of our characterization of the discounted shadow price process by examining some specific cases.

5. Known, Fixed Resource Stock

In the rest of the paper we shall study the optimal value function V, the optimal policies c^* and e^*, and the discounted shadow price process P for several special cases. We begin by analyzing the benchmark case of certainty. In this classic case studied by Hotelling (1931), the given initial stock X_0 is to be consumed optimally over $[0,\infty)$ when no additional stock or a substitute is anticipated, so $T = \infty$ and $\lambda \equiv 0$. While the usual approach to analysis of this case employs variational calculus, our optimality equation (1) specializes to

$$(12) \quad \alpha V(x) = \underset{c}{\mathrm{Max}} \left\{U(c) - cV'(x)\right\}, \; x > 0.$$

It can be shown that, with U concave increasing, the unique solution V is concave and increasing in resource stock size x. Thus, the shadow price of the resource, $V'(x)$, is positive and decreasing in the amount on hand. As to the optimal consumption rate $c^*(x)$, an interior optimum in (12) requires $U'(c^*(x)) = V'(x)$. Thus, the marginal utility of consumption is equated with the marginal worth of unit consumption postponed. Since, by concavity, $U'(\cdot)$ and $V'(\cdot)$ are decreasing, optimal consumption rate $c^*(x)$ is increasing in the stock size.

By Theorem (11) and the fact that $\lambda \equiv 0$, we see that the discounted shadow price is a martingale. But this case is deterministic, so P is constant, i.e., the shadow price, $V'(X_t)$, rises at the social rate of discount α. By the first order optimality condition (7), therefore, the market price, which is the optimal marginal utility of consumption, $U'(c^*(X_t))$, also rises at the rate of discount. This is the "fundamental theorem of economics of exhaustible resources" and has the following economic interpretation (see Hotelling (1931), Solow (1974)). In the competitive resource market, equilibrium requires that the resource holders be indifferent between supplying at different points in time. This requires that the discounted prices must be the same at each point in time; otherwise profits could be increased by changing the supply pattern. Alternatively, the resource stock can be viewed as an asset and the equilibrium in the assets market requires that all assets yield the same rate of return, equal to the interest rate α. Hence the value of the resource stock must grow at rate α. If it grows slower, more will be supplied earlier and the resource will be exhausted too quickly. If the price grows at a rate faster than α, then it is better for the suppliers to hold the stock as an investment that yields a rate of return higher than α. Finally, if the resource is owned by a monopolist, the corresponding statement is that his marginal profit must rise at the rate of interest.

Given the above price dynamics, the optimal consumption pattern over time can be derived. Since $c^*(\cdot)$ is increasing in the stock size which is depleting over time (in absence of new discoveries), the optimal consumption rate declines through time, and under the commonly made assumption $U'(0) = -\infty$, exhaustion occurs only asymptotically. More precisely, since $P_t = \exp(-\alpha t)U'(c_t^*)$ is a constant, upon differentiating with respect to time, one obtains

$$\dot{c}_t^*/c_t^* = -\alpha/\eta(c_t^*),$$

where $\eta(c) = -c\, U''(c)/U'(c)$ is the elasticity of marginal utility. For example, if $U(c) = c^{(1-\varepsilon)}$ for $0 < \varepsilon < 1$, then $\eta(c) = \varepsilon$ and hence $c_t^* = c_0^*\, e^{-\alpha t/\varepsilon}$, i.e., optimal consumption decreases exponentially. The initial rate of c_0^* is then chosen so that $\int_0^\infty c_t^*\, dt = X_0$, i.e., $c_0^* = \alpha\, X_0/\varepsilon$.

6. Extraction of Fixed Uncertain Stock

This is the case of optimally "eating a cake of unknown size" studied by Kemp (1976,77), Cropper (1976), Loury (1978) and Gilbert (1979). The total stock size is a random variable S with the distribution function $F(\cdot)$ and the density function $f(\cdot)$, and no additional discoveries of the resource or a substitute are expected. The resource state X_t is the cumulative amount consumed by time t and the "event" corresponds to exhaustion, so that $T = \min\{t \geq 0;\ X_t = S\}$. In addition, the terminal reward $W \equiv 0$, and the only control variable is the consumption rate.

