DISCUSSION PAPER NO. 491

THE INDETERMINACY OF INTEREST RATES */

by

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August 1981

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*/ The research reported here was done with the support of National Science Foundation Grant No. SOC 79-06672

1. Introduction

The main idea I wish to convey is that interest rates should be viewed as determined by government policy. This idea conflicts with a prejudice which seems to be part of the neoclassical tradition and which assumes that interest rates are determined solely by irresistible market forces. The theoretical justification for this prejudice has been explained carefully by Irving Fisher (1930). Nevertheless, this prejudice is not consistent with results to be found in the literature on Samuelson's consumption loan model. It has often been observed that his model has many stationary equilibria, each with a different interest rate and a different level of government debt. This interest rate can be any non-negative number, and, as long as it is not less than the rate of population growth, the equilibrium is efficient. (See especially Diamond (1965).)

There is an obvious and banal way to reconcile these results with the neoclassical prejudice -- market forces determine interest rates once the level of government debt is given. But this reconciliation does not say enough, for all interest rates which are not less than the rate of population growth are equally attractive from the neutral point of view traditional in welfare economics. All are the outcome of a Pareto optimal equilibrium. All these equilibria maximize a social welfare function. In effect, the government chooses the interest rate and any choice can be rationalized using a social welfare function.

The literature on the consumption loan model has tended to obscure the indeterminacy of the interest rate by treating the golden rule allocation as the sole optimal allocation. (In the golden rule equilibrium, the interest rate equals the rate of population growth.) The golden rule maximizes a social welfare function, which, roughly speaking, gives equal weight to all generations. But this function is just one of many possible welfare functions. There seems to be no reason to favor it, other than personal preference. For this reason, I will emphasize that the nominal rate of interest may be viewed as the rate of discount or pure time preference in a social welfare function.

It is this social rate of time preference that the government determines. In a stationary equilibrium, this rate equals the real rate of interest. But if society's capital stock is smaller than the steady state level, then the real interest rate exceeds the social rate of time preference. As the capital stock grows, the real rate converges to society's rate of time preference. Under normal circumstances, lower rates of time preference correspond to higher rates of growth. Thus, the government also chooses the economy's rate of growth and all rates of growth are equally acceptable from the point of view of welfare economics.

One might try to make the rate of interest determinate by insisting that the government not intervene. I say that the government intervenes if its aggregate net indebtedness is not zero. However, Samuelson (1958) has shown that if the government does not intervene, then there may exist no Pareto optimal equilibrium.

The views just expressed make sense in Samuelson's overlapping generations model. However, they can be challenged easily by introducing a bequest motive in the way that Barro (1974) does. Barro assumes that parents, when deciding on bequests to their children, weigh their own utility against that of their children. He shows that if the bequests are positive, then issuing government debt has no effect at all on interest rates. Because of the obvious importance of Barro's observation, much of what follows is devoted to a critical examination of the theory of interest in models with bequests. First of all, I analyze the theory of interest in a special case of Barro's model. I assume that all consumers have the same rate of pure time preference, ρ, and prove that there exist stationary equilibria with interest rates anywhere in the interval $[0,\rho]$. All these equilibria are Pareto optimal, and a stationary allocation is Pareto optimal only if its associated interest rate falls in the interval $[0,\rho]$. (Throughout the paper, I assume that the population is constant.) If one insists that the government not intervene, then the interest rate becomes determinate. Thus, Barro's insight leads to a theory which makes the interest rate determinate.

One could criticize this theory simply by dismissing the model. A key assumption is that a parent's utility depends on that of his child. This implies interpersonal comparison of utility, which is usually thought of as highly suspect. However, the idea that parents care about their children is obviously attractive, so that I do not reject the model outright.

The interest rate has been made determinate by excluding government intervention. If intervention is permitted, the only effect of the bequest motive is to confine the interest rate to the interval $[0,\rho]$ rather than to the half line $[0,\infty)$. One could, quite reasonably I believe, claim that it is silly to exclude government intervention. However, such is the appeal of an equilibrium without intervention that I feel obliged to use a more elaborate argument to criticize the theory of interest based on Barro's model.

What I do is to make his model slightly more realistic by allowing the number of children born to each parent to vary among parents. When this is done, government intervention is necessary in order to obtain a Pareto optimal stationary equilibrium. Without intervention, equilibria either do not exist or are not Pareto optimal, depending on the assumptions made about the determination of family size. Thus, government intervention is necessary for the usual nicer properties of equilibrium. With these arguments, I do not mean to advocate intervention but to break up the construction built on Barro's model. The difficulties that make intervention necessary arise because parents experience the utility of their children and because I have included this vicarious utility in the definition of Pareto optimality. Both of these assumptions seem questionable.

I also discuss very briefly Barro's gift motive, which is the desire of children to support their parents in their old age. When formulated in a certain way, the motive makes ρ a lower bound on the interest rate. If both a gift motive and a bequest motive are included, the theory falls apart when the rate of pure time preference varies among individuals.

After discussing the effects of variability in family size, I make some arguments which I claim are based on common sense. The theme of the paper is that general equilibrium does not determine interest rates. One of the concepts closely associated with equilibrium theory is Pareto optimality. Therefore, it is important to establish which definition of Pareto optimality is best associated with equilibrium theory. I try to distinguish between such a definition and a notion of Pareto optimality appropriate for the theory of public choice. I claim that a notion of Pareto optimality associated with equilibrium theory should not include the utility for bequests. This utility should be included only in a notion of optimality associated with public choice, and such a notion could just as well include each consumer's attitudes towards the welfare of all people other than himself.

When the utility for bequests is excluded from the definition of Pareto optimality, then the conclusions are the same as in the case of Samuelson's consumption loan model. Any non-negative interest rate corresponds to a Pareto optimum, and government intervention can achieve any non-negative interest rate, (though the intervention might have to include an inheritance tax).

I also argue that the utility for bequests should be treated as arbitrary, just as the utility for consumption is usually treated as arbitrary. Barro's specification of the utility for bequests should be treated as one of many possibilities. Only empirical work can tell us which specification is most realistic.

In conclusion, I think it is most appropriate and useful to think of the government as determining implicitly the social rate of pure time preference. It is then possible to think of the economy as adapting to this rate in a manner not very different from that described in the theory of optimal growth. (See McKenzie (1980) for a survey of this theory.)

The plan of this paper is as follows. In section 2, I discuss the theory of interest in a consumption loan model with no bequests. In section 3, I discuss briefly the theory which obtains when consumers are immortal. (The interest rate is determined in such a model.) Section 4 describes the theory of interest in a version of Barro's model. In section 5, I discuss what happens in this model when the number of children born to each parent varies. Section 6 contains a general discussion of the theory of interest. In section 7, I relate my work to the literature.

Many of the arguments found in this paper are not original. This is especially true of section 2. What is new here, I believe, is that I present a coherent argument that general equilibrium does not determine interest rates.

John Geanakoplos (1980), is the only author I have found who has made a connection between the theory of the consumption loan model and the indeterminacy of interest rates. His work is discussed in section 7.

Two secondary matters discussed in the paper may interest some readers. I show that if one allows the social rate of time preference to fluctuate, then one obtains a kind of Pareto optimal trade cycle (example 2.14). Also, at the end of section 2, I note that the model I use provides no justification for the monetary policy espoused by Friedman, which increases the money supply at a constant rate.

2. Interest Rates Without Bequests

In this section, I elaborate ideas that have already appeared in the literature. However, I want to make sure that these ideas are thoroughly understood and so I try to make them clear. Also, I believe I make a small contribution by pointing out that nominal interest rates may be interpreted as discount rates in a social welfare function. Related literature is discussed in section 7.

The model I use is very similar to that of Diamond (1965) in that I adjoin a production sector to the Samuelson consumption loan model. All consumers have identical utility functions and endowments. One consumer is born each period. Each consumer lives $A+1$ periods, where $A \geqq 1$. There are two goods, labor and one produced good, which I call food. L_a denotes the labor supply of a consumer in his a^{th} period of life, where $a = 0,1,\ldots,A$. The utility function of a consumer is $\sum_{a=0}^{A} (1+\rho)^{-a}u(x^a)$, where x^a is his consumption of food in his a^{th} period of life. The number ρ is his rate of pure time preference. I assume that

2.1) u is continuously differentiable, strictly increasing, and strictly concave.

There is one industry. The production function of the industry is $y = f(k,L)$, where k is the input of food, L is the input of labor, and y is the output of food. I assume that

2.2) f is continuously differentiable, concave, homogeneous of degree one, non-decreasing and such that $f(k,L) = 0$ if either k or L is zero. Also $\lim_{k\to 0} \frac{\partial f(k,1)}{\partial k} = \infty$.

These assumptions imply that $\lim_{k\to\infty} \frac{\partial f(k,1)}{\partial k} = 0$. For if $\frac{\partial f(k,1)}{\partial k} \geq a > 0$, for all k, then $f(k,1) \geqq ak$, so that $f(1,k^{-1}) \geqq a$, which is impossible since $\lim_{k\to\infty} f(1,k^{-1}) = 0$.

An allocation for the economy is of the form $((x_t^a),(k_t))$, where (k_t) represents the sequence $k_0, k_1, \ldots$ and (x_t^a) denotes the numbers x_t^a, for $a = 0,1,\ldots,A$ and t such that $a + t \geqq 0$. x_t^a denotes the consumption of food during period $t+a$ by the consumer born in period t. k_t denotes the quantity of food set aside during period t by firms in order to be used in production during period $t+1$.

The allocation $((x_t^a),(k_t))$ is feasible with initial stock k_{-1} if $\sum_{a=0}^{A} x_{t-a}^a + k_t = f(k_{t-1}, \sum_{a=0}^{A} L_a)$, for $t = 0,1,\ldots$. Assumption (2.2) implies that all feasible allocations are bounded. The allocation is stationary if x_t^a and k_t do not depend on t. A stationary allocation is denoted by $((x^a),k)$.

An allocation $((x_t^a),(k_t))$ which is feasible with initial stock k_{-1} is said to be Pareto optimal if there is no other allocation which is feasible with the same initial stock and which makes every consumer at least as well off and at least one better off.

The description of an equilibrium requires numbers p_t, w_t, r_t and τ_t, for $t = 0,1,\ldots$, where p_t is the price of food at time t, w_t is the wage, r_t is the nominal interest rate, and τ_t is the tax. Interest is paid at rate r_t during period $t+1$ on funds held from period t to period $t+1$. The quantity τ_t is a lump-sum tax paid by each consumer alive in period t. The tax may be negative, in which case it represents a subsidy. The description of an equilibrium also requires numbers k_{-1} and C_a, for $a = 1,\dots,A$. C_a is the net credit position of the consumer of age a at the beginning of period 0. k_{-1} is the initial capital stock.

An equilibrium consists of $((x_t^a),(k_t);\ (p_t),(w_t),(r_t),(\tau_t);\ k_{-1},(C_a))$. These must satisfy the following conditons.

(i) $((x_t^a),(k_t))$ is a feasible allocation with initial stock k_{-1}.

(ii) For $t \geqq 0$, k_t and $\sum_{a=0}^{A} L_a$ solve the problem

$$\max_{k\geqq 0, L\geqq 0} [p_{t+1}f(k,L) - (1+r_t)p_t k - w_{t+1}L].$$

(iii) $\sum_{a=0}^{A} L_a$ solves the problem

$$\max_{L\geqq 0} [p_0 f(k_{-1},L) - w_0 L].$$

(iv) If $t \geqq 0$, then $(x_t^a)_{a=0}^{A}$ solves the problem $\max \sum_{a=0}^{A} (1+\rho)^{-a}u(x^a)$

subject to $x^a \geqq 0$, all a, and

$$\sum_{a=0}^{A} [\prod_{s=0}^{a-1} (1+r_{t+s})^{-1}][p_{t+a}x^a - w_{t+a}L_a + \tau_{t+a}] \leqq 0.$$

(v) If $-A \leqq t < 0$, then $(x_t^a)_{a=-t}^{A}$ solves the problem

$$\max \sum_{a=-t}^{A} (1+\rho)^{-a}u(x^a) \text{ subject to } x^a \geqq 0, \text{ all } a, \text{ and}$$

$$\sum_{a=-t}^{A} [\prod_{s=-t}^{a-1} (1+r_{t+s})^{-1}][p_{t+a}x^a - w_{t+a}L_a + \tau_{t+a}] \leqq C_{-t}.$$

(vi) The sequence $p_0, p_1, \dots$ is bounded away from zero and bounded from above. $p_0 = 1$.

An equilibrium is <u>stationary</u> if none of the variables $x_t^a, k_t, p_t, w_t, r_t$ or τ_t depend on t. In a stationary equilibrum, the p_t may be suppressed since they all equal 1. Also, k_{-1} and the C_a may be suppressed, since they are determined by transactions in the stationary state. Therefore, a stationary equilibrium is written as $((x^a), k; w, r, \tau)$. In such an equilibrium, r is the real as well as the nominal interest rate.

In interpreting an equilibrium, it may be helpful to keep in mind the following description of economic life. Time is divided into days and nights. Days correspond to the enumerated time periods. All business activity takes place during the day. During each day, firms accumulate the supplies they will use in production during the following day. The labor of a given day transforms materials into output available during the same day. All payments are made with credit balances at a central bank. The bank pays interest at rate r_t on balances held overnight from day t to day $t+1$. The interest is paid on day $t + 1$. Negative balances pay interest at the same rate r_t. The only budget constraint on consumers is that they have a non-negative balance at the end of their lives. Otherwise, consumers may borrow and lend freely.

Condition (vi) above guarantees that real interest rates are not influenced by a persistent inflation or deflation. If one eliminates condition (vi), it is possible to eliminate the nominal interest rates r_t from the description of equilibrium. If $((x_t^a), (k_t); (p_t), (w_t), (r_t), (\tau_t); k_{-1}, (C_a))$ is an equilibrium, then $((x_t^a), (k_t); (\hat{p}_t), (\hat{w}_t), (0), (\hat{\tau}_t), k_{-1}, (C_a))$ satisfies conditions (i) - (iv) above, where, for all t, $\hat{p}_t = b_t p_t, \hat{w}_t = b_t w_t$, and $\hat{\tau}_t = b_t \tau_t$, and where $b_t = \prod_{s=0}^{t-1} (1 + r_s)^{-1}$.

I have chosen to include the (r_t) and condition (vi) for several reasons. First of all, the equilibria are easier to interpret if the p_t are bounded and are bounded away from zero. Secondly, if the p_t are bounded, then the r_t may be interpreted as discount rates in a social welfare function.

Thirdly, the inclusion of the r_t makes it possible to distinguish monetary policy (the r_t) from fiscal policy (the τ_t). This distinction is not entirely artificial, for there exist many different equilibria for the same sequence r_t, each of the equilibria having a different sequence τ_t. (See remark (2.7) below.)

Another reason for including condition (vi) is that it eliminates a source of indeterminacy of real interest rates. In a model in which consumers live two periods, it is easy to see that once the sequences (r_t) and (τ_t) and the initial conditions k_{-1} and C_1 are given, then the entire equilibrium is determined by the numer p_0. It may happen that an equilibrium exists for each value of p_0 in an entire interval and yet only one of these equilibria satisfies condition (vi). The other equilibria would display persistent inflation in that the sequence p_t would diverge to infinity. Hence, the real rates of interest depend on the choice of p_0. By excluding these inflationary equilibria, I strengthen my argument that interest rates are indeterminant.

There is a large literature on these inflationary equilibria, initiated by Gale (1973) and Black (1974). Others who have discussed these equilibria include Geanakoplos (1980), Calvo (1978,1979), Brock and Scheinkman (1980), Scheinkman (1980a), Cass, Okuno and Zilcha (1980), and Hahn (1980).

