DISCUSSION PAPER NO. 489

A STOCHASTIC CALCULUS MODEL OF CONTINUOUS TRADING: COMPLETE MARKETS

J. Michael Harrison, Stanford University

Stanley R. Pliska, Northwestern University

July, 1981

Abstract. A paper by the same authors in the 1981 volume of Stochastic Processes and Their Applications presented a general model, based on martingales and stochastic integrals, for the economic problem of investing in a portfolio of securities. In particular, and using the terminology developed therein, that paper stated that every integrable contingent claim is attainable (i.e., the model is complete) if and only if every martingale can be represented as a stochastic integral with respect to the discounted price process. This paper provides a detailed proof of that result as well as the following: The model is complete if and only if there exists a unique martingale measure.

1. Introduction

In our (1981) paper we presented a general stochastic calculus model for the buying and selling of a portfolio of securities. To recapitulate, let $(\Omega, \mathcal{F}, P)$ be a probability space, let $T < \infty$ be a fixed time horizon, and let $\mathbb{F} = \{\mathcal{F}_t;\ 0 \le t \le T\}$ be a filtration satisfying les conditions habituelles with $\mathcal{F}_o$ containing only Ω and the null sets of P and with $\mathcal{F}_T = \mathcal{F}$.

Let $S = \{S_t;\ 0 \le t \le T\}$ be a vector valued stochastic process whose components $S^o, S^1, \ldots, S^K$ are adapted, right continuous with left limits, and strictly positive. Moreover, it is assumed that S^o is a semimartingale with $S^o_o = 1$. Here S^k_t represents the time t value of the kth security. Upon setting $\beta = 1/S^o$, one defines the discounted price process $Z = (Z^1, .., Z^K)$ by setting $Z^k = \beta\, S^k$ for $k = 1, \ldots, K$.

Let $\mathbb{P}$ be the set of probability measures Q on $(\Omega, \mathcal{F})$ that are equivalent to P and such that Z is a (vector) martingale under Q. It is assumed that $\mathbb{P}$ is nonempty, so S and Z are actually semimartingales under P. An arbitrary element $P^* \in \mathbb{P}$ is selected and called the reference measure. Let E^* denote the corresponding expectation operator. The assumption that $\mathbb{P}$ is nonempty is made to rule out arbitrage opportunities that would permit investors to make unreasonable profits.

Let L(Z) denote the set of all vector valued, predictable processes $H = (H^1, .., H^K) = \{H_t;\ 0 \le t \le T\}$ that are integrable with respect to the semimartingale Z (see Jacod (1979, p. 52) for details about L(Z)). An admissible trading strategy is any vector valued, predictable stochastic process $\Phi = (\Phi^o, \Phi^1, .., \Phi^K) = \{\Phi_t;\ 0 \le t \le T\}$ such that

(i) $(\Phi^1, \ldots, \Phi^K) \in L(Z)$,

(ii) $V^*(\Phi) \ge 0$, where $V^*(\Phi) = \beta\, \Phi\, S = \beta \sum_{k=o}^{K} \Phi^k S^k$,

(iii) $V^*(\Phi) = V_o^*(\Phi) + G^*(\Phi)$, where $G^*(\Phi) = \int \Phi\, dZ = \sum_{k=1}^{K} \int \Phi^K dZ^k$, and

(iv) $V^*(\Phi)$ is a martingale under P^*.

Here Φ_t^k represents the number of shares or units of security k held by the investor at time t, $V^*(\Phi)$, the discounted value process, represents the discounted value of the portfolio, and $G^*(\Phi)$, the discounted gains process, represents the discounted net profit or loss due to the transactions by the investor. Thus (ii) says admissible trading strategies cannot permit the value of the portfolio to become negative, (iii) says that all changes in the value of the portfolio are due to the investment rather than due to infusion or withdrawal of funds, and (iv) serves to rule out certain foolish strategies that throw away money. Note that condition (iv) is the only one that might depend on the choice of the reference measure.

A contingent claim X is simply defined as a positive random variable (recall $\mathcal{J} = \mathcal{J}_T$). Such a claim is said to be attainable if there exists an admissible trading strategy Φ such that $V_T^*(\Phi) = \beta_T X$, in which case Φ is said to generate X. A claim X is said to be integrable if $E^*(\beta_T X) < \infty$. The model is said to be complete if every integrable claim is attainable. Contingent claims are useful as models of various financial entities such as stock options, and knowing the model is complete facilitates the computation of the price of a claim. See our (1981) paper for more about this.

The sole purpose of this paper is to rigorously prove and extend an important result in our (1981) paper. The new result is presented as the following

Theorem. The following statements are equivalent:

(a) The model is complete under P^*.

(b) Every martingale M can be represented in the form

$$M = M_o + \int H \, dZ$$

for some $H \in L(Z)$.

(c) $\mathbb{P}$ is a singleton.

We call (b) the representation property. By a martingale, here, we mean the real valued stochastic process $M = \{M_t;\ 0 \le t \le T\}$ satisfies the usual definition of a martingale under the filtration $\mathbb{F}$ and reference measure P^*. Theorem (3.35) and its corollary in our (1981) paper only asserted that (c) $\Rightarrow$ (a) $\Leftrightarrow$ (b), and the proof was only sketched.

2. Proof of the Theorem

The proof that (a) and (b) are equivalent is straightforward.