Given $X_t = x < S$, the failure rate of S is $\lambda(x) = f(x)/[1 - F(x)]$. It follows that if the consumption rate is $c_t = c$, then the hazard rate of the time of exhaustion is $c\lambda(x)$. Thus, the optimality equation (1) specializes to

(13) $\alpha V(x) = \underset{c}{\mathrm{Max}}\ \{U(c) + cV'(x) - c\lambda(x)\ V(x)\},\ x > 0.$

The optimality condition (7) now becomes

(14) $- V'(x) = U'(c^*(x)) - \lambda(x)V(x),\ x > 0.$

Substituting (14) into the specialization of (2) yields

(15) $V(x) = [U(c^*(x)) - c^*(x)U'(c^*(x))]/\alpha,\ x > 0,$

By concavity of U, the numerator on the right hand side of (15) is nonnegative and is interpreted as the consumer surplus (i.e., the difference between the amount the consumer is willing to pay and what he actually pays). Thus, the optimal value of the stock is the discounted value of the current consumer surplus. Combining (14) and (15) yields the following relationship between the shadow price and the consumption rate when $X_t = x$:

(16) $-V'(x) = U'(c^*(x)) - \lambda(x)[U(c^*(x)) - c^*(x)U'(c^*(x))]/\alpha.$

To analyze the price process, we make the reasonable assumption that F is an increasing failure rate distribution, i.e., $\lambda'(x) > 0$ for all x, so that the likelihood of immediate exhaustion increases as more resource is consumed. This would be the case, for example, if S is a uniform random variable. Since $\frac{\partial\lambda(x,c,e)}{\partial x} = c\lambda'(x)$, $W = 0$, and $V > 0$, this means by the remark following Theorem (11) that the discounted shadow price process is a supermartingale. In other words, and speaking loosely, the (undiscounted) shadow price is expected to rise at a rate slower than the rate of discount. In fact, the price might even fall during intervals of time. By analyzing (16), it can be shown that the consumption rate must either always fall or first fall and then rise over time. If the discount rate α is small (i.e., if the society is more future-oriented), then the consumption rises through time (i.e., it is postponed). See Kemp (1976), Cropper (1976), and Loury (1978) for details and additional economic interpretations.

We close this section by considering a special case in which $\lambda(x)$ is a constant λ or, equivalently, when the distribution of the resource stock size is exponential, i.e., $F(x) = 1 - e^{-\lambda x}$. From the memorylessness property of the exponential distribution, it is clear that, given no exhaustion yet, the optimal value $V(x)$ is a constant $\overline{V}$ independent of the cumulative consumption x. Hence, (13) becomes $\alpha \overline{V} = \underset{c}{\text{Max}} \{U(c) - c\lambda\overline{V}\}$ and the optimal consumption rate is a constant c^* which satisfies $U'(c^*) = \lambda\overline{V}$. Also $\overline{V} = U(c^*)/(\alpha + \lambda c^*)$, which is the expected discounted utility from the constant consumption rate c^* until the moment of exhaustion. The resource uncertainty may thus be viewed as raising the discount rate from α to $\alpha + \lambda c^*$.

7. Exploration and Uncertain Discovery of Additional Stock

In the previous section, learning about the uncertain stock size was accomplished through extraction alone; the probability distribution of the stock size was then updated over time by merely using the fact that the true stock has to be at least as large as the cumulative amount already extracted. In this section, exploration is considered as a distinct activity of learning that involves expenditures to search for and discover the existence of additional stocks. Pindyck (1978) has considered the exploration activity under certainty, MacQueen (1961,64) and Heal (1978) have studied related models involving uncontrolled stochastic discoveries, while Arrow and Chang (1980) and Deshmukh and Pliska (1980) have analyzed optimal consumption and exploration decisions when the latter controls the uncertainty about timings and/or magnitudes of discoveries.

In this section, X_t denotes the size of proven reserves on hand at time t and the "event" refers to the discovery of a new stock. We assume that only one discovery is possible and it occurs at a random time T which can be controlled through the exploration expenditure rate $e \in [0,\bar{e}]$. The probabilistic rate of discovery $\lambda(e)$ is (now independent of x and c) assumed to be increasing in e, with $\lambda(0) = 0$. Let the nonnegative random variable Z denote the size of the stock discovered at T and suppose $H(\cdot)$ is the probability distribution of Z.