(2.3) Theorem Let the sequence $r_0, r_1, \ldots$ and the numbers k_{-1} and $C_1, \ldots, C_A$ be given. Suppose that there is $\bar{r}$ such that $0 \leq r_t \leq \bar{r}$, for all t. Suppose also that $\sum_{t=0}^{\infty} \prod_{s=0}^{t-1} (1+r_s)^{-1} < \infty$. Finally, suppose that $k_{-1} > 0$. If assumptions (2.1) and (2.2) apply, then there exists an equilibrium with nominal interest rates

$r_0, r_1, \ldots$ and initial conditions $k_{-1}, C_1, \ldots, C_A$. Any equilibrium with interest rates $r_0, r_1, \ldots$ is Pareto optimal.

Proof I first prove that there exists an equilibrium. Consider the social welfare function

$$\sum_{t=0}^{\infty} \left[\prod_{s=0}^{t-1} (1+r_s)^{-1}\right] \sum_{a=0}^{A} \left[\prod_{s=1}^{\min(a,t)} (1+r_{t-s})\right] (1+\rho)^{-a} u(x_{t-a}^a) .$$

This is the welfare function obtained by taking the weighted average of the utilities of all consumers, where the weight given to the utility of the consumer born in period t is $\prod_{s=0}^{t-1} (1+r_s)^{-1}$. Since feasible allocations are bounded and $\sum_{t-0}^{\infty} \prod_{s=0}^{t-1} (1+r_s)^{-1} < \infty$, it follows that there exists a feasible allocation with initial stock k_{-1} which maximizes the above welfare functions among all feasible allocations. Let this allocation be $((\underline{x}_t^a), (\underline{k}_t))$.

I now define the sequences p_t, w_t, and τ_t. I define the sequence p_t by induction on t. By assumption, $p_0 = 1$. Given p_t, let $p_{t+1} = p_t\left(\frac{\partial f(\underline{k}_t, \Sigma L_a)}{\partial k}\right)^{-1} (1+r_t)$. Let $w_t = p_t \frac{\partial f(\underline{k}_{t-1}, \Sigma L_a)}{\partial L}$. I define the tax τ_t so that it balances the budget of the consumer of age A alive in period t. The definition of the τ_t is by induction on t. $\tau_0 = C_A + w_0 L_A - p_0 x_{-A}^A$. Suppose that $\tau_0, \ldots, \tau_{t-1}$ have been defined.

If $t \leqq A-1$, then $\tau_t = \left[\prod_{s=0}^{t-1} (1+r_s)\right] (C_{A-t} + w_0 L_{A-t} - p_0 \underline{x}_{t-A}^{A-t}$

$$- \tau_0) + \sum_{n=1}^{t-1} \left[\prod_{s=n}^{t-1} (1+r_s)\right] (w_n L_{n-t+A} - p_n \underline{x}_{t-A}^{n-t+A} - \tau_n) + w_t L_A - p_t \underline{x}_{t-A}^A .$$

If $t \geqq A$, then $\tau_t = \sum_{n=t-A}^{t-1} \left[\prod_{s=n}^{t-1} (1 + r_s) \right] (w_n L_{n-t+A} - p_n \underline{x}_{t-A}^{n-t+A} - \tau_n)$

$+ w_t L_A - p_t \underline{x}_{t-A}^{A}$.

I now show that $((\underline{x}_t^a), (\underline{k}_t); (p_t), (w_t), (r_t), (\tau_t); k_{-1}(C_a))$ is an equilibrium. It is clear that these variables satisfy conditions (i) - (v) of the definition of an equilibrium. It remains to be shown that the sequence $p_0, p_1, \ldots$ is bounded and bounded away from zero.

Let $U_t(x) = \max \{ \sum_{a=0}^{A} \left[\prod_{s=1}^{\min(a,t)} (1+r_{t-s}) \right] (1 + \rho)^{-a} u(x^a) \mid x^0 + \ldots + x^A = x \}$. Let $\underline{x}_t = \sum_{a=0}^{A} \underline{x}_{t-a}^a$, for $t = 0, 1, \ldots$. Because the allocation $((\underline{x}_t^a), (\underline{k}_t))$ maximizes the welfare function specified earlier, there exists $\lambda > 0$ such that

2.4) $\quad \dfrac{dU_t(\underline{x}_t)}{dx} \leqq \lambda\, p_t$, for all t, with equality if $\underline{x}_t > 0$.

Since feasible allocations are bounded, the sequence $\underline{x}_t$ is bounded and so inequality (2.4) implies that the p_t are bounded away from zero.

I now show that the p_t are bounded. First of all, inequality (2.4) implies that $p_t \leqq \lambda^{-1} \dfrac{dU_t(0)}{dx}$, if $\underline{x}_t > 0$. Since the r_t are bounded, it follows that there is $\bar{p}$ such that $\dfrac{dU_t(0)}{dx} \leqq \lambda \bar{p}$, for all t. Hence,

2.5) $\quad p_t \leq \bar{p}$, if $\underline{x}_t > 0$.

In order to show that the p_t are bounded, I use the fact that

2.6) for all $\varepsilon > 0$, $k(\bar{r}) - \varepsilon \leqq \underline{k}_t \leq k(0) + \varepsilon$, for all sufficiently large t,

where $\bar{r}$ is the upper bound on the r_t and $k(r)$ is the solution of the equation $\frac{\partial f(k, \Sigma L_a)}{\partial k} = 1 + r$. Inequality (2.6) follows from the following facts. The optimal capital stock is non-increasing in the discount rates applied to future utility. When the discount rate is constant (and equals r), then the optimal capital stock converges monotonically to $k(r)$. (See Brock and Mirman (1972), section 2.)

Inequality (2.6) implies that there is an upper bound on the number of successive periods during which $\underline{x}_t$ may be zero. (Choose ε so that $k(0) + \varepsilon$ is less than the maximum sustainable capital stock. If $\underline{x}_t$ were zero for too long, $\underline{k}_t$ would eventually exceed $k(0) + \varepsilon$.) Let T be the bound on the number of periods in succession in which $\underline{x}_t$ may be zero. The definition of the p_t and inequality (2.6) imply that for some $B > 0$, $p_t \leqq B\, p_{t+1}$, for all t. Hence, by (2.5) $p_t \leqq B^T \bar{p}$, for all t.

This completes the proof that an equilibrium exists.

I now show that an equilibrium $((x_t^a), (k_t); (p_t), (w_t), r_t), (\tau_t); k_{-1}(C_a))$ is Pareto optimal. I define numbers $b_0, b_1, \ldots$ such that the equilibrium allocation maximizes the welfare function

$$\sum_{t=0}^{\infty} \left(\prod_{s=0}^{t-1} (1+r_s)^{-1} \right) \sum_{a=0}^{A} \left(\prod_{s=1}^{\min(a,t)} (1 + r_{t-s}) \right) (1+\rho)^{-a} b_{t-a}\, u(x_{t-a}^a).$$

This is the welfare function which gives weight $b_t \prod_{s=0}^{t-1} (1 + r_s)^{-1}$ to the utility of the person born in period t. The above infinite sum converges if the b_t are bounded.

I define the b_t as follows. If $x_t^a > 0$, for some a, let $b_t = p_t \prod_{s=0}^{a-1} (1 + r_{t+s})^{-1} (1+\rho)^a \left(\frac{du(x_t^a)}{dx}\right)^{-1}$. (The choice of a does not affect the value of b_t, by the first order conditions of equilibrium.) If $x_t^a = 0$, for all a, let $b_t = \min_{a=0,\ldots,A} p_{t+a} \left(\prod_{s=0}^{a-1} (1 + r_{t+s})^{-1}\right) (1+\rho)^a \left(\frac{du(0)}{dx}\right)^{-1}$.

Clearly, $b_t > 0$, for all t. Since the p_t and x_t^a are bounded, the b_t are bounded, so that the welfare function is well-defined.

It is easy to see that the equilibrium allocation satisfies the first order conditions for maximization of the above welfare function over feasible allocations. Since the welfare function is concave, it follows that the equilibrium allocation maximizes the welfare function. Hence, since the b_t are all positive, the equilibrium allocation is Pareto optimal.

□

2.7) Remark For any sequence of interest rates, r_t, and for any sequence of positive numbers b_t which is bounded and bounded away from zero, there exists an equilibrium with interest rates r_t which maximizes a welfare function which gives weight $b_t \prod_{s=0}^{t-1} (1 + r_s)^{-1}$ to the utility of the person born in period t. The effect of varying the b_t's is to change the taxes τ_t.

Remark The sequence of equilibrium taxes τ_t may be unbounded. In order to become unbounded, it must oscillate between positive and negative numbers.

Remark If $r_t = r > 0$, for all t, then the social welfare function maximized by an equilibrium takes the simple form

$$\sum_{t=0}^{\infty} (1+r)^{-t} \sum_{a=0}^{A} \left(\frac{1+r}{1+\rho}\right)^{a} b_t u(x^a_{t-a}) \quad .$$

2.8) Theorem Let the numbers k_{-1} and $C_1, \ldots, C_A$ be given and suppose that $k_{-1} > 0$. If assumptions (2.1) and (2.2) apply, then there exists an equilibrium with initial conditions k_{-1} and $C_1, \ldots, C_A$ and with $r_t = 0$, for all t. Any such equilibrium is Pareto optimal.

I do not give the proof of this theorem. The only difficult part is the proof of Pareto optimality. It may be found in the literature. See,for instance, proposition 5 in Wallace (1980).

I also do not prove the following theorem. The proof is quite easy.

2.9) Theorem Suppose that assumptions 2.1 and 2.2 apply. For every $r \geqq 0$, there exists a stationary equilibrium with interest rate r. All such equilibria are Pareto optimal.

I now describe the financial flows associated with equilibrium. Consider an equilibrium $((x^a_t),(k_t);(p_t),(r_t),(\tau_t);\ k_{-1},(C_a))$. Think of firms as borrowing the money they spend on supplies. They use their sales to pay wages and to repay their loans and the interest. Since production functions are homogeneous of degree one, firms never earn any profit or suffer a loss , except perhaps in period zero. Thus, the aggregate credit balance of firms at the beginning of day $t \geqq 1$ is $-(1+r_{t-1})p_{t-1}k_{t-1}$. At the end of the day their balance is $-(1+r_{t-1})p_{t-1}k_{t-1} - w_t \sum L_a + p_t f(k_{L-1}, \sum L_a) - p_t k_t = -p_t k_t$. I assume that any profit or loss in period zero is absorbed by the central bank.

Let K_t^a be the credit balance at the end of day $t+a$ of the consumer born in period t. If $t \geqq 0$, then $K_t^a = \sum_{s=0}^{a} \left[\prod_{n=s}^{a-1} (1+r_{t+n}) \right] \left[w_{t+s} L_s - p_{t+s} x_t^s - \tau_{t+s} \right]$. If $-A \leqq t < 0$, then $K_t^a = C_{-t} + \sum_{s=-t}^{a} \left[\prod_{n=s}^{a-1} (1+r_{t+n}) \right] \left[w_{t+s} L_s - p_{t+s} x_t^s - \tau_{t+s} \right]$. The aggregate balance of all consumers at the end of day t is $K_t = \sum_{a=0}^{A} K_{t-a}^a$. Notice that $K_t = (1+r_{t-1}) K_{t-1} + w_t \sum_{a=0}^{A} L_a - p_t \sum_{a=0}^{A} x_{t-a}^a - (A+1)\tau_t$. Since $w_t \sum_{a=0}^{A} L_a + (1+r_{t-1}) p_{t-1} k_{t-1} = p_t k_t + p_t \sum_{a=0}^{A} x_{t-a}^a$, it follows that

$K_t = (1+r_{t-1}) K_{t-1} + p_t k_t - (1+r_{t-1}) p_{t-1} k_{t-1} - (A+1)\tau_t$, or

$$2.10) \quad r_{t-1} K_{t-1} - r_{t-1} p_{t-1} k_{t-1} - (A+1)\tau_t = K_t - K_{t-1} - (p_t k_t - p_{t-1} k_{t-1}),$$

$$\text{for } t = 1,2,\ldots \quad .$$

This equation describes how the financial position of the government evolves. The left-hand side is the deficit of the government during period t. The right hand side is the increase in the indebtedness of the government.

Suppose that the equilibrium is stationary, so that it may be written as $((x^a), k; w, r, \tau)$. Then, equation (2.10) becomes

$$2.11) \quad r(K-k) = (A+1)\tau ,$$

where K is the aggregate credit position of consumers in every period. This equation simply says that the aggregate tax payments equal the interest on the government debt. Both $K - k$ and τ may be negative. In this case, the government has a surplus in its capital account and distributes the interest it earns as a lump-sum subsidy.

Remark Stationary equilibria with higher interest rates may require higher taxes in order to pay the higher interest on government debt. The passage from a stationary state with a high interest rate to one with a lower interest rate may require an increase in taxes during the transition period in order to create a government surplus. The surplus would reduce the government debt. (See the discussion of example (2.13) below.)

I say that the government does not interfere or that government policy is neutral if the government's net credit position is zero. This definition is more arbitrary than it might seem. It requires that the government's monetary and other obligations be exactly offset by loans to the private sector. However, I adopt this definition of neutrality because it describes the situation that would exist if there were no government at all.

Another possible definition of neutrality would be an absence of taxes or subsidies. However, I have rejected this definition because it gives the false impression that stationary equilibria with interest rate zero involve no government interference. It may be seen from equation (2.11) that taxes are zero in such equilibria. But these equilibria involve no taxes only because I have assumed that labor is the only non-produced resource. If land were included in the model as a factor of production, then a government subsidy would be required in order to sustain an equilibrium with interest rate zero. The subsidy would enable consumers to pay the part of the price of food attributable to rent on land.

One might be tempted to try to create a theory with a deterministic interest rate by requiring that the government not intervene. However, government interference may be necessary for the existence of a stationary equilibrium. For instance, in the following example, the aggregate credit psoition of consumers is non-positive at any interest rate (i.e., consumers are never creditors). Since firms are debtors at any interest rate, the government is always a net creditor.

<u>Example</u>: $A = 1$, $u(x) = \sqrt{x+1}$, $\rho = 0$, $L_0 = 0$, $L_1 = 1$, $f(k,L) = 2\sqrt{kL}$.

The key property of the above example is that consumers earn nothing in the first period of life. If in a two period example , $L_0 = 1$ and $L_1 = 0$, then it is possible that consumers would be large creditors, so that the only stationary equilibrium with no government intervention would have a negative interest rate and so be inefficient. (See example (4.14) below.) This problem was pointed out by Samuelson (1958) in a model with no production. Scheinkmen (1980b) has pointed out that if there is some non-produced factor of production other than labor, such as land, then in a model in which consumers are creditors there is an equilibrium without government intervention and with non-negative interest rates. Cass and Yaari (1966b) have pointed out that if money itself has positive utility or if there is some commodity money such as land which has positive utility, then equilibrium is Pareto optimal in a model in which consumers are creditors. However, introducing land would not cure the problem illustrated by the example given above. It is not clear to me which of the conditions $(L_0 = 0, L_1 = 1)$ or

$(L_0 = 1, L_1 = 0)$ better represents reality. If there were no bequests and no government intervention, then people would have to finance their own education, and it might well be that the population as a whole would be debtors at all reasonable interest rates. For this reason, I find that a theory which assumes no government intervention makes the most sense in models with bequests. Such a theory is described in section 4.

Remark Throughout this paper, I ignore the deadweight loss from taxes. If this loss were included, then the only Pareto optimal equilibria would be those without taxes. But it does not follow that such equilibria are the only correct ones. The only conclusion we can draw is that if society could somehow settle on a social welfare function for present and future generations, then the deadweight loss from taxes should be included in the welfare calculus. For instance, if the welfare of future generations were discounted at the rate 0.1 and if 0.05 were the no tax stationary equilibrium interest rate, then the stationary equilibrium which maximized welfare would probably have an interest rate between 0.1 and 0.05 .