(a) $\Rightarrow$ (b): Let M be an arbitrary martingale. Since any martingale can be expressed as the difference of two positive martingales, we shall assume, without loss of generality, that M is positive. Setting $X = S_T^o M_T$, we know there exists an admissible trading strategy Φ such that $V_T^*(\Phi) = M_T$. Moreover, the martingale $V^*(\Phi) = V_o^*(\Phi) + \int H \, dZ$ by our definition of admissible trading strategies, where $H = (\Phi^1, \ldots, \Phi^K)$. Thus M has the same representation, because $M_t = E^*(\beta_T X | \mathcal{F}_t) = V_t^*(\Phi)$.

(b) $\Rightarrow$ (a): Let X be an arbitrary integrable contingent claim. Define a martingale M by setting $M_t = E^*(\beta_T X | \mathcal{F}_t)$, and let $H \in L(Z)$ be such that $M = M_o + \int H \, dZ$. Set $\Phi^1 = H^1, \ldots, \Phi^K = H^K$, while for Φ^o put $\Phi^o = M_o + \int H \, dZ - HZ$. This yields an admissible trading strategy Φ with $V^*(\Phi) = M$. Thus $V_T^*(\Phi) = \beta_T X$, X is attainable, and the model is complete.

The proof that (b) and (c) are equivalent is more involved, for it relies on a theory (see Jacod (1979, Ch. XI)) relating the representation property to a condition involving a certain set of probability measures.

Let $M(Z)$ denote the set of all probability measures on $(\Omega, \mathcal{F})$ such that Z is a local martingale under each $Q \in M(Z)$, and note $\mathbb{P} \subset M(Z)$. An element Q of $M(Z)$ is said to be an extreme point if it cannot be expressed as a strictly convex combination of two distinct elements of $M(Z)$. Let $M_e(Z)$ denote all the extreme points of $M(Z)$.

According to Theorem (11.2) in Jacod (1979), $Q \in M_e(Z)$ if and only if Z can represent every H^1 martingale (under Q). By localization, this means $Q \in M_e(Z)$ if and only if Z can represent every local martingale (under Q). Consequently, $P^* \in M_e(Z)$ if and only if the representation property (b) holds. We shall use this important result to show that (b) and (c) are equivalent.

(b) $\Rightarrow$ (c): This now follows immediately from Corollary (11.4) in Jacod (1979), which says (see also condition (iv) in Theorem (11.3)) that if $P^* \in M_e(Z)$, then there cannot exist another $Q \in M(Z)$ with Q equivalent to P^* (in particular, with $Q \in \mathbb{P}$).

(c) $\Rightarrow$ (b): It suffices to show $P^* \in M_e(Z)$. Suppose not. Then there exists some $\alpha \in (0,1)$ and $Q', Q'' \in M(Z)$ such that $P^* = \alpha Q' + (1-\alpha) Q''$. Following the idea in the proof of Proposition (11.14) in Jacod (1979), because $Q' \leq P^*/a$ one can show Z is a martingale under Q', and similarly for Q''. Thus Z is a martingale under $Q_\beta \equiv \beta Q' + (1-\beta) Q''$ for every $\beta \in (0,1)$. Since Q_β is equivalent to P^* for all $\beta \in (0,1)$, this means $Q_\beta \in \mathbb{P}$ for all $\beta \in (0,1)$. But this contradicts the fact that $\mathbb{P}$ is a singleton.

3. Concluding Remarks

The presentation of Theorem (3.35) in our (1981) paper was followed by a brief discussion of cases when the martingale representation property (b) holds. Here we shall make some supplementary comments.

The martingale representation property holds for any diffusion process that's a martingale and for which the Stroock-Varadhan problem (see page 4 of their 1979 book) has a unique solution (e.g., if the diffusion coefficients are Lipschitz). This follows from Yamada and Watanabe (1971).

The martingale representation property is also satisfied by a diffusion process that's a martingale if the diffusion matrix is non-degenerate and the coefficients are continuous. This was mentioned in Jacod and Yor (1977) and in Yor (1977).

In the appendix of Yor (1978), written with J. de Sam Lazaro, it is shown that the only one-dimensional martingales that have stationary increments and satisfy the representation property are the Wiener and Poisson martingales. In their proof they did not assume the increments are independent, although this turns out to be implied by the other conditions.

Acknowledgments

We are grateful to M. Yor for providing the information in Section 3 and to the anonymous referee of our (1981) paper for prompting us to scrutinize our "$\mathbb{P}$ is a singleton" condition.

References

J. M. Harrison and S. R. Pliska (1981), "Martingales and Stochastic Integrals in the Theory of Continuous Trading," Stochastic Processes and Their Applications, Vol. 11, No. 3, pp. 215-260.

J. Jacod (1979), Calcul Stochastique et Problemes de Martingales, Notes in Math, 714, Springer-Verlag, New York.

J. Jacod and M. Yor (1977), "Etudes des Solutions Extremales et Representation Integrale des Solutions pour Certains Problemes de Martingales, Z. Wahr., Vol. 38, pp. 83-125.

D. W. Stroock and S. R. S. Varadhan (1979), Multidimensional Diffusion Processes, Springer-Verlag, New York.

T. Yamada and S. Watanabe (1971), "On the Uniqueness of Solutions of Stochastic Differential Equations," J. Math. Kyoto Univ., Vol. 11, No. 1.

M. Yor (1977), "Remarques sur la Representation des Martingales comme intégrales stochastiques," Sem. de Prob. XI, Lecture Notes in Math. 581, pp. 502-517, Springer-Verlag, New York.

M. Yor (1978), "Sous-Espaces Denses dans L^1 ou H^1 et Representation des Martingales," Sem. de Prob. XII, Lecture Notes in Math 649, pp. 265-309, Springer-Verlag, New York.