If the resource stock just before the discovery is $X_{T-} = x$ and if the discovery is of size Z=z, then the post-discovery deterministic problem is that of optimally consuming the total resource stock $X_T = (x + z)$ on $[T, \infty)$. This problem was analyzed in Section 5; let $\hat{V}(\cdot)$ denote the concave increasing function that is the solution of (12). Consequently, the terminal reward at T for the problem in the present section is

$$(17) \quad W(x) = \int_0^\infty \hat{V}(x + z)\, dH(z), \; x \geq 0.$$

Note that $W(\cdot)$ is concave and increasing with $W(\infty) = U(\bar{c})/\alpha$.

With this $W(\cdot)$, the optimality equation (1) now becomes

$$(18a) \quad \alpha V(x) = \max_c \{U(c) - cV'(x)\} + \max_e \{-e + \lambda(e)\,[W(x) - V(x)]\}, \; x > 0,$$

with the boundary condition $c^*(0) = 0$ yielding

(18b) $\alpha V(0) = \underset{e}{\mathrm{Max}} \left\{-e + \lambda(e)\ [W(0) - V(0)]\right\}.$

In Deshmukh and Pliska (1980) it was shown for a similar model (with an unlimited number of discoveries permitted) that V is concave increasing, that the optimal consumption policy c^* is increasing, and that the optimal exploration policy e^* is decreasing. The following provides some similar results for the present model.

(19) Theorem. The optimal value V is a concave increasing function with $V \leq W$ and $V(\infty) = U(\bar{c})/\alpha$. The optimal consumption policy c^* is increasing.

Proof. To see why $V' \geq 0$, compare the two optimal control problems corresponding to two arbitrary starting points $x_1 < x_2$. Use c^* starting at x_1, but use $c^*(x - x_2 + x_1)$ starting at x_2, so the consumption rate, and thus the utility rate $U(c_t)$, will be exactly the same at each time until time T, when the discovery is made. Similarly, use the optimal exploration policy e^* starting at x_1, but use $e^*(x - x_2 + x_1)$ starting at x_2, so the random variable T will be the same in both cases. Thus, the expected discounted return up to time T will be the same in the two cases. Since X_T for the case starting at x_2 will exceed X_T for the case starting at x_1 by exactly $x_2 - x_1$, the terminal reward $W(X_T)$ will be greater for the case starting at x_2. Since the optimal policy was used starting at x_1 but not at x_2, this all means $V(x_2) \geq V(x_1)$.

The argument why $V \leq W$ is similar. Let the starting point x_1 be arbitrary. In the first case, use the optimal consumption policy c^* until time T when a discovery of size Z occurs, and then use the optimal consumption policy from the Hotelling model (see Section 5). The optimal exploration policy e^* is also used, so the expected discounted return is simply $V(x_1)$. In the second case, start at the random level $x_1 + Z$ and use the consumption policy $c(x - Z + x_1)$ until time T, where T has the same distribution as in the first case, but no exploration is going on here. After T, use the optimal consumption policy from the Hotelling model. This policy is not necessarily the optimal way to consume the quantity $x_1 + Z$, so the expected discounted return for the second case is less than $W(x_1)$. Now, compare the two cases. The consumption rate at each time before T is the same. At time T, the stock level will be the same, even though the quantity Z was added at different times, so the consumption rate at each time after T is the same. Thus, the expected discounted utility due to consumption is the same in both cases. But the first case involves exploration costs, so $V(x_1)$ is less than the expected discounted return for the second case, which in turn is less than $W(x_1)$.

With an infinite supply of stock, the consumption policy $c(\cdot) \equiv \bar{c}$ yields a return of $U(\bar{c})/\alpha$. Since $W(\infty) = U(\bar{c})/\alpha$ it is clear no higher return can be achieved, so $e^*(\infty) = 0$ and $V(\infty) = U(\bar{c})/\alpha$.