Remark There exist equilibria with negative interest rates. In fact, there exists a stationary equilibrium with interest rate r, for each r such that $-1 < r < 0$, but these equilibria are not Pareto optimal. The reason that they are not optimal has nothing to do with externalities or the absence of markets. The lack of Pareto optimality stems from the unboundedness of the present value of all present and future consumption. The finiteness of this quantity is essential to the standard proof that equilibria are Pareto optimal. If $r > 0$, then the present value of all consumption is indeed finite and the standard proof applies. The case $r = 0$ is borderline and requires a special proof. Cass (1972 a,b) has discussed related issues in depth.

Remark The indeterminacy of interest rates has little to do with the infiniteness of the horizon. Interest rates are also indeterminate in a model with overlapping generations and a finite horizon. Moreover, if the horizon is finite, equilibria with negative interest rates are Pareto optimal. A model with a finite horizon is really just an ordinary general equilibrium model. The taxes are the usual transfer payments that make it possible to achieve any Pareto optimum as an equilibrium. Different interest rates correspond to different distribution of welfare between generations. The situation is illustrated by the following example.

Example There are two periods, two consumers, one good and no production. The first consumer lives in both periods and the second consumer lives only in the second period. Each consumer is endowed with one unit of the good in each period in which he lives. The good is not storable. The utility function of the first consumer is $u_0(x^0,x^1) = \sqrt{x^0} + \sqrt{x^1}$, where x^a is the quantity consumer in period a. The utility function of the second consumer is $u_1(x) = x$, where x is the quantity consumer in his single period of life.

The feasible allocations of this example correspond to one edge of the Edgeworth box. All of these allocations are Pareto optimal. All but one may be achieved as equilibria with transfer payments. These equilibria may be interpreted as equilibria with taxes and interest. Such an equilibrium is described as follows. The first consumer consumes one unit in the first period and x in the second period, where $0 < x \leqq 2$. The second consumer consumes $2 - x$ and in the second period. The price of the good is one in each period. The interest rate is $\sqrt{x-1}$. Each consumer pays a tax of $x - 1$ in the second period. The first consumer pays a tax of $(2 - 2x)x^{-1/2}$ in the first period. (The allocation with $x = 0$ is Pareto optimal but cannot be achieved as an equilibrium.)

Remark In equilibrium, consumers and firms behave as Irving Fisher (1930) said they would, but their behavior does not determine the interest rate. Marginal rates of substitution and transformation all correspond to the interest rate.

Remark Theorems (2.3),(2.8) and (2.9) may easily be generalized to economies with many commodities, with depreciation, with irreversible investment, and with utility functions which are not separately additive with respect to time.

Remark It is possible to modify the model so that the population grows at rate n. In this case, stationary equilibria with interest rate r would be Pareto optimal if and only if $r \geqq n$. The interest rate r is both the nominal and the real interest rate and is also the rate of discount in a social welfare function. The social welfare function is

$$2.12) \qquad \sum_{t=0}^{\infty} (1+r)^{-t} \sum_{a=0}^{A} \sum_{h=1}^{N(t-a)} \left(\frac{1+r}{1+\rho}\right)^{a} u(x_{t-a}^{ah}),$$

where $N(t)$ is the population at time t and x_t^{ah} is the consumption of the h^{th} person, born at time t, and during his a^{th} period of life. If $N(t+1) = (1+n)N(t)$, for all t, and if one takes into account the fact that x_{t-a}^{ah} does not depend on h in a social optimum, then the welfare function becomes

$$\sum_{t=0}^{\infty} \left(\frac{1+n}{1+r}\right)^t \sum_{a=0}^{A} \left(\frac{1+r}{(1+n)(1+\rho)}\right)^a u(x_{t-a}^a)$$

In the literature, it is standard practice to imitate Samuelson (1958) and to call equilibria inefficient if the interest rate is less than the rate of population growth. However, this statement is misleading, for in reality the population must be bounded, so that equilibria with small interest rates are efficient, even if the bound is not approached for thousands of years. If the numbers $N(t)$ in formula (2.12) are bounded, then the social welfare function is well-defined for any $r > 0$, and one can even speak of an optimum when $r = 0$. Suppose that the population were to grow at a steady rate n for a long time and then to approach an asymptote. Suppose also that the social discount rate were r, where $0 \leqq r < n$. Then, the real interest rate would be r during the period of population growth, it would decrease when the growth slowed, and would be r again when the population was stagnant.

The steady state growth path with interest rate n has been called the golden rule because it corresponds to a social welfare function which gives the _per capita_ utility of each generation equal weight (see Starrett (1972)). However, the equilibrium just described with interest rate zero gives equal weight to the utility of all people in all generations. It, therefore, is also golden. To be more precise, in the social welfare function corresponding to the usual golden rule, the weight attached to a person born at time t is $(1+n)^{-t}$. In the social welfare function corresponding to an interest rate of zero, the weight attached to the utility of someone born in period t is one.

Remark I have assumed no technical change. I have not done so because technical change complicates matters considerably and it seems clear that it does not tend to fix interest rates. One complication is that it is impossible to imagine stationary states with technical change, unless the utility functions, production functions, and technical change all take very special forms. There is a possibility that technical change would put a positive lower bound on efficient interest rates, but this possibility requires very special assumptions. Suppose that because of technical change output can grow at the rate g forever. (The growth must really be forever and not a mere one million years.) If the utility function were $u(x) = \sqrt{x}$, and the interest rate were less than $\sqrt{1+g} - 1$, then equilibrium would be inefficient, and the social welfare function $\sum_{t=0}^{\infty} (1+r)^{-t} \sum_{a=0}^{A} \left(\frac{1+r}{1+\rho}\right)^a u(x_{t-a}^a)$ would not converge. However, if $u(x)$ were bounded or like $\log x$, then the social welfare function would converge for any $r > 0$. In fact, all equilibria with non-negative interest rates would be efficient. Notice that technical change can put a positive lower bound on interest rates only if output can be unbounded. This possibility alone strikes me as absurd, given that we live on a planet of finite size.

Remark It is possible to have random nominal interest rates. If interest rates were random, they would have to be insurable if equilibria were to be Pareto optimal. Only the government could provide such insurance, since an insurer could not benefit from the law of large numbers. The insurance could be provided through the taxes, τ_t, provided all people had identical tastes and endowments. Theorem (2.3) remains true, provided the random r_t are non-negative, bounded, and such that $E \sum_{t=0}^{\infty} \prod_{s=0}^{t-1} (1+r_s)^{-1} < \infty$, where E denotes expected value.

Remark It is possible to allow taxes to depend on the age of the tax-payer as well as on the date. Doing so allows one to study social security, for the effect of social security is to increase taxes for the young and to reduce them for the old. Some care must be taken in interpreting the effect of social security. I discuss this matter in another paper (1981).

I now give two examples of equilibria in order to clarify the concept. The first example is such that the nominal interest rate is fixed and the real interest rate varies but converges monotonically to the nominal rate as time goes to infinity.

2.13) Example $A = 1$, $u(x) = \log x$, $\rho = 1/2$, $L_0 = 1, L_1 = 0$, $f(k,L) = 2\sqrt{k,L}$.

For each $r \geqq 0$, the formulas which follow describe an equilibrium for the example with $r_t = r$, for all t. In these formulas, I parameterize the initial stock so that $k_{-1} = b(1+r)^{-2}$, where $b > 0$. The other initial condition is $C_1 = (4 + 7r + 6r^2)(1+r)^{-1}(2+r)^{-1}(5+2r)^{-1}\sqrt{b}$. The other variables are as follows: for all t,

$$x_t^0 = 3(1+2r)(1+r)^{-2}(5+2r)^{-1}b^{2^{-t-1}},$$

$$x_t^1 = 2(1+2r)(1+r)^{-1}(5+2r)^{-1}b^{2^{-t-2}},$$

$$k_t = (1+r)^{-2}b^{2^{-t-1}}, p_t = b^{2^{-1}-2^{-t-1}}, w_t = (1+r)^{-1}\sqrt{b}, r_t = r,$$

and

$$\tau_t = r(2r-3)(1+r)^{-1}(2+r)^{-1}(5+2r)^{-1}\sqrt{b}.$$

Notice that p_t converges monotonically to $\sqrt{b}$ and k_t converges monotonically to $(1+r)^{-2}$ as t goes to infinity. The real interest rate in terms of food prices is $\rho_t = (1+r)p_t p_{t-1}^{-1} - 1 = (1+r)b^{-2^{-t-2}} - 1$. It converges monotonically to r, but exceeds r if $b < 1$ and is less than r if $b > 1$. The real interest rate calculated in terms of labor is always one.

Notice also that the tax does not depend on time. In fact, the government's budget always balances and its debt never changes in nominal value. If $0 < r < 3/2$, then the government's debt is negative and the tax is in fact a subsidy. If in addition $b < 1$, so that the economy is growing, then the price of food falls over time and the real value of the government's net credit balances and its subsidy increase.

I now show that one way to induce economic growth in the above example is by means of a capital levy followed by lower interest rates. Suppose that up to period -1 the economy has been in a steady state with interest rate 1 and food price 1 and suppose that the government then decides to move the economy to the equilibrium just described and with interest rate 1/2. In order to do so, the government should announce in period 0 the new interest rates and subsidies for period 0 and thereafter. But those announcements alone would not suffice, for consumers would have too many assets. The government would have to levy a lump-sum tax on the old consumers of period 0. This tax could be interpreted as a capital levy. It is not hard to calculate what it should be. Let $C_1(r) = (4 + 7r + 6r^2)(1+r)^{-1}(2+r)^{-1}(5+2r)^{-1}$. In the new equilibrium, old consumers should start period zero with assets $C_1(1/2)\sqrt{b}$. Since $k_{-1} = 1/4$, $b = 9/16$. Therefore, $C_1(1/2)\sqrt{b} = 3/10$. The old consumers would in fact have assets $C_1(1) = 17/42$. Therefore, the capital levy should be $\frac{17}{42} - \frac{3}{10}$. (If the government is a debtor, one way to engineer a capital levy is through rapid inflation.)

Notice that firms would make the transition between interest rate regimes without any loss or gain, for prices do not change from period -1 to period 0. In both periods, the wage is 1/2 and the price of food is 1. Firms' sales in period 0 would exactly cover their wage and interest payments and debt repayments. However, firms would increase their borrowings in period 0, from $p_{-1}k_{-1} = 1/4$ to $p_0k_0 = 1/3$.

The combined effect of the capital levy on old consumers and the increased borrowings of firms would be to increase the government's net credit balance (or reduce its liabilities). The increase in the government's credit balance leads to an increase in its subsidy (or to a reduction in taxes).

I calculated the equilibrium for the above example by maximizing the social welfare function $\sum_{t=0}^{\infty} (1+r)^{-t} \left(\log x_t^0 + \frac{2}{3}(1+r) \log x_{t-1}^1 \right)$. It is not hard to calculate the solution to this problem once one notices that the total consumption of food in any period is a constant multiple of the stock available. That is, $x_t^0 + x_{t-1}^1 = c\,2\sqrt{k_{t-1}}$, for some $c > 0$. This fact was pointed out to me by Leonard Mirman and is proved in Mirman (1979, pp. 45,6).

The next example is such that the interest rate fluctuates, so that the economy exhibits a kind of trade cycle (with no unemployment, of course).

2.14) <u>Example</u> $A = 1, u(x) = \log x, \rho = 0, L_0 = 1, L_1 = 0$, if $(k,L) = 2\sqrt{kL}$.

The following formulas describe an equilibrium for the above example with $r_t = 1$, for t even and $r_t = 0$, for t odd. (I have rounded off to the third decimal place.)

$x_{2t}^0 = 0.365$, $x_{2t}^1 = 0.514$, $x_{2t+1}^0 = 0.375$,

$x_{2t+1}^1 = 0.534$, $k_{2t} = 0.506$, $k_{2t+1} = 0.494$,

$p_{2t} = 1$, $p_{2t+1} = 1.423$, $w_{2t} = 0.703$,

$w_{2t+1} = 1.012$, $\tau_{2t} = 0$, $\tau_{2t+1} = -0.056$, for $t = 0,1,\ldots,$ and

$k_{-1} = k_{2t+1}, C_1 = 0.534$.

The real interest rate calculated in terms of the price of food is $\rho_{2t} = 0.405$ and $\rho_{2t+1} = 0.423$. Notice that real interest rates fluctuate much less than nominal interest rates and in the opposite direction.

I have calculated the above equilibrium so that the government's budget balances in every period. In fact, it is not necessary for the budget to balance every period. It need balance only over a cycle. If taxes are constant over time, one obtains a very different equilibrium in which real interest rates move the same way as do nominal rates. The real interest rate is 0.628 in even periods and 0.228 in odd periods. Both monetary and fiscal policy count.

It is not hard to make the calculations for the above example. One simply writes down all the equations characterizing an equilibrium, one set for even periods and one set for odd periods. One then expresses all variables in terms of the tax rate and the capital stock at the end of even periods. Substitution into the feasibility constraints for even and odd periods then gives two equations which may be solved for the two unknowns.

The next example shows that the government's net debt position in a stationary equilibrium is not necessarily a monotone function of the interest rate r, so that there might be more than one equilibrium corresponding to a given level of the government's net debt position. Thus, it is not necessarily appropriate to imagine that the interest rate is determined by the market once the government fixes its net credit position.

2.15) Example $A = 1$, $u(x) = \log x$, $\rho = 0$, $L_0 = 1$, $L_1 = 0$, f is homogeneous of degree one and such that

$$g(k,1) = \begin{cases} 4\sqrt{k}, & \text{if } 0 \le k \le 1 \text{ and} \\ 4, & \text{if } k \geqq 1 . \end{cases}$$

The function f is not differentiable in this example, but if it were approximated by a differentiable production function, the qualitative results about to be described would remain true. If $0 \leq r \leq 1$, then the equilibrium capital stock is one and the firm is at a corner. Thinking of f as differentiable, it is appropriate to assume that all the income not going to capital goes to labor. That is, $w = f(1,1) - (1+r)$ if $k = 1$ and $0 \leq r \leq 1$. Under these hypotheses, it is not hard to calculate that the stationary equilibrium quantity of government debt when the interest rate is r is

$$G(r) = \begin{cases} \dfrac{2(1-r^2)}{(2+r)^2}, & \text{if } 0 \leqq r \leqq 1, \text{ and} \\ \\ \dfrac{8(r-1)}{(2+r)^2(1+r)}, & \text{if } r \geqq 1. \end{cases}$$

$G(r)$ is decreasing in r for $0 \leq r < 1$, is increasing in r for $1 < r < \frac{1+\sqrt{13}}{2}$, and is again decreasing for $r > \frac{1+\sqrt{13}}{2}$. When $r < 1$, $G(r)$ is decreasing because increases in r decrease labor income rapidly and so reduce consumers' net asset position. This concludes the discussion of example (2.15).

In trying to relate the results of this section with issues in monetary economics, one should keep in mind that the model represents a world in which the government has put into practice Friedman's recommendation that money earn interest at the same rate as do other assets. All financial assets are equivalent in the model so that there is nothing which corresponds exactly to the money supply. Perhaps the government's net debt position is the best analogue to the money supply.

It is also important to keep in mind that the model assumes perfect foresight or rational expectations. It would be naive to assume that the public could form such expectations immediately after a sudden change in government policy. To the extent that lags in expectations are important, the model is not a good model for analyzing policy changes.

In connection with monetary policy, it should be noticed that the model provides no justification for Friedman's recommendation that the money supply grow at a fixed rate (see Friedman (1968)). The fluctuating equilibria of example (2.14) is Pareto optimal, even though interest rates fluctuate. Also if taxes are constant, the government's net credit position fluctuates. A rigorous defense of Friedman's position can be made by including some friction, or market imperfection, as Lucas (1972) has done. (Friedman also spoke of frictions in the form of information lags.) Friction makes it difficult for the economy to adapt to change. Friction does not make one interest rate more "natural" than another. It simply makes changes in interest rates costly.

The model of this section provides no justification for Friedman's recommendation that the rate of interest equal consumers' pure rate of time preference. A justification of this recommendation may be made by assuming that consumers are subject to uninsurable risk. I have discussed this matter in two other papers (1980a, 1980b) and hope soon to synthesize that discussion with the results of this paper.

3) Immortal Consumers

There is one kind of model in which the interest rate is deterministic, and that is a model with immortal consumers. I describe such a model here, for the model with bequests discussed subsequently may be viewed as an attempt to incorporate the properties of a model with immortal consumers into one with overlapping generations.

In order to simplify the exposition, I assume that there is a single immortal consumer. Otherwise, the model is as in the previous section. The consumer's utility for an infinite stream of food $(x_1, x_2, \dots)$ is $\sum_{t=0}^{\infty} (1+\rho)^{-t} u(x_t)$, where $\rho > 0$ and u satisfies assumption (2.1). He can supply one unit of labor in each period and $f(k,L)$ is the production function where f satisfies assumption (2.2). The consumption stream $(x_1, x_2, \dots)$ is feasible with initial capital k_{-1}, if there exists a sequence of capital stocks $k_0, k_1, \dots$ such that $x_t + k_t = f(k_{t-1}, 1)$, for $t = 0,1,\dots$. A stationary equilibrium consists of $(x,k;w,r)$, where $x + k = f(k,1)$ and $(K,L) = (k,1)$ maximizes $f(K,L) - (1+r)K - wL$ and where $(x_0, x_1, \dots)$ maximizes $\sum_{t=0}^{\infty} (1+\rho)^{-t} u(x_t)$ subject to $\sum_{t=0}^{\infty} (1+r)^{-t} x_t \leq \sum_{t=0}^{\infty} (1+r)^{-t} w$. There exists a unique stationary equilibrium (x,k,w,r) and it has interest rate $r = \rho$. The equilibrium allocation is the unique allocation which maximizes the consumer's utility among all feasible allocation with initial capital k.

It is also possible to define non-stationary equilibria. These converge to stationary equilibria, so that the asymptotic real rate of interest is ρ .

It is easy to generalize the above results in the case of several consumers, provided they all have the same rate of pure time preference ρ .

If consumers have different rates of pure time preference, there exists no stationary equilibrium. In fact, in any equilibrium, a consumer eventually consumes nothing if he has a rate of time preference greater than the minimum among all consumers.

It is possible to generalize the above results to a model with many commodities. However, in this case, the convergence of non-stationary equilibria to a stationary state may occur only if consumers' rate of pure time preference is very small. All this is explained in Bewley (1979).

Remark The model just discussed is best interpreted as a model with infinitely many Arrow-Debreu forward markets. These markets are defined in Arrow (1953) and in Chapter 7 of Debreu (1959).

Remark Some equilibrium models of the business cycle assume that there is a single immortal consumer with a utility function of the form $\sum_{t=0}^{\infty} (1 + \rho)^{-t} u(x_t)$. (See, for instance, Kydland and Prescott (1980).) It should be realized that such models fix society's rate of pure time preference and hence exclude what may be one of the elements of the business cycle. Fluctuations in society's implicit rate of pure time preference may well play a role in the business cycle.

4. A Simple Bequest Model

In this section, I describe a theory which may seem to be plausible and which determines the interest rate. The theory is based on a model which is a special case of that of Barro (1976). Each consumer lives two periods, has exactly one child, and attaches equal weight to his own utility and to that of his child. The lifetime utility of a consumer is $u(x^0) + (1 + \rho)^{-1}[u(x^1) + V(h)]$, where $V(h)$ is the lifetime of the child if he receives inheritance h and where x^a is the parent's consumption in the a^{th} period of life. I assume that $\rho > 0$. Apart from the inclusion of bequests and the restriction that people live only two periods, the model is as in section 2. Assumptions (2.1) and (2.2) still apply. I show the following. The stationary allocation associated with interest rate r is Pareto optimal if and only if $0 \le r \le \rho$. If the government may intervene by maintaining a net debit or credit position, then any of these Pareto optimal stationary allocations is the allocation of a stationary equilibrium. But if no intervention is allowed, then the bequest motive tends to fix the interest rate r. If bequests may be negative, then there is only one stationary equilibrium with no government intervention, and it has interest rate ρ. If bequests are required to be non-negative, then there may be no stationary eqilibrium unless land is included in the model. If land is included and bequests are non-negative, then there is at least one stationary equilibrium and all equilibrium interest rates are in the interval $(0, \rho]$. Under reasonable conditions, this equilibrium is unique.

I now turn to the analysis. As before, an <u>allocation</u> is of the form $((x_t^a, (k_t))$ and it is feasible with initial stock k_{-1} if $x_t^0 + x_t^1 = f(k_{t-1}, L_0 + L_1)$, for $t = 0,1,\ldots$. To every allocation

$((x_t^a),(k_t))$, there correspond numbers v_t, for $t \geq -1$, where $v_{-1} = u(x_{-1}^1) + v_0$ and $v_t = u(x_t^0) + (1+\rho)^{-1}[u(x_t^1) + v_{t+1}]$. For $t \geqq 0$, v_t is the lifetime utility of the consumer born in period t and v_{-1} is the utility experienced in period 0 by the consumer born in period -1. Observe that

$$v_{-1} = \sum_{t=0}^{\infty} (1+\rho)^{-t}[u(x_t^0) + u(x_{t-1}^1)] \quad \text{and}$$

$$v_t = u(x_t^0) + \sum_{n=1}^{\infty} (1+\rho)^{-n}[u(x_{t+n}^0) + u(x_{t+n-1}^1)], \quad \text{for } t \geq 0.$$

A feasible allocation $((x_t^a),(k_t))$ with initial capital k_{-1} is Pareto optimal if there exists no feasible allocation $((\hat{x}_t^a),(\hat{k}_t))$ with the same initial capital k_{-1} and such that $\hat{v}_t \geqq v_t$, for all t, with strict inequality for some t, where the numbers v_t and $\hat{v}_t$ are associated with the allocations $((x_t^a),(k_t))$ and $((\hat{x}_t^a),(\hat{k}_t))$, respectively. Notice that the inclusion of the bequest motive strengthens the definition of Pareto optimality. Fewer allocations are Pareto optimal.

4.1) <u>Theorem</u> A feasible stationary allocation $((x^a),k)$ is Pareto optimal if and only if

i) $0 \leqq r \leqq \rho$, where $r = \frac{\partial f}{\partial K}(k, L_0 + L_1) - 1$, and

ii) there exists $\lambda > 0$ such that $\frac{\partial u(x^0)}{\partial x} \leqq \lambda$, with equality if $x^0 > 0$ and $\frac{du(x^1)}{dx} \leqq \frac{1+\rho}{1+r}\lambda$, with equality if $x^1 > 0$.

<u>Proof</u>: In order to avoid the inconvenience of corner conditions, I assume in the proof that $\lim_{x \to 0} \frac{du(x)}{dx} = \infty$, so that condition (ii) above

becomes $\frac{du(x^0)}{dx} = \frac{1+r}{1+\rho} \frac{du(x^1)}{dx}$. If this assumption is removed, the proof changes in an obvious and mechanical way.

Suppose that $((x^a),k)$ is Pareto optimal. The marginal rate of substitution of food between periods must equal the marginal rate of transformation. Hence $\frac{du(x^0)}{dx} = \frac{1+r}{1+\rho} \frac{du(x^1)}{dx}$. Also, $\frac{du(x^1)}{dx} \geqq \frac{du(x^0)}{dx}$, for if $\frac{du(x^1)}{dx} < \frac{du(x^0)}{dx}$, then an old consumer would gain by giving a little food to his child. Thus, $\frac{du(x^1)}{dx} \geqq \frac{du(x^0)}{dx} = \frac{1+r}{1+\rho} \frac{du(x^1)}{dx}$, which implies that $r \leqq \rho$. If $r < 0$, then all consumers would consume more in every period if the consumers in period zero reduced the capital stock by consuming more. Therefore, $r \geqq 0$.

Now suppose that $0 \leqq r \leqq \rho$ and that $\frac{du(x^0)}{dx} = \frac{1+r}{1+\rho} \frac{du(x^1)}{dx}$. I consider separately the cases $r = \rho, \rho > r > 0$, and $r = 0$.

<u>$r = \rho$</u> The stationary allocation $((x^a),k)$ is the unique feasible allocation with initial stock k which maximizes v_{-1} among all such allocations. Hence, any change in this allocation would harm the person born in period -1.

<u>$\rho > r > 0$</u> The stationary allocation $((x^a),k)$ maximizes the social welfare function

$$\frac{1+r}{\rho - r} v_{-1} + \sum_{t=0}^{\infty} (1+r)^{-t} v_t = \sum_{t=0}^{\infty} \sum_{n=0}^{\infty} (1+r)^{n-t} (1+\rho)^{-n} [u(x_t^0) + \frac{1+r}{1+\rho} u(x_{t-1}^1)].$$

In order to see that this is so, observe that $((x^a),k)$ satisfies the first order conditions of this maximization problem. Since the welfare function is concave, the first order conditions imply optimality.

<u>$r = 0$</u> I here imitate arguments in Gale (1967). I normalize u so that

4.2) $\quad u(x^0) + (1+\rho)^{-1} u(x^1) = 0$ and

4.3) $\quad \dfrac{du(x^0)}{dx} = 1.$

Suppose that $((x^a),k)$ is not Pareto optimal and let $((x_t^a),(k_t))$ be the feasible allocation with initial capital k which Pareto dominates $((x^a),k)$. For $t \geqq -1$, let v_t be the utility level associated with $((x_t^a),(k_t))$ in the manner described in the paragraph preceding the statement of the theorem. Observe that

$$(1+\rho)^{-1}v_{-1} + \rho(1+\rho)^{-1} \sum_{t=0}^{T} v_t = \sum_{t=0}^{T} [u(x_t^0) + (1+\rho)^{-1}u(x_{t-1}^1)]$$

$+ \sum_{n=1}^{\infty} (1+\rho)^{-n}[u(x_{T+n}^0) + (1+\rho)^{-1}u(x_{T+n-1}^1)]$. Since $((x_t^a),(k_t))$ Pareto dominates $((x^a),k)$, it follows, using (4.2), that

4.4) for all T, $\sum_{t=0}^{T} [u(x_t^0) + (1+\rho)^{-1}u(x_{t-1}^1)] + \sum_{n=1}^{\infty} (1+\rho)^{-n}[u(x_{T+n}^0) + (1+\rho)^{-1}u(x_{T+n-1}^1)]$ is non-negative, non-decreasing in T, and is strictly positive for T sufficiently large.

Since feasible allocations are bounded, the sum $\sum_{n=1}^{\infty} (1+\rho)^{-n}[u(x_{T+n}^0) + (1+\rho)^{-1}u(x_{T+n-1}^1)]$ remains bounded as T varies. Hence, (4.4) implies that $((x_t^a),(k_t))$ is a good program in the sense of Gale (1967), p. 11). That is, $\inf_{T\geqq 0} \sum_{t=0}^{T} [u(x_t^0) + (1+\rho)^{-1} u(x_{t-1}^1)] > -\infty$. Goodness implies that $\lim_{t\to\infty} k_t = k$ (see theorem 8 in Gale (1967)).

Since f is concave and $\frac{\partial f}{\partial k}(k,L_0+L_1) = 1$, it follows that

$$4.5)\qquad x_t^0 + x_{t-1}^1 + k_t - k_{t-1} = f(k_{t-1}, L_0 + L_1) - k_{t-1}$$

$$\leqq f(k, L_0 + L_1) - k = x^0 + x^1.$$

Also $(1+\rho)^{-1}\dfrac{du(x^1)}{dx} = \dfrac{du(x^0)}{dx} = 1$, where the second equation is (4.3). Since u is concave, it follows that

$$4.6)\qquad u(x_t^0) + (1+\rho)^{-1}u(x_{t-1}^1) - x_t^0 - x_{t-1}^1$$

$$\leqq u(x^0) + (1+\rho)^{-1}u(x^1) - x^0 - x^1 = - x^0 - x^1,$$

where the equation follows from (4.2).

Combining (4.5) and (4.6), I obtain $u(x_t^0) + (1+\rho)^{-1}u(x_{t-1}^1) \leqq k_{t-1} - k_t$, for $t = 0,1,\ldots$. It follows that $\sum_{t=0}^{T} [u(x_t^0) + (1+\rho)^{-1}u(x_{t-1}^1)]$
$+ \sum_{n=1}^{\infty}(1+\rho)^{-n}[u(x_{T+n}^0) - (1+\rho)^{-1}u(x_{T+n-1}^1)] \leqq k - k_t + \sum_{n=1}^{\infty}(1+\rho)^{-n}(k_{T+n-1} - k_{T+n})$.
Since $\lim_{t\to\infty} k_t = k$, the right hand side of this inequality converges to zero. This contradicts (4.4) □

I now turn to equilibria. First of all, suppose that bequests must be non-negative. The definition of equilibrium involves a <u>valuation function</u> $V(H)$. The number $V(H)$ is the lifetime utility of a consumer with inheritance H. The function V is defined by the equation

$$4.7)\qquad V(H) = \max\,\{u(x^0) + (1+\rho)^{-1}[u(x^1) + V(h)] \mid h \geqq 0, x^0 \geqq 0, x^1 \geqq 0,$$

$$\text{and } x^0 + (1+r)^{-1}(x^1 + h) = wL_0 + (1+r)^{-1}\, wL_1 + H + S\}\ ,$$

where h is the bequest and S is the government subsidy. I suppose that the subsidy is paid when consumers are young. The subsidy could be negative, in which case it is a tax. Notice that the treatment of taxes and subsidies is slightly different from that of section 2.

Equation (4.7) defines V, provided $0 \leq r \leq \rho$. I say that $((\underline{x}^a),H)$ solves this equation if $((x^a),h) = ((\underline{x}^a),H)$ solves the maximization problem on the right hand side .

4.8) <u>Definition</u> A <u>stationary equilibrium</u> consists of $((x^a),k;H,w,r,S)$, where

i) $((x^a),k)$ is a feasible stationary allocation,

ii) $(k,L_0 + L_1)$ solves the problem $\max\, [f(K,L) - (1+r)K - wL]$,

and

iii) $((x^a),H)$ solves equation (4.7).

Now suppose that bequests may be negative. A consumer's maximization problem is not well-defined if there is no lower bound on bequests whatsoever, for a parent would be willing to leave an arbitrarily large debt to his child, knowing that his child could leave a still larger debt in turn to his own child. For this reason, I assume that bequests must be at least $\underline{h}(r)$ when the interest rate is r, where $\underline{h}(r)$ is defined as follows

4.9) If $r > 0$, $\underline{h}(r) = -(1+r)r^{-1}w(r)\, L_0 - r^{-1}w(r)L_1$, where $w(r)$ is the wage rate when the interest rate is r. $\underline{h}(0)$ is any number less than $-w(0)\, L_0$.

Notice that if a consumer were to inherit less than $\underline{h}(r)$, when $r > 0$, then the interest on his inherited debt would exceed his income. When bequests may be negative, the valuation function is defined by equation (4.7) with the inequality $h \geqq 0$ replaced by $h \geqq \underline{h}(r)$. A stationary equlibrium is defined as in (4.8).

4.10) <u>Theorem</u> There exists a stationary equilibrium with interest rate r for each r such that $0 \leq r \leq \rho$, and each of these equilibria is Pareto optimal. There exists no stationary equilibrium with an interest rate exceeding ρ. These statements are true whether or not bequests may be negative.

<u>Proof</u> First of all, I define the equilibrium allocation when the interest rate r is such that $0 \leqq r \leqq \rho$. Let k be the unique solution to the equation $\dfrac{\partial f(K, L_0 + L_1)}{\partial K} = 1 + r$, let $w = \dfrac{\partial f(k, L_0 + L_1)}{\partial k}$, and let $x = f(k, L_0 + L_1) - k$. For $y \geqq 0$, let $(x^0(y), x^1(y))$ be the unique solution to the problem $\max \{u(x^0) + (1+\rho)^{-1}u(x^1) \mid x^0 + (1+r)^{-1}x^1 = y\}$. Clearly, there exists a unique value of y, call is $\underline{y}$, such that $x^0(\underline{y}) + x^1(\underline{y}) = x$. Let $x^a = x^a(\underline{y})$, for $a = 0,1$. Clearly, $((x^a), k)$ is a feasible allocation and $(k, L_0 + L_1)$ maximizes profit at the wage rate w and interest rate r. Hence, $((x^a), k)$ satisfies conditions(i) and (ii) of the definition of a stationary equilibrium.