To see why V is concave, suppose not. Since W is concave, V is nondecreasing and $V \leq W$, there exist points $x_1 < x_2$ with $W(x_1) - V(x_1) = W(x_2) - V(x_2)$ and $V'(x_1) < V'(x_2)$. Rewrite the functional equation (15a) in the form

$$V'(x) = \max_{c,e} \left\{\frac{1}{c} [U(c) - e - \alpha V(x) + \lambda(e)[W(x) - V(x)]\right\}$$

(see Deshmukh and Pliska (1980) for a demonstration of a similar transformation). Now compare $V'(x_1)$ and $V'(x_2)$. Since $W(x_1) - V(x_1) = W(x_2) - V(x_2)$ and $V(x_1) \leq V(x_2)$, this equation says $V'(x_1) \geq V'(x_2)$, a contradiction.

Finally, to see that c^* is increasing, differentiate the first order optimality condition (7) to show that V is concave if and only if c^* is increasing.

Thus, the resource stock has a decreasing marginal value, and it is optimal to consume more rapidly at higher levels of this stock. Unfortunately, unlike our previous model, we are unable to show that the optimal exploration rate e^* is decreasing in the level of stock. Note, however, that with a reasonable assumption that $\lambda(\cdot)$ is nondecreasing in the exploration effort e, upon differentiating the first order optimality condition (8), one sees that $e^*(x)$ is decreasing in x if and only if $W(x) - V(x)$ is decreasing in x.

In other words, higher resource levels are associated with reduced exploration if and only if the post-event shadow price, $W'(x)$, is lower than the pre-event one, $V'(x)$. This would be another criterion for saying that the event is a favorable one. However, for a related model in Deshmukh and Pliska (1981) we have provided a counterexample in which $W > V$ but $W' > V'$.

Since $\frac{\partial \lambda}{\partial x} = 0$, it follows from Theorem (11) that the discounted shadow price P is a martingale, i.e., the expected shadow price rises at the rate of discount. This is the stochastic analog of Hotelling's (1931) result discussed in Section 5, wherein no additional discoveries were possible. Note that if W-V is decreasing as discussed above, the discounted shadow price process P makes a downward jump at time T. Since this process is a martingale, this means it increases before time T in just the right way to compensate for the jump at time T.

8. R & D and Uncertain Development of a Substitute

In the previous section, the occurrence of the favorable event of discovery of new stock relaxed the resource constraint temporarily, that is, it postponed the moment of exhaustion. In this section, we consider the possibility of an extremely favorable event (a technological change) that permanently eliminates the resource constraint as a result of the development of a producible perfect substitute. The substitute development process may be expedited by allocating higher R & D expenditures. The special case of uncontrolled development was analyzed by Dasgupta and Heal (1974), Dasgupta and Stiglitz (1981) and Hoel (1978), while Dasgupta, Heal and Majumdar (1977) and Kamien and Schwartz (1978) have also permitted the development process to be controlled endogenously.

Let X_t be the size of the natural resource stock on hand at time t and suppose T corresponds to the random time at which the perfect substitute becomes available. If the substitute can be produced from then on at a unit cost of k and if $X_T = x$, the planner's problem on $[T, \infty)$ is to determine the substitute production rate $s_t \; \varepsilon [0,\bar{s}]$ and the resource consumption rate $c_t \varepsilon [0,\bar{c}]$, $t > T$, so as to maximize

$$\int_T^\infty e^{-\alpha t}[U(c_t + s_t) - ks_t] \text{ subject to } \int_T^\infty c_t dt = x.$$

Let $W(x)$ be the optimal value of this program, given $X_T = x$. Then, the dynamic programming argument yields the following optimality equation

$$(20a) \quad \alpha\, W(x) = \underset{c,s}{\text{Max}} \left\{U(c + s) - ks - cW'(x)\right\}, \; x > 0$$

with

(20b) $\alpha\, W(0) = \underset{s}{\text{Max}} \{U(s) - ks\}$.