I now define the equilibrium bequest H and subsidy S. If bequests must be non-negative, let $H = 0$ if $r < \rho$ and let $H = \frac{1+\rho}{\rho}\max(0, x^0 + (1+\rho)^{-1}x^1 - wL_0 - (1+\rho)^{-1}wL_1)$ if $r = \rho$. Also, let $S = x^0 + (1+\rho)^{-1}x^1 - wL_0 - (1+\rho)^{-1}wL_1$, if $r < \rho$ and let

$S = \min\ (0, x^0 + (1+\rho)^{-1}x^1 - wL_0 - (1+\rho)^{-1}wL_1)$ is $r = \rho$. If bequests may be negative, let $H = \underline{h}(r)$ if $r < \rho$ and $H = \frac{1+\rho}{\rho}(x^0 + (1+\rho)^{-1}x^1 - wL_0 - (1+\rho)^{-1}wL_1)$ if $r = \rho$. (Notice that $H \geqq \underline{h}(r)$, if $r = \rho$.) Let $S = x^0 + (1+\rho)^{-1}x^1 + r\underline{h}(r) - wL_0 - (1+\rho)^{-1}wL_1$, if $r < \rho$ and let $S = 0$ if $r = \rho$.

Standard results from dynamic programming imply that V exists. (See Denardo (1967), theorem 3.) It is not hard to show that $((x^a), H)$ solves equation (4.7). If $r < \rho$, then $((x^a), H)$ is a corner solution to the consumer's maximization problem in that he would like to bequeath less than H but is prevented from doing so by the constraints $h \geqq 0$ or $h \geqq \underline{h}(r)$.

This completes the proof that there exists a stationary equilibrium for each interest rate r such that $0 \leqq r \leqq \rho$. The fact that all these equilibrium are Pareto optimal follows from theorem (4.1).

I now show that there is no equilibria with interest rates $r > \rho$. Suppose that one did exist. The first order conditions of the consumer's maximization problem imply that $\frac{dV(h)}{dh} \leqq \frac{1+\rho}{1+r}\frac{dV(H)}{dH}$, where H is a consumer's inheritance and h is his bequest. Since $r > \rho$, this means that $\frac{dV(h)}{dh} < \frac{dV(H)}{dH}$. Since $h = H$, this inequality is impossible.

□

<u>Remark</u> Suppose that $\underline{h}(r)$ were defined to be $-(1+r)r^{-1}(w(r)L_0 + S) - r^{-1}w(r)L_1$, for $r > 0$. Then if bequests could be negative, there would be no equilibrium for interest rates r such that $0 < r < \rho$.

<u>Remark</u> If the equilibrium interest rate is ρ, then the size of the

government subsidy is somewhat arbitrary as long as the inequality constraint on bequests is not binding. In this case, the only effect of a change in subsidies is to change the size of bequests. This fact was observed by Barro (1974). However by increasing its net credit position far enough, the government could make the constraints on bequests binding and so reduce the interest rate.

4.11) Remark I now sketch what happens if the rate of time preference is allowed to vary. Suppose that there are many families with various rates of time preference and that the ancestors and descendents of each individual have the same rate. Then, there exists a stationary equilibrium for each interest rate from 0 to the smallest rate of pure time preference of any family, and the equilibrium allocation is Pareto optimal. There are no stationary equilibria and no Pareto optimal stationary allocations with interest rates exceeding the smallest rate of pure time preference. The same conclusions apply if the rate of pure time preference of each child is a random variable, perhaps independent of the parent's rate of pure time preference. One might imagine that in this case it would be possible to have an equilibrium interest rate between the minimum and maximum rates of time preference, but this is not so. If the interest rate exceeded the minimum rate of pure time preference and if prices were constant, then the distribution of marginal utilities of wealth among the population would tend to concentrate at zero as time went on. It follows that the distribution of wealth would wander off toward infinity.

I now consider which of the stationary equilibria of theorem (4.10) involve no government intervention. (Recall that the government intervenes when it has a non-zero net credit position.) Let $((x^a),k;H,w,r,S)$ be a stationary equilibrium. Suppose that bequests may be negative. If $r = \rho$, then the equilibrium bequest is such that the consumer's budget balances without a subsidy. Since the subsidy is zero and the interest rate is positive, the government's net credit position is zero. If $0 < r < \rho$, then subsidies are positive, so that the government intervenes. If $r = 0$, then the government's net credit position is $x^0 + k - wL_0 - \underline{h}(0)$, which is positive by assumption (4.9).This proves the following theorem.

4.12) <u>Theorem</u> If bequests may be negative, then there is only one stationary equilibrium with a non-negative interest rate which involves no government intervention, and that is the equilibrium with interest rate ρ.

If bequests must be non-negative the situation is more complicated.

4.13) <u>Theorem</u> If bequests must be non-negative, then there may exist no stationary equilibrium which has a non-negative interest rate and no government intervention.

<u>Proof</u> The following example demonstrates the theorem.

4.14) <u>Example</u> $u(x) = \log x$, $\rho = 1/2$, $L_0 = 1$, $L_1 = 0$, and $f(k,L) = k^{1/4}L^{3/4}$.

Consider a stationary equilibrium for this example with interest rate r. If $r = 0$, the government's credit position is $-5^{-1}4^{-4/3}$, so that this equilibrium requires intervention.

I now show that there is no stationary equilibrium with a positive interest rate unless the government imposes a tax. Let $r > 0$ be the equilibrium interest rate. The equilibrium supply of food available for consumption is $(3 + 4r)(4 + 4r)^{-4/3}$. In a stationary equilibrium with a positive interest rate, consumers would benefit financially from receiving a positive inheritance and bequeathing the same quantity. Therefore, one obtains a lower bound on a consumer's equilibrium consumption by solving $\max \{ \log x^0 + \frac{2}{3} \log x^1 \mid x^0 + (1+r)^{-1}x^1 = w\}$. If x^0 and x^1 solve this problem, then $x^0 + x^1 = (\frac{15 + 6r}{20})(4 + 4r)^{-1/3}$. Since $(\frac{15 + 6r}{20})(4 + 4r)^{-1/3} > (3 + 4r)(4 + 4r)^{-4/3}$ when $r > 0$, it follows that the demand for food exceeds the supply.

□

The non-existence of equilibrium just described is an artifact of the model. Equilibria do exist if land is included is as a factor of production. In order to see that this is so, let the production function be $f(k,L,T)$, where T is the quantity of land used. Assume that the appropriate analogue of assumption (2.2) applies and that $\frac{\partial f(K,L,T)}{\partial T} > 0$ if $k > 0$, $L > 0$, and $T > 0$. Suppose that society is endowed with one unit of land so that a stationary allocation $((x^a),k)$ is feasible if $x^0 + x^1 + k = f(k,L_0 + L_1,1)$. A stationary equilibrium is defined by $((x^a),k;H,w,q,r,S)$, where q is the rent on land. In order for this vector to be an equilibrium, $((x^a),k)$

must be a feasible allocation $((x^a),H)$ must solve equation (4.7), and $(k,L_0 + L_1,1)$ must solve the problem $\max\ [f(k,L,T) - (1+r)K - wL - qT]$.

The government's net credit position in a stationary equilibrium is $r^{-1}q + k + x^0 - wL_0 - H$. In order to interpret this expression, notice that $r^{-1}q$ is the value of the land. One should imagine that a government either owns the land or lends to individuals enough money to buy it.

4.15) Theorem In the model just described, there exists a stationary equilibrium with no government intervention and an interest rate in the interval $(0,\rho]$. This is so even when bequests must be non-negative. Such an equilibrium is Pareto optimal

Proof: Assume that bequests must be non-negative.

Clearly, there exists a stationary equilibrium with government intervention for each r in $[0,\rho]$, call it $((x^a(r)),k(r)\,;H(r),w(r),q(r),r,S(r))$. Let the equilibrium for $r = \rho$ be such that $S(\rho) = 0$, if such a choice of S is possible. $C(r) \equiv r^{-1}q(r) + k(r) + x^0(r) - w(r)L_0 - H(r)$ is the government's net credit position in the equilibrium. It is enough to prove that $C(r) = 0$, for some r in $(0,\rho]$. If $0 \leq \rho^{-1}q(\rho) + k(\rho) + x^0(\rho) - w(\rho)L_0$, then $H(\rho)$ equals this number and so $C(\rho) = 0$. Suppose that $\rho^{-1}q(\rho) + k(\rho) + x^0(\rho) - w(\rho)L_0 < 0$. Since q is continuous and $q(r) > 0$, for all r, $\lim_{r\to 0} r^{-1}q(r) = \infty$. If $0 < r < \rho$, then $H(r) = 0$. Also, $w(r)$ is bounded for r in $(0,\rho)$. Hence, $\lim_{r\to 0} C(r) = \infty$. Since C is continuous on $(0,\rho)$, $C(r) < 0$, for r near ρ. Hence $C(r) = 0$, for some r in $(0,\rho)$. This proves that there exists a stationary equilibrium with no government intervention. It should be clear that the equilibrium allocation is Pareto optimal.

□

Under reasonable conditions on f, the equilibrium of the previous theorem is unique. One sufficient condition is that $\frac{\partial f(k,L,T)}{\partial k \partial T} > 0$, whenever $k > 0$, $L > 0$, and $T > 0$. Therefore, one can think of the equilibrium as determinate.

Scheinkman (1980b) has proved results in the same spirit as the above theorem for a model without bequests.

Remark One can further restrict the interest rate by introducing what Barro (1974) calls the gift motive. A gift motive exists when individuals care about the welfare of their parents. The gift motive may be introduced by making the utility of an individual be $V_p(g) + u(x^0) + (1+\rho)^{-1}(u(x^1) + V_c(h))$, where V_p is the utility of the individual's parent experienced by the parent in his old age, g is the gift to the parent, V_c is the lifetime utility of the individual's child, and h is the bequest. In this model, the only Pareto optimal stationary allocation is the one with interest rate ρ, and ρ is also the only stationary equilibrium interest rate, even if one allows government intervention. The equilibrium I have in mind is a Nash equilibrium of a game between individuals and their descendents. Parents must think what their children will give them before making lifetime economic plans.

Barro (1976) has mentioned the effect of the gift motive on interest rates. Buiter (1979) and Carmichael (1981b) have studied models with the gift as well as the bequest motive.

I have chosen not to include the gift motive partly because I find the model with gifts less interesting and convincing. Also, the theory based on the model disintegrates as soon as rates of pure time preference vary. No such problems are met when only the bequest motive is included (see remark (4.11) above).

I sketch briefly what goes wrong if rates of time preference do vary. Suppose that there are many families and that all the descendents of an individual share his rate of time preference. Then, there exists no stationary equilibrium and no Pareto optimal stationary allocation. The problem is that in a stationary equilibrium or Pareto optimal allocation, $\lambda_p = (1+r)(1+\rho)^{-1} \lambda_c$, where λ_p and λ_c are the marginal utilities of of food (in youth) of the parent and child, respectively. But stationarity implies that $\lambda_p = \lambda_c$. Hence, $r = \rho$, for all families, which is impossible if ρ varies among families. Related problems arise if the rate of time preference of each child is a random variable.

I say no more about the gift motive in the rest of the paper.

5) A Critique of the Simple Bequest Model

In this section, I criticize the theory of interest which was obtained in the previous section by assuming that the government does not intervene. I make a single change in the model of that section, which is to make family size vary. I show that after this change is made the essential results of the theory are no longer valid. By the essential results, I mean that stationary equilibria exist and are Pareto optimal. The results are essential in the sense that from a neoclassical point of view they are the results one would expect of a market theory of interest. I also show that if one allows government intervention, then there do exist Pareto optimal stationary equilibria in the new model, but the government can choose its interest rate to be any number in the interval $[0,\rho]$. To summarize, the bequest motive does not rid one of the need for government intervention, but it does put a lid on the interest rate.

In order to give the bequest model the best chance possible, I consider many institutional arrangements having to do with testaments, insurance, and government policy. In order to subject the model to a severe test, I allow family size to be determined in a number of different ways. For instance, I allow people to choose the size of their family and I allow family size to be determined randomly by exogenous forces. It turns out that no institutional arrangement other than government intervention guarantees in all cases the existence of a Pareto optimal stationary equilibrium.

It is possible to say briefly (and somewhat inaccurately) why government intervention is required. Pareto optimality implies that the marginal utility of wealth for a child be no more than $(1 + r)^{-1}(1 + \rho)$ times that of his parent, where r is the real rate of interest and ρ is the pure rate of time preference of the parent. The inequality is strict only if parents are at a corner solution and leave nothing (or negative amounts) to their children. If the above marginal condition is satisfied with equality, then usually testaments favor children with larger families, for such children are in turn responsible for bequests to their own children. Such favoritism is difficult to arrange in a way which is consistent with stationarity. However, if the government takes over the role of legator by subsidizing consumers, there is no need for bequests. In fact, the marginal condition stated above can be satisfied with equality even when bequests are zero.

All the discussion which follows analyzes variants of the following simple example.

5.1) Example In all aspects except the process of reproduction, this example is a special case of the model of the previous section with the following specifications:

$$u(x) = \log x,\ \rho = 1/2,\ L_0 = 1, L_1 = 0,\ f(k,L) = 2\sqrt{kL}\ .$$

Reproduction is as follows. Each person either has two children or none. I consider two possibilities, people choose whether to have children or nature chooses the parents. If nature chooses, then each person has children with probability one-half. Also, the fertilities of different individuals are mutually independent. When people choose whether to have children, they make the choices which maximizes their expected utility. If a person is indifferent between having children and having none, he makes each choice with equal probability.

I assume that there are a continuum of consumers, so that the population can remain constant in a steady state. The set of consumers alive at any time can be identified with the unit interval [0,1], though I do not label distinct consumers.

The lifetime utility of a consumer with no children is simply $u(x^0) + \frac{2}{3} Eu(x^1)$, where E denotes expected value. The lifetime utility of a consumer with children is $u(x^0) + \frac{2}{3}[Eu(x^1) + u_1 + u_2]$, where u_i is the expected lifetime utility of the i^{th} child.

I consider the following cases regarding the selection of parents.

Case 1 Nature chooses the parents, and no young person knows whether he will have children.

Case 2 Nature chooses the parents and young people know whether they will have children.

Case 3 People decide whether to have children and all people have identical attitudes toward children.

Case 4 People decide whether to have children and differ in their attitudes toward children.

Each of these cases is meant to reflect a set of circumstances that arises in reality. Case 2 may seem strange. In interpreting this case, one should imagine that people live three periods. In the first period of life, people are dependent children and so do not appear in the model. Hence, young people have children, and the children become adult when the parents are old.

I will consider an institutional arrangement to be acceptable only if it gives a Pareto optimal equilibrium in all four cases, I, in fact, consider more cases than are logically necessary in order to eliminate all institutions except government intervention. Case 4 alone would suffice for this purpose. I consider the other three cases in order to reinforce the argument and to make clear the nature of the problems that arise.

I list below the institutions I consider.

Simple testaments. A simple testament is one which gives the same bequest to each child, regardless of whether he has children.

Conditional testaments. Such a testament makes the bequest to each child depend on the number of children had by each of the legator's children. Conditional bequests may also depend on the identity of the children. Conditional testaments can be used to influence the decisions of children as to whether to have children.

Remark Simple testaments are special cases of conditional testaments, but it does not follow that conditional testaments are socially better. It is conceivable that restricting freedom could be necessary for Pareto optimality.

Insured testaments. Such a testament is like a conditional testament, except that bequests are paid by an insurance institution. The parent pays a fixed fee to the insurer. An equilibrium condition is that the expected value of the payments to the children equal the sum received by the insurer.

Child insurance. Child insurance is a contract entered into between a young person and an insurer. When the insured person is old, he receives a certain sum if he has children and pays a certain sum if he has none. Again, an equilibrium condition is that the insurer break even on the average.

Government financial intervention. As before, the government intervenes if it maintains a net credit or debit position.

The institutional arrangements I will consider each involve one of the forms of testament, perhaps together with child insurance or government intervention.

The full definition of a stationary equilibrium varies from case to case. However in each case, I insist on (5.2) - (5.6) below.

5.2) The population, wage rate, interest rate, and capital stock are constant over time.

Notation. The equilibrium interest rate, wage rate, and *per capita* capital stock are denoted by r, w, and k, respectively.

Constancy of the population implies that half the population has children. This is automatically true in cases 1 and 2. When people choose whether to have children, there are equilibria in which everyone has children and the population grows, but such equilibria would not have a variable family size.

5.3) The statistical distribution in the population of bequests and consumption remains constant over time.

5.4) The production sector maximizes profits.

Given the specifications in example (5.1), this last condition implies that $w = (1+r)^{-1}$, $k = 2^{-1}(1+r)^{-2}$, and that the quantity of food *per capita* available for consumption each period is $2^{-1}(1+r)^{-2}(1+2r)$.

5.5) Consumers' choices maximize their expected utility.

5.6) The allocation is feasible.

A stationary equilibrium is called *symmetric* if all the parameters associated with a single individual depend only on whether he has children.

The definition of *Pareto optimality* is standard. An allocation is Pareto optimal if it is feasible and cannot be Pareto dominated by another feasible allocation. I never make welfare comparisons between economic states with different sets of people.

I consider both the case where bequests may be negative and where they are required to be non-negative. If bequests may be negative, I require that they must not be less than $\underline{h}(r) = -w(L_0 + L_1) = -(1+r)^{-1}$.

I now turn to the analysis of the specific cases.

Case 1. Nature chooses the parents and no young person knows whether he will have children .

First of all, I prove the following.

5.7) Suppose that there is no government intervention. Then whether or not bequests are required to be non-negative and no matter what the institution arrangement, all Pareto optimal stationary equilibria have interest rate $\rho = 1/2$. Also, everyone consumes the same quantity of food when he is young as when he is old and every parent consumes the same quantity of food as does each of his children. Finally, every parent receives the same inheritance as did his parent.

Notice that the statement implies that in a Pareto optimal stationary equilibrium, the population may be divided into classes according to consumption in each period. All descendents and ancestors of someone in a class belong to the same class. Pareto optimal equilibria are not necessarily symmetric.

Proof of (5.7) I give the proof for the case where bequests must be non-negative. If bequests may be negative, the argument is similar.

Suppose that a stationary equilibrium is Pareto optimal. Then, the consumption of an individual cannot depend on whether he has children or on whether his sibling has children. I now show that this fact implies that uninsured conditional testaments are simple. Suppose that a conditional testament made someone's bequest depend on whether his sibling had children. Payments from child insurance would not depend on whether the sibling had children. Therefore, the individual's income and so his consumption would depend on whether his sibling had children. This contradicts Pareto optimality. So, the inheritance of an individual can depend only on whether he has children. A testament that was optimal from a parent's point of view would give the same bequest to each child if both had children. Also, the total amount to be distributed to both children is the same no matter how many children the children have. It follows that a bequest cannot depend on whether either the heir or his sibling has children. That is, testaments are simple. In conclusion, I may restrict attention to simple and to insured testaments.

Suppose that testaments are simple and that a consumer may buy child insurance. In this case, the maximization problem is

$$5.8)\qquad V(H) = \max \{\log x^0 + \frac{1}{3}\log x_0^1 + \frac{1}{3}\log x_2^1 + \frac{2}{3}V(h) \mid h \geqq 0, x^0 + (1+r)^{-1}x_0^1 = w + H - (1+r)^{-1}A, \text{ and } x^0 + (1+r)^{-1}(x_2^1 + 2h) = w + H + (1+r)^{-1}A\}.$$

$V(H)$ is the expected lifetime utility of a person who inherits H. A is the quantity of child insurance bought by the consumer. He bequeaths h to each of his children. His food consumption when he is young is x^0. His food consumption when he is old if he has i children is x_i^1. The above equation defines V if $0 \leq r \leqq \frac{1}{2}$.

Let λ_0 and λ_2 be the Lagrange multipliers associated with the first and second constraints, respectively, in the above maximization problem. Then, the first order conditions of the maximization problem imply that $\frac{dV(H)}{dH} = \lambda_0 + \lambda_2 = (x^0)^{-1}$, $(1+r)^{-1}\lambda_0 = (3\,x_0^1)^{-1}$, $(1+r)^{-1}\lambda_2 = (3\,x_2^1)^{-1}$, and $\frac{2}{3}\frac{dV(h)}{dh} \leqq 2(1+r)^{-1}\lambda_2$, with equality if $h > 0$. Since the equilibrium is Pareto optimal, $x_0^1 = x_2^1$ and so $\lambda_0 = \lambda_2$. Hence,

$\frac{dV(h)}{dh} \leqq \frac{3}{2}(1+r)^{-1}\frac{dV(H)}{dH}$, with equality if $h > 0$. If $r > \frac{1}{2}$ and if V exists, then $\frac{dV(h)}{dh} < \frac{dV(H)}{dH}$, so that any child has a smaller marginal utility of wealth than his parent. This is not consistent with the stationarity of equilibrium. If $r < \frac{1}{2}$, then $h = 0$, so that $H = 0$. By adding the two budget constraints and using the fact that $w = (1+r)^{-1}$, I obtain that $x^0 = 3/5(1+r)$ and $x_0^1 = x_2^1 = 1/5$. Therefore, the average <u>per capita</u> demand for food each period is $(5 + 2r)/10(1+r)$. The supply <u>per capita</u> available for consumption is $(1+2r)/2(1+r)^2$. If $0 < r < 1/2$, then $(5+2r)/10(1+r) < (1+2r)/2(1+r)^2$, so that the economy is not in equilibrium. If $r = 0$, the economy is in equilibrium, but the net credit position of the government is $2/5$. Since by assumption the government does not intervene, it follows that $r = 1/2$.

Since $r = 1/2$, it follows that $x^0 = x_0^1 = x_2^1$. Also $h = H$. If $h > 0$, then $\frac{dV(H)}{dH} = \frac{dV(h)}{dh}$, so that $h = H$ by the strict concavity of V. If $h = 0$, then $\frac{dV(H)}{dH} \geqq \frac{dV(h)}{dh} = \frac{dV(0)}{dh}$, so that by the strict concavity of V, $H = 0 = h$. This completes the proof of (5.7) for the case of simple testaments with child insurance.

If testaments are simple and there is no child insurance, then the variable A must be dropped from equation (5.8). Otherwise, the argument remains unchanged.

Suppose that testaments are insured and that there is no child insurance. In this case a consumer's maximization problem is

$$\text{5.9)} \qquad V(H_0,H_2) = \max \{\log x^0 + \tfrac{1}{3}\log x_0^1 + \log x_2^1 + \tfrac{2}{3}V(h_0,h_2) \mid$$

$$h_0 \geqq 0,\ h_2 \geq 0,\ x^0 + (1+r)^{-1}x_0^1 = w + (1+r)^{-1}H_0$$

$$x^0 + (1+r)^{-1}x_2^1 = w + (1+r)^{-1}H_2\}.$$

H_i is the inheritance received if the individual has i children, h_i is the bequest made to a child with i children. (The bequest is received when the child is old.) The rest of the notation is as before.

Pareto optimality implies that

$$\frac{\partial V(H_0,H_2)}{\partial H_0} = \frac{\partial V(H_0,H_2)}{\partial H_2} \quad \text{and}$$

$$\frac{\partial V(h_0,h_2)}{\partial h_0} = \frac{\partial V(h_0,h_2)}{\partial h_2} \quad .$$

Using these facts, the argument is much as in the case of simple statements. Instead of proving $h = H$, one proves $h_2 = H_2$. The argument does not change if one includes child insurance.

□

Remark. There exists one symmetric stationary equilibrium with simple testaments and no government intervention. It has interest rate 5/6 and is not Pareto optimal.

I now show that there are no Pareto optimal stationary equilibria without government intervention or one of the two forms of insurance. Suppose there were some Pareto optimal equilibrium with no insurance or intervention. Since there is no insurance, parents are restricted to simple and conditional testaments. In the proof of (5.7), it was shown that the conditional testament must be simple. By (5.7), each person consumes the same quantity, x, in youth as in old age, and he bequeaths to each child the same quantity, H, which he inherits. His budget equation is $x + \frac{2}{3}x = w + H$, if he has no children and $x + \frac{2}{3}(x + 2H) = w + H$, if he has children. These equations imply that $H = 0$. By (5.7), the interest rate is 1/2. It follows that $x = 2/5$, which is less than the quantity 4/9 which is available *per capita* for consumption. Hence, the economy is not in equilibrium. This contradiction implies that government intervention or insurance are necessary for Pareto optimality.

Now suppose that people can buy child insurance, that testaments are simple, and that there is no government intervention. In this situation, there exist Pareto optimal stationary equilibria, whether or not bequests must be non-negative. In fact, all equilibria are Pareto optimal, though I do not prove this fact. Statement (5.7) asserts that in a Pareto optimal stationary equilibrium, the interest rate is 1/2 and the population may be divided into classes. Everyone in one class consumes the same quantity of food. The distribution of the population over the classes must be such that average *per capita* consumption per period is 4/9. The equilibrium bequest of someone in the class that consumes x per period is $H = 5x - 2$. These people buy $A = H$ units of insurance. That is, when they are old, they receive $5x - 2$ if they have children and they pay $5x - 2$ if they have none.

Similar results obtain if testaments may be insured. All stationary equilibria are Pareto optimal. An equilibrium with testament insurance and no child insurance may be described as follows. Consider the class of people who consume x per period. Each parent in the class pays $H = 2(5x-2)$ to the insurer. The insurer pays $H_0 = \frac{5}{2}x - 1$ to childless children and $H_2 = \frac{25}{2}x - 5$ to children with children. The average of x over society must be 4/9.

By intervening, the government can arrange a Pareto optimal stationary equilibrium with any interest rate r such that $0 \leqq r \leqq 1/2$. Suppose that bequests must be non-negative. Then, one may let testaments be simple. In this case, a consumer's maximization problem is as below.

$$V(H) = \max \{\log x^0 + \frac{1}{3} \log x_0^1 + \frac{1}{3} \log x_2^1 + \frac{2}{3} V(h) \mid h \geqq 0,$$

$$x^0 + (1+r)^{-1} x_0^1 = w + H + S, \text{ and } x^0 + (1+r)^{-1}(x_2^1 + 2h) = w + H + S\},$$

where $V(H)$ is the expected lifetime utility of a consumer with inheritance H, h is the bequest to each child, and S is the subsidy.

There is a symmetric stationary equilibrium for each interest rate r such that $0 \leqq r \leqq \frac{1}{2}$ and each such equilibrium is Pareto optimal. These equilibria are just as in Theorem (4.10). The bequests may be chosen to be zero in all these equilibria. The subsidy replaces bequests in the role of passing wealth from generation to generation.

If bequests may be negative, then the equilibria with interest rates less than 1/2 require subsidies which must be conditional in the sense that they depend on whether the recipient has children.

In summary, the only ways to achieve Pareto optimality in case 1 are government intervention, child insurance, or testament insurance.

Case 2. Nature chooses the parents and young people know whether they will have children.

The analysis of this case closely parallels that of case 1. One difference is that child insurance is impossible, since young people know whether they are to have children. Statement (5.7) remains valid, though the proof is slightly different. The key to the proof of (5.7) in case 1 is that in a Pareto optimal stationary equilibrium, each consumer's consumption must be independent of whether he has children. In case 2, the key step is to notice that in a Pareto optimal stationary equilibrium, the marginal utility of wealth of a parent must be at least $\frac{2}{3}(1+r)$ times the marginal utility of wealth of each of his children. I leave out the details of this proof.

The conclusion is as in case 1. Insurance (in this case, just testament insurance) leads to a Pareto optimal equilibrium with interest rate 1/2. If no such insurance is available, then government intervention is needed in order to achieve Pareto optimality. Government interference can lead to any interest rate in the interval [0,1/2].

<u>Remark</u>. There exists one symmetric stationary equilibrium with simple testaments and no government intervention. It has interest rate $7/10$ and is not Pareto optimal.

<u>Case 3.</u> People decide whether to have children and all people have identical attitudes toward children.

The overall conclusions in this case are just the same as those just listed for case 2. Nevertheless, I discuss case 3 in some detail because it seems particularly attractive.

The requirement that the population be constant leads to a silly difficulty when people choose whether to have children. The difficulty can be illustrated most vividly by changing the utility for food from $\log x$ to, say, $101 - (1+x)^{-1}$. If this is the utility function, then a parent gains more than $(1+r)^{-2}100 - 2$ units of utility by having children, even if he is punished for having children by being deprived of food and having all his descendents be deprived of food. Hence, unless r were very large, everyone would have children and the population would grow. The same sort of problem arises when the utility function for food is $\log x$.

There are a number of ways to avoid this problem. I do so by making the lifetime utility of a parent depend on the difference between his utility and that of his child. More precisely, I assume that the lifetime utility of a parent is of the form

5.10) $$v = \log x + \frac{2}{3}(\log x^1 + u_1 + u_2 - 2\bar{v}),$$

where u_1 and u_2 are the expected lifetime utilities of the first and second children, respectively, and where $\bar{v}$ is the expected lifetime utility of the parent. $\bar{v}$ should be thought of as a parameter determined by custom. $\bar{v}$ does not depend on the parent's choices.

I require that in a stationary equilibrium

5.11) $\bar{v} = v.$

That is, a parent's lifetime utility equals $\bar{v}$.

In order to make sense of $\bar{v}$, I require that in a stationary equilibrium

5.12) all the ancestors of any one person have the same lifetime utility level, and this utility level is the parameter $\bar{v}$ in his lifetime utility function as a parent.

Recall that in example (5.1), I assume that if a person is indifferent between having and not having children, then he makes either choice with probability 1/2. Therefore, if everyone is indifferent between these choices, the population remains constant.

I now prove the following.

5.13) If there is no government interference and no insurance, then there exists no stationary equilibrium with a non-negative interest rate.

Since testament insurance is excluded, only simple and uninsured conditional testaments are possible. First of all, I prove (5.13) when testaments are simple. I make the argument for the cases of non-negative and negative bequests simultaneously.

Suppose that a stationary equilibrium exists with interest rate $r \geqq 0$. By conditions (5.11) and (5.12), all ancestors of a parent must enjoy the same lifetime utility as that enjoyed by the parent. I now concentrate on the set or class of people whose parents had expected lifetime utility level $\bar{v}$. Let $\bar{x}$ be defined by $\bar{v} = \log \bar{x} + \frac{2}{3} \log \frac{2}{3} (1+r) \bar{x}$ and let $\bar{H} = \frac{5}{3} \bar{x} - \bar{v}$. $\bar{H}$ is the inheritance that just enables a childless person to enjoy lifetime utility level $\bar{v}$,

I now show that any grandparent of someone in the given class has consumption pattern $(x^0, x^1) = (\bar{x}, \frac{2}{3}(1+r)\bar{x})$. Observe that both children of one parent have the same lifetime utility level, for each receives the same inheritance and each makes the decision regarding having children which maximizes his utility. Parents of people in the given class enjoy lifetime utility $\bar{v}$, by definition. Hence, both children of a grandparent of someone in the class enjoy lifetime utility level $\bar{v}$, and this is the grandparent's lifetime utility level. It follows from the form of the lifetime utility function (5.10) that the consumption pattern of the grandparent is $(\bar{x}, \frac{2}{3}(1+r)\bar{x})$.