It can be shown that the optimal value function $W(x)$ is concave increasing in x. Optimal consumption and production rates $c^*(x)$ and $s^*(x)$ are then obtained as the maximizers in (20) and can be characterized as follows. If $W'(x) < k$ (i.e., the imputed price of the resource is less than the cost of producing the substitute) then $s^*(x) = 0$ and no substitute production takes place. Also $U'(c^*(x)) = W'(x)$, so that, as before, $c^*(\cdot)$ is increasing. As the resource stock depletes over time, the consumption rate decreases and the shadow price rises at rate α (as in Section 5) until $U'(c^*(x)) = W'(x) = k$. At that time the resource stock is just exhausted and the optimal production rate s^* from then on is determined by $U'(s^*) = k$, with $W(0) = [U(s^*) - ks^*]/\alpha$.

With an infinite level of stock, it is optimal to consume at the maximum rate $\bar{c}$. If also $U'(\bar{c}) < k$, then it is optimal for no substitute production to take place, in which case $W(\infty) = U(\bar{c})/\alpha$.

Prior to the development instant T, the control variables are c (the resource consumption rate) and e (the R & D expenditure rate). The former depletes the resource and the latter increases the rate $\lambda(e)$ of discovery of the substitute. The resulting optimal value function V then satisfies the same optimality equation (18) as in the previous section, except now $W(\cdot)$ is given by (20) instead of (17). The same kinds of results can be obtained, namely, if $U'(\bar{c}) < k$ (so $W(\infty) = U(\bar{c})/\alpha$), then V is concave increasing with $V \leq W$ and $V(\infty) = U(\bar{c})/\alpha$, and the resource consumption rate prior to discovery of the substitute is increasing in the resource stock. The only difference from the proof of Theorem (19) is the argument why $V \leq W$. Briefly, in the first case you start at x_1 and use the optimal policy both before and after T, so the expected discounted return is $V(x_1)$. In the second case, you start at x_1 and use exactly the same consumption policy, but do no substitute production, before T, and then you consume and produce the substitute in the same way as in the first case. Thus, the expected discounted return in the second case is greater, because there are no R&D expenditures. Finally, upon considering the problem corresponding to (20), one sees that the expected discounted return for the second case is less than or equal to $W(x_1)$. Unfortunately, again we are unable to demonstrate that $W' < V'$, so that the optimal R&D expenditure rate is decreasing in the level of stock.

As to the prices, since $\frac{\partial \lambda}{\partial x} = 0$, the discounted shadow price process is again a martingale. If at the time of invention of the substitute $X_T = x$, then the undiscounted price changes from $V'(x)$ to $W'(x)$. Then it rises at rate α until it becomes $k = W'(0)$ and stays at that level from then on.

In summary, as the resource level falls over time, the stock price rises, the consumption rate is reduced and the intensity of the R & D activity is increased until a substitute is discovered. At that time the stock price changes, the consumption rate is increased and R & D expenditures become unnecessary. Then the price rises and only the resource is consumed until it is exhausted. At that point, the resource price just equals the cost of producing the substitute. From then on, the constant rate of consumption is sustained only through the substitute production.

9. Remarks

We have presented a general model of natural resource decisions that involves uncertainty regarding the time of occurrence of some significant event of interest. The consumption rate decision depletes the resource stock and the exploration rate decision expedites the occurrence of a favorable event. Dynamic programming and probability theory were employed to characterize the optimal value function, optimal decision policies and the behavior of prices. The model was then specialized to the analysis of three cases involving the events that are most unfavorable (exhaustion), somewhat favorable (discovery of a new stock) and most favorable (development of a substitute). The analysis was mostly heuristic and the emphasis was on intuitive arguments and interpretations rather than on the technical details involved. The related literature was also reviewed within the context of the general model and its four special cases.

The model could be extended along two significant directions. It may be important to allow for the possibility of occurrence of multiple random events, such as a sequence of discoveries of new stocks (as in Deshmukh and Pliska (1980)) or a sequence of partial substitutes developed. Secondly, the probabilistic rate of occurrence of the events should depend not only on current decisions but also on some aspect of the environment and the past history (such as the time elapsed, the cumulative amount of stock discovered or the cumulative R & D expendutures) which may expedite or delay the occurrence of the event; we have incorporated such factors in a model involving discoveries of new stocks and a random environment in a companion paper (Deshmukh and Pliska (1981)).

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