I now show that the equilibrium interest rate is 1/2. Consider a great grandparent of someone in the given class. By what was said in the previous paragraph, the great grandparent and both of his children have consumption pattern $(x^0, x^1) = (\bar{x}, \frac{2}{3}(1+r)\bar{x})$. This means that the marginal utility of food for each of them (when young) is $(\bar{x})^{-1}$.

Suppose that bequests must be non-negative. The great grandparent adjusts his bequest until his own marginal utility for food (when young) is at least $\frac{2}{3}(1+r)$ times that of his children. This means that $r \leqq 1/2$ and $r = 1/2$

if the bequest is positive. In proving (5.7), I have already argued that if $0 < r < 1/2$, then bequests are zero and consumers cannot afford to buy the entire output. Hence, this case is impossible. There is an equilibrium with interest rate zero, but it involves government interference, which has been excluded. This leaves $r = 1/2$ as the only possibility. Similar arguments apply if bequests may be negative, so that $r = 1/2$ in both cases.

I now show that every parent of someone in the given class receives an inheritance $-3\bar{H}$. Consider a grandparent of someone in the given class. Let H be his inheritance and let h be his bequest to each of his children. I show that $h = \frac{3}{4}(H - \bar{H})$. By what has been said in the previous two paragraphs, the grandparent's consumption pattern is $(\bar{x},\bar{x})$ so that $H = x^0 + \frac{2}{3}x^1 - w + \frac{4}{3}h = \frac{5}{3}\bar{x} - w + \frac{4}{3}h = \bar{H} - \frac{4}{3}h$. That is $h = \frac{3}{4}(H - \bar{H})$. The difference equation $H_{t+1} = \frac{3}{4}(H_t - \bar{H})$ is stable with rest point $-3\bar{H}$. Therefore, in the stationary equilibrium, the bequest to a parent in the given class is $-3\bar{H}$.

I now show that $\bar{H} \leqq 0$. If bequests must be non-negative, then $-3\bar{H} \geqq 0$, so that $\bar{H} \leqq 0$. Suppose that bequests may be negative. I show that $\bar{H} = 0$. If everyone in the given class were indifferent between having children and not doing so, then it would follow that $-3\bar{H} = \bar{H}$, so that $\bar{H} = 0$. But it is conceivable that no one would ever be indifferent between having and not having children. Parents might so adjust their bequests that either both children or neither would choose to have children. Suppose this was so. Since the population of the class must remain constant, some parent in the class must set his bequests so that both his children choose not to have children. Fix attention on one such parent. If he gave one of his children the minimum bequest, which is $-w$, the child would consume nothing and have an infinite marginal utility for food. Hence, the parent would not leave the minimum bequest. Since $r \geq 1/2$, it follows that both the parent and his children have the same marginal utility for food. This means that both parent and children have consumption pattern (x,x), for some x. The lifetime utility of the parent is $\log x + \frac{2}{3} \log x + \frac{2}{3} (2 \log x + \frac{4}{3} \log x - 2 \bar{v})$, which by condition (5.11) must equal $\bar{v} = \log \bar{x} + \frac{2}{3} \log \bar{x}$. Hence, $x = \bar{x}$, so that the parent must bequeath $\bar{H}$ to each of his children and his own inheritance must be $\frac{7}{3} \bar{H}$. But his inheritance is also $-3\bar{H}$, so that $\bar{H} = 0$. This completes the proof that $\bar{H} \leqq 0$.

Since $\bar{H} \leqq 0$, it is not hard to see that the _per capita_ consumption of food per period by people in the class is at most 2/5. Since the class was arbitrary, the same is true for the whole population. But the quantity of food available _per capita_ is $4/9 > 2/5$. Hence, the economy is not in equilibrium. This completes the proof that no stationary equilibrium exists when testaments are simple.

I now turn to the analysis of testaments which are conditional and not insured. Such testaments can be used to influence the decisions of heirs as to whether to have children. I assume that a testament need not distribute the entire sum set aside for bequests if the heirs make decisions other than those desired by the parent. If bequests can be negative, then children can even be punished with debts. The testament establishes a bimatrix game between the children in which the strategies are to have or not to have children and the payoffs are expected lifetime utilities. The solution of such a game might be ambiguous. I make the following assumption in order to avoid ambiguity.

5.14) The game established by a testament either has a saddle point, or the best choice of each child is independent of the choice of the other child.

This assumption is not a serious restriction in that a parent could maximize his own utility while writing his testament so as to satisfy the assumption. If the difference between what the parent could bequeath and the minimum bequest (0 or $\underline{h}(r)$) is large enough to influence the children's decisions, then he could simply select the pair of decisions by his children and the distribution of bequests which maximized his own utility. He could guarantee that these decisions would be made by specifying that the children be punished if they made decisions other than those specified.

It is easy to see that if a testament is optimal from the point of view of the parent, then

5.15) $$\frac{dV(H)}{dH} \geqq \frac{2}{3}(1+r) \max\left(\frac{du_1(h_1)}{dh_1}, \frac{du_2(h_2)}{dh_2}\right), \text{ with equality if}$$

$\max(h_1, h_2)$ exceeds the minimum bequest, 0 or $\underline{h}(r)$,

where $V(H)$ is the parent's expected lifetime utility, H is his inheritance, $u_k(h_k)$ is the expected lifetime utility of the k^{th} child, and h_k is his bequest if both children make the decisions specified by the testaments.

I now prove that there exists no stationary equilibrium. I give the argument for the case in which bequests must be non-negative. If bequests may be negative, the argument is very similar, except one proves that the parameter $\bar{h}$, appearing below, is non-positive, rather than proving that it is zero.

Suppose that a stationary equilibrium exists and that its interest rate is $r \geqq 0$. First of all, $r \leqq 1/2$, for inequality (5.15) implies that the marginal utility for food of each child (when he is young) is at most $\frac{3}{2}(1+r)^{-1}$ times that of his parent. Hence, if $r > 1/2$, the marginal utility of each child is less than that of his parent, and this is not consistent with stationarity.

As in the case of simple testaments, I consider the class of people defined by the fact that the parent of each person in the class has expected lifetime utility $\bar{v}$. The population of the class must remain constant, so that there must be parents in the class. Let $\bar{h}$ be the inheritance which just enables a parent to enjoy expected lifetime utility $\bar{v}$. By (5.11) and (5.12), every parent in the class receives inheritance $\bar{h}$.

I now show that $\bar{h} = 0$. Fix attention on one parent, and let v_k be the expected lifetime utility of his k^{th} child and let h_k be his inheritance.

I first show that if $v_1 \leqq \bar{v}$ and $v_2 \leqq \bar{v}$, then $h_1 = h_2 = 0$. Suppose that $v_1 \leqq \bar{v}$, $v_2 \leqq \bar{v}$, and max $(h_1,h_2) > 0$. Then, because of the nature of the parent's utility function (5.10), the parent would increase his utility level if he had no children and spent the money so saved, $h_1 + h_2$, on his own consumption. The parent's parent would have foreseen that this would be the situation and would have changed his testament so that his child would have no children. This proves that $h_1 = h_2 = 0$ when $v_1 \leqq \bar{v}$ and $v_2 \leqq \bar{v}$. It follows that if ever both children of one parent in the class have children, then $\bar{h} = 0$.

I next show that if exactly one child of the given parent has children, then $h_1 = h_2 = 0$. Suppose that the first child has no children, that the second one does, and that max $(h_1,h_2) > 0$. Then, $v_2 = \bar{v}$, and by the result of the previous paragraph $v_1 > \bar{v}$. Also, $h_1 > h_2$, for if $h_1 \leqq h_2$, then the second child would want to have no children so as to increase his utility to v_1 or more. The parent would have foreseen this and would have permitted him not to have children. Since the first child has a larger inheritance than the second and need make no bequests, he consumes more food than the second child. It follows that $\frac{du_1(h)}{dh_1} < \frac{du_2(h_2)}{dh_1} = \frac{3}{2}(1+r)\frac{dV(H)}{dH}$, where the notation is as is (5.15). Hence, it would be advantageous to the parent to reduce h_1 slightly and increase his own consumption. He could prevent the first child from having children after the reduction in his inheritance, for the parent could deny him any inheritance if he had children. If the first child had no inheritance and had children, he would have a lifetime utility of at most $\bar{v} < v_1$. I have reached a contradiction and so it cannot be that exactly one child has children if $\max(h_1, h_2) > 0$.

The results of the previous two paragraphs imply that whenever at least one of the parent's children has children, then $h_1 = h_2 = 0$. Hence, $\bar{h} = 0$, and the given class is the whole population. It now follows easily, as before, that the economy has no stationary equilibrium with an interest rate in [0,1/2]. This completes the proof of (5.13).

The effects of insurance and government intervention are similar to those in case 2. Child insurance is impossible, since people choose whether to have children. However, testament insurance and government subsidies lead to Pareto optimal stationary equilibria, and these equilibria are the same as those discussed in case 1. All the Pareto optimal equilibria in case 1 give the same allocation to a person whether he has children or not. It follows that in each of those equilibira, the people of the model now being discussed would be indifferent between having children and not doing so. It follows that each individual would have children with probability 1/2, and so the population would remain constant.

<u>Case 4</u> People decide whether to have children and differ in their attitudes toward children.

There are many ways to model differences in attitudes toward having children. I discuss two models, which are intended to reflect the following possibilities: 1) all parents attach full weight to the utility of their children, but differ in the pleasure derived from parenthood itself, and 2) parents differ in the weight they attach to their children's utility. In both models, there are two types of people, type 0 and type 2. Type 2 people are the more favorably disposed toward being parents. Each child has an equal chance of being of either type and his chances are independent of the type of his parent and his sibling. I assume that the type of an individual cannot be observed directly by anyone but himself. This seems to be a reasonable assumption, since types are distinguished only by their utility functions.

In model 1, the lifetime utility of a consumer is as follows:

$$\log x^0 + \frac{2}{3}(\log x^1 + u_1 + u_2 - 2\bar{v}) - A, \quad \text{if he is a parent and of type } 0,$$

$$\log x^0 + \frac{2}{3}(\log x^1 + u_1 + u_2 - 2\bar{v}) + A, \quad \text{if he is a parent and of type } 2,$$

and

$$\log x^0 + \frac{2}{3}\log x^1, \text{ if he is of either type and has no children.}$$

In these formulas, u_k is the expected lifetime utility of the k^{th} child. The parameter $\bar{v}$ is as in (5.10). Conditions (5.11) and (5.12) now become the following statement.

All ancestors of a given parent enjoy the same expected lifetime utility level as the parent, and this utility level equals the parameter $\bar{v}$ in the parent's utility function.

In model 2, the lifetime utility function of a person is as follows:

$\sqrt{x^0} + \frac{2}{3}\sqrt{x^1}$, if he is of type 0, whether he has children or not, of if he is of type 2 and has no children, and

$\sqrt{x^0} + \frac{2}{3}\sqrt{x^1} + u_1 + u_2$, if he is of type 2 and is a parent.

Notice that type 0 people attach no weight at all to the utility of their children. I have changed the utility for food from $\log x$ to the non-negative function $\sqrt{x}$, so that people of type 2 would want to have children.

The production sector of models 1 and 2 is assumed to be as in example (5.1).

Analysis of model 1 leads to the same conclusions as in case 2. The only ways to achieve a Pareto optimal stationary equilibrium are through government intervention or testament insurance. In order to prove this statement, it is sufficient to notice that statement (5.7) applies to model 1. Thus, the inclusion of differences in attitudes towards having children does not necessarily eliminate the difficulties that have been met already.

Testament insurance makes sense in model 1, but it is hard to imagine that such insurance would work in reality if differences in attitudes toward having children were important. For one thing, the attitudes and values of children are influenced by those of their parents. If children tended to share their parents' attitudes toward having children, testament insurance would lead to an adverse selection problem. Also, if parents invested in testament insurance, they might try to persuade their children to have children. This is moral hazard. It is well-known that adverse selection and moral hazard make insurance difficult if not infeasible. This point can be illustrated by an example similar to model 1 in which the type of a parent and his child are correlated (and there are at least three types).

In model 2, testament insurance meets another difficulty. If an insured testament left more to a child with children than to one with none, then a type 0 child would have children simply in order to increase his inheritance and then would leave nothing to his own children. It follows that testament insurance does not lead to a Pareto optimal stationary equilibrium. I leave out the details of the analysis. As in all the previous cases, government interference can lead to a stationary equilibrium with an interest rate anywhere in $[0,1/2]$.

In the context of case 4, I have just pointed out the difficulties associated with testament insurance. In my opinion, these difficulties would be of sufficient importance to make such insurance impossible in reality. This leaves government intervention as the only means of achieving a Pareto optimal stationary equilibrium in example (5.1).

I do not mean to imply that government intervention could guarantee Pareto optimality in general. I do not prove such a statement, for I do not wish to make a case for government intervention. I merely wish to indicate how Barro's model could be used to make such a case.

Remark It is easy to make it impossible for the government to achieve Pareto optimality. Return to the model of section 4 in which each individual has exactly one child. Imagine that consumers vary in the intensity of their utility. (For instance, $u(x) = \log x$ for some and $2 \log x$ for others.) Imagine that the utility function of a child is a random variable independent of the utility function of the parent. Then, consumers with low utility would accumulate assets for the benefit of descendents with high utility. In effect, family lines would self-insure against a member with a big appetite. (Laitner (1979a) has discussed this phenomenon.) The self-insurance would make equilibrium impossible at interest rate ρ . Since the self-insurance would never be perfect, Pareto optimality could not be achieved at any interest rate. A formally equivalent problem arises in monetary theory. (See Bewley, (1980a).) It does not seem that this problem should be taken seriously in the context of bequest models. That it can arise at all indicates that there is something wrong with the bequest model of section 4.

Remark . The utility function used in this section, $\log x$, does not satisfy assumption (2.1) in that it is not well-defined at zero. I chose $\log x$ for computational convenience. The results of the section remain valid if $\log x$ is replaced by a function, such as $\log (x+1)$, which does satisfy (2.1).

6. An Attempt at Common Sense

I now explain what I believe should be the role of the bequest motive in the theory of interest. I argue that a distinction should be made between an equilibrium theory of interest and a theory which treats the interest rate as an object of public choice. In an equilibrium theory of interest, the bequest motive should not appear in the criteria for Pareto optimality. That is, the utility for bequests should not be included in the individual utility functions used for making Pareto comparisons. In a public choice theory of interest, consideration might be given to individuals' attitudes toward many motives besides the welfare of their own descendents. I also argue that the utility for bequests should be allowed to be arbitrary and that one should not assume that the utility for bequests depends on the utility of the legatee. Finally, I discuss the nature of the theory of interest one obtains if one treats bequests in the way I recommend.

In order to understand the distinction between general equilibrium and public choice theories of interest, it is helpful to think of the usual static theory of allocation. In the usual foundation of this theory, the preference ordering of each individual depends only on his own consumption. But we all have opinions about and reactions to many aspects of economic life that have nothing to do with our own consumption. This would be so even if there were no such thing as a public good. For instance, some feel that is is unjust that many people are poor and a few are rich. If we introduce into our model such opinions and feelings about external matters, we obtain the model of social choice theory, in which each individual has a preference ordering over the entire set of social states. An outcome which is Pareto optimal with respect to the larger preference orderings may not be so when each individual's ordering is restricted to his own consumption, and vice versa. I do not mean to imply that one or the other of these criteria is superior. I merely wish to emphasize that they are different. One corresponds to the engineering point of view often associated with general equilibrium theory. The other reflects an interest in ethical and political questions.

Returning to the intergenerational model, it seems that the utility for bequests should be treated as an opinion about an external matter, and therefore should be excluded from the Pareto criteria associated with general equilibrium theory. If we included the welfare of descendents in each person's utility function, why not include the welfare of other people's descendents and of the person's own contemporaries, as Sen (1969) and Marglin (1963) have suggested should be done? It is not obvious that people do not have strong feelings about the welfare of people who are not related to them.

One might argue that one should include in the Pareto criterion only objects about which individuals can make decisions. But does this argument not lead to the conclusion that we should expand people's field of action by creating a social mechanism similar to that of Groves and Ledyard (1980) and which would permit people to choose the rate of interest collectively?

One way to try to settle on a Pareto criterion is to make a distinction between felt utility and utility used in making judgments or decisions, and then to include only felt utility in the Pareto criterion associated with equilibrium theory. The problem with this approach is that it is hard to imagine how one could determine what utility a person feels and what he uses only in order to make decisions. Nevertheless, for clarity of thought, it is helpful to keep the distinction in mind. In order to understand the distinction, consider the following simple example of a Robinson Crusoe economy with two time periods and one good. Crusoe has one unit of the good in the first period and none in the second, but the good may be stored. His utility function is $u_1(x_1) + u_2(x_2)$, where x_t is the quantity consumed in period t and u_t is the utility felt in period t. Crusoe's maximization problem is, therefore, max $\{u_1(x_1) + u_2(x_2) \mid x_1 \geqq 0,\ x_2 \geqq 0, \text{ and } x_1 + x_2 = 1\}$. Observe that if $x_2 > 0$, then Crusoe decides in period 1 to save some of the good until period 2, even though he does not experience the utility from consuming the savings until period 2. That is, he makes decisions based on utility which is not experienced at the time of the decisions. If the anticipation of pleasure gave him the same amount of pleasure, then his total utility would be $u_1(x_1) + 2u_2(x_2)$. In conclusion, one can make a distinction between utility used in making a decision and utility which is felt at the time the decision is made.

There seems to be no reason to assume that the utility for bequests is felt utility. Is it not reasonable to assume that people make provisions for their children's future in the same way that they provide for their own future? One might argue that no one would ever make a bequest if it did not give him pleasure, but it seems to me that to adopt such a view is to take the pleasure principle or the economic man too seriously. The argument amounts to saying that no one is ever truly generous.

In conclusion, it seems that the Pareto criterion which is most naturally associated with the point of view of general equilibrium theory does not include the utility for bequests. Thus in the model of section 4, only the utility $u(x_1) + (1+\rho)^{-1} u(x_2)$ should be used to define Pareto optimality, even though consumers make decision using the function $u(c_1) + (1+\rho)^{-1}[u(c_2) + V(h)]$.

I now turn to the basic assumption of the Barro model, which is that the utility of a parent is an increasing function of the utility of a child. It seems that this assumption is untenable. If it or something like it were true in reality, then parents and children would never disagree. Who can deny that relations between parents and children are diverse, complicated, and often difficult, and that the presence of inheritable wealth often worsens them? Furthermore, the assumption does not seem to find support in observed economic behavior. In a Barro model like that of section 4, the bequest motive affects interest rates by making personal savings very sensitive to interest rates. But econometricians have had a very hard time determining whether interest rates have any effect on personal saving at all. (Recent work on the subject includes Boskin (1978), who claims to observe a strong effect, and Howrey and Hymans (1978), who criticize Boskin and find no significant effect.)

It seems wisest then to suspend judgment when constructing models with bequests and to leave the exact nature of each consumer's bequest motive as an unknown datum. We should leave it to empirical work to determine the nature of the bequest motive in the population. I do not mean to imply that the bequest motive is not important. Recent empirical work suggests that inherited wealth accounts for a large part of private wealth. The latest paper on this subject that I have seen is Kurz (1981).

If one adopts the views expressed in the above paragraph, little remains of a general equilibrium theory of interest. If the bequest motive is excluded from the Pareto criterion, then any non-negative interest rate is the interest rate of a Pareto optimal equilibrium. This is so even in the models of section 5. Most of the difficulties with Pareto optimality met with there occurred only because the utility for bequests appeared in the Pareto criterion.

It is true that the bequest motive might make it impossible for the government to achieve very high interest rates by means of the monetary and fiscal policy considered up to now. But a tax on inheritances would make it possible to achieve high interest rates. For instance, if the bequest motives were as in section 4, a tax of size $\frac{r-\rho}{1+r}h$ on a bequest of size h would make it possible to achieve an interest rate $r > \rho$. But it is not clear _a priori_ that such a tax would be necessary. Whether one would be necessary would depend on the population's preferences for bequests and the distribution of preferences in the population.

Remark. The population distribution of the bequest motive is important. For instance, suppose that in the model of section 4, most people were as described there, but some few were spendthrift and never left any bequest. If each child could be a spendthrift with a small probability of at best some $\varepsilon > 0$, then there would exist a stationary equilibrium with an arbitrarily high interest rate and no inheritance tax. In such an equilibrium, there would be a stationary distribution of bequests in the population. Families would accumulate assets until a spendthrift consumed them all.

One might try to fix the interest rate by insisting that the government not intervene. But it seems arbitrary to do so. In reality, we do have a government, and it does borrow and lend and collect taxes, even taxes on gifts and bequests.

If one agrees that neither equilibrium forces nor Pareto optimality fix a natural interest rate, then the conclusion is nearly inescapable that the interest rate is determined by national (or foreign) governments and perhaps by the weight of history and the inertia of the public's expectations. At this point, one can either stop and say that the choice of interest rate must be made by the political process, or one can try to analyze the matter as a problem in the theory of public choice. This problem is more difficult than the problem of public goods. In the theory of public goods, the task is to elicit consumers preferences over their own consumption bundle, when this bundle includes public goods. But we have seen that consumers' preferences over their own consumption bundles do not determine the interest rate. One would need, therefore, to elicit other information relevant to the choice of interest rate, such as individuals' attitudes towards progress, industrialization, rapid change, and so on and how individuals weigh the prosperity of future generations against that of their contemporaries. Even if one could obtain all this information, so many issues are involved that Arrow's impossibility theorem (Arrow, 1951) should make one hesitate before choosing any one way to use the information to select an interest rate.

7. Relation to the Literature

The discussion which follows is divided into four sections. In these, I discuss the literature on, respectively, the natural rate of interest, overlapping generations models, models with bequests, and the optimal rate of investment.

The Natural Rate of Interest

Milton Friedman refers to a natural rate of interest in his presidential address (1968). He does not say exactly what he means by this term. He simply refers to Wicksell. (See page 7 of Friedman (1968).) However, Friedman does seem to have in mind that the natural rate is a constant determined by equilibrium of the underlying real forces in the economy. He asserts that the monetary authority can keep the nominal rate of interest below the natural rate only at the cost of more and more rapid inflation. Similarly, it can keep the nominal rate above the natural rate only at the cost of more and more rapid deflation. (Wicksell made similar statements. See (1937), p. 107.) It is just this sort of idea that I wish to argue against in this paper. Friedman later seems to have changed his mind, for in "The Optimum Quantity of Money," (1969), he argues that the rate of interest should equal consumers' rate of pure time preference. I have seen no attempt on his part to reconcile this recommendation with his views on the natural rate of interest.

Wicksell does not seem to have given a clear definition of the natural rate of interest. But after reading chapters 8 and 9 of his Interest and Prices, I believe he had in mind what we call the marginal efficiency of capital. In the models of this paper, the marginal efficiency of capital can limit the interest rate only in the short-run. If investment is reversible, as I have assumed, then the current real rate of interest equals the marginal efficiency of capital. The government can set this rate as high as it likes, even in the short-run, unless the bequest motive puts an upper bound on interest rates. The current availability of capital puts a lower bound on the current real rate of interest. But by persistently maintaining a low nominal rate of interest, the government can induce capital accumulation which would lower the real rate of interest. If investment were irreversible, investment would stop once the real rate of interest reached a certain level. Nevertheless, the government could set the current real rate as high as it liked, unless, of course, it were prevented from doing so by the bequest motive. The high interest rate would bring about a gradual decumulation of capital until the marginal efficiency of capital equaled the real interest rate.

In volume I of A Treatise on Money, Keynes adopts a notion of the natural rate of interest (page 155). He defines this level to be the level at which savings equals investment. In terms of the models of this paper, the natural rate of interest is the rate at which the government's budget balances. In a comparative static sense, this condition does not define any interest rate, for the government has a balanced budget in every stationary equilibrium, no matter what the rate of interest. (See equation (2.11).) However, in a dynamic sense, Keynes' condition does define an interest rate. The initial asset holdings of consumers and the initial price level do define an interest rate if the government budget must balance. For if the government were to change the nominal interest rate, it would normally have to run a deficit or surplus during the transition period. The budget imbalance would allow people to accumulate or decumulate enough government debt so that the new interest rate could be sustained with a balanced government budget. (See the discussion following example (2.13).)

In The General Theory, Keynes changed his definition of the natural rate, asserting that there is "a different natural rate of interest for each hypothetical level of employment" (p. 242). He defines "the neutral rate of interest" to be the rate which is consistent with full employment (p. 243). In terms of the models used in this paper, it is hard to see how this condition could define any interest rate. One possibility is to hold fixed investor's expectations as to future prices and interest rates, but it does not seem reasonable to assume that expectations are fixed except in the very short-run.

Remark. It seems adviseable to separate the notions of the natural rate of interest and the natural rate of unemployment, though Friedman (1968) does associate the two. Nothing I have said pertains to discussions of the natural rate of unemployment or the Phillips curve.

Overlapping Generations Models

Many authors have pointed out that there are multiple equilibria in overlapping generations models. Samuelson (1958) himself pointed out that there are two stationary equilibria in the model he studied. Cass and Yaari (1966a,b) and Gale (1973) clarified the nature of these equilibria. Gale also pointed out that there are a continuum of non-stationary equilibria, one for each choice of the initial price. (As I pointed out just before stating theorem (2.3), my definition of equilibrium excludes all but one of these equilibria.) Geanakoplos (1980) pointed out that the multiplicity of equilibria observed by Samuelson and Gale meant that the rate of interest is indeterminate in the consumption loan model. He is the only author I have found who has spoken of the interest rate as indeterminate. He related this indeterminancy to the assertions of Sraffa (1960) that the interest rate is determined by non-market forces. Geanakoplos also asserted that the government may be thought of as determining interest rates. (See Geanakoplos, p. 224.)

Samuelson, Cass and Yaari, Gale, and Geanakoplos did not allow the government to collect taxes or pay subsidies, so that the models they study normally have only finitely many stationary equilibria. This means that the indeterminacy discussed by them is no more serious than the indeterminacy of equilibrium in the usual general equilibrium model. (Equilibrium in the usual model is unique only in special cases, but Debreu (1970) has proved that generically there are only finitely many equilibria.)

Diamond (1965) used a model almost identical to that used above in section 2. He studied the stationary equilibria that result from various levels of government debt. He did a careful comparative static study of the relation between govenment debt and other variables, including the utility level of a typical consumer in stationary equilibrium. Diamond dealt with a whole range of equilibrium interest rates, and mentioned that any interest rate is economically efficient as long as it is at least as great as the rate of population growth. However, Diamond did not speak of the interest rate as indeterminate. He probably believed that the correct rate was the rate of population growth, since he emphasized the utility level of a typical consumer. This emphasis was natural since he was interested in the burden of a national debt. The national debt nearly ceases to be a burden if one adopts the attitude of this paper, which is that all Pareto optimal equilibria are equally acceptable from the point of view of an economist. From this point of view, the national debt could be a burden only if it led to over-accumulation of capital. But national debt might not be a burden in this sense, since the debt could tend to decrease the capital stock. Of course, national debt can be a burden in that the taxes needed to pay interest on it cause dead weight losses.

Phelps and Shell (1969) have extended Diamond's analysis to an overlapping generations model in which saving is a fixed fraction of disposable income.

Starrett (1972) has pointed out that in the consumption loan model there is a stationary allocation associated with each interest rate. He showed that such an allocation is efficient if and only if the interest rate is at least as large as the rate of population growth. He did not describe these allocations as equilibria.

Wallace (1980) proved that there exists a continuum of Pareto optimal, stationary equilibrium, real interest rates in the consumption loan model. (See his proposition 5.) He has no nominal interest payments, so that his equilibria involve steady deflation. But otherwise, his result is the same as theorem (2.9) above. Similar results have been proved by Brock and Scheinkman (1980). (See their propositions 1 and 2.)

In many other papers on overlapping generations models, the interest rate may be seen to be arbitrary. For instance, it is implicitly so in

Balasko and Shell (1981). It is also arbitrary in the model of Bryant and Wallace (1979).

My own contribution to this literature, made in section 2 above, seems to be simply to emphasize the arbitrariness of real interest rates in overlapping generations models. Also, I have found no reference where it is observed that the nominal interest rate may be viewed as the discount rate in a social welfare function, though this fact may be inferred from discussion in Carmichael (1981b).

Models With Bequests

There seems to be a long tradition of papers dealing with the bequest motive. Early works include Yaari (1964,5), Meade (1966), Stiglitz (1969), and Atkinson (1971). Yaari and Atkinson studied the behavior of one consumer. They assumed that the utility for bequests was an arbitrary function, unrelated to the utility for wealth of the heir. Meade and Stiglitz studied growth models with bequests. In their papers, there is no utility for bequests. Bequests are determined by a fixed rule. The equilibrium interest rate depends on the nature of this rule.

The papers of Becker (1974) and Barro (1974) are the earliest I have found which assume that the utility for bequests is the same as the heir's utility for wealth. Barro's model is practically identical to that of section 4 above. He showed that in such a model, changes in government debt do not affect consumers' spending on consumption, provided that bequests are positive. It is not hard to see that the same is true of the models of section 5, provided that the government uses conditional taxes or subsidies which depend on the size of the taxpayer's family. The conditional taxes must be such that the welfare of a consumer does not depend on whether he has children.

Barro's work has been analyzed, criticized, and extended by Feldstein (1976), Buchanan (1976), Drazen (1978), Buiter (1979), Carmichael (1981a), and Stokey (1980). Barro's reply (1976) to Feldstein and Buchanan is also interesting. Buiter gives a fairly thorough study of overlapping generations models, with and without bequests. It is clear in his study that the equilibrium interest rate is arbitrary or can be influenced by government intervention. However, the extent of the arbitrariness is obscured in his paper because he views the golden rule interest rate as the optimum one. Both Buiter and Carmichael allow gifts by children to parents as well as bequests by parents to children. Carmichael studies overlapping generations models with bequests and gifts in a second paper (1981b). Stokey has criticized Barro's work on the grounds that in his model an equilibrium with positive bequests is unstable.

Tobin (1978) has criticized Barro's model, and some of his arguments are the same as ones I have used. For instance, he mentions that the infinite chain from generation to generation would be broken by people with no children or by parents who do not care about their children.

I have seen no work other than my own which discussed at length the Pareto optimality of equilibria in models with bequests. Carmichael (1981b) does have some discussion of the subject.

The Optimal Rate of Investment

Sen (1961) and Marglin (1963) have argued that the social rate of investment should be chosen by the political process. Their argument is based on what Sen calls the isolation paradox. Suppose that each individual has a social welfare function which gives positive weight to the welfare of all his contemporaries and of all people in the future, not just to those who are the individual's descendents. Thus, an individual might be willing to save and sacrifice more for the future if the saving were collective than if made only by the individual. That is, the individual might be willing to be taxed for sake of stimulating growth. Marglin elaborated Sen's argument. Marglin's paper has been discussed and criticized by Tullock (1964) and Lind (1969).

In arguing for public determination of the rate of growth, Sen and Marglin contrasted the optimum rate of growth with that which would be selected by the market mechanism.

My own reaction to the work of Sen and Marglin is that the rate of investment (or the rate of discount) is in the public domain, whether or not people are conscious of this fact or care about future generations. It makes no sense to contrast a politically determined optimum rate of discount with a market rate. Of course, once it is recognized that the discount rate is a public issue, then it is appropriate to take into account the considerations associated with the isolation paradox.

ACKNOWLEDGEMENT

I would like to thank Robert Eisner, Morton Kamien, John Ledyard, Michael Magill, Leonard Mirman and Nancy Stokey for helpful discussions.

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