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SELLING TO RISK AVERSE BUYERS WITH UNOBSERVABLE TASTES

by

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ABSTRACT

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Schemes maximizing the expected profit from selling indivisible units to risk averse, heterogeneous buyers are studied. For the polar cases of an unlimited and a unit supply, optimal schemes are constructed for buyers who exhibit constant risk aversion and who differ only in their monetary evaluations. With a unit supply, the optimal scheme resembles a first-price auction in which buyers are paid to submit high bids, but must pay to submit low bids. The results are interpreted in terms of insurance provision and the use of risk to weaken incentive compatability constraints. Journal of Economic Theory, , (English). Northwestern University, Evanston, Illinois.

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1. Introduction

In order to practice perfect price discrimination, a monopoly must know the maximum price each customer is willing to pay. However, in many environments customers cannot be distinguished on the basis of their preferences. The monopoly's task is then to practice so-called second degree price discrimination, i.e., to construct a choice setting in which customers willing to pay more are induced, by their own decisions, to pay greater prices.

The first such environments to be studied were ones in which the empirical distribution of buyers' tastes is known. Studies of monopoly in these environments, such as Spence [23], Goldman, Leland and Sibley [3], Mussa and Rosen [16], Roberts [21], and Mirman and Sibley [14], characterized profit-maximizing selling schemes within the class of schemes in which each buyer is presented the same nonuniform price schedule. This class of selling schemes is not the largest, e.g., it excludes schemes that result in random allocations. However, it can be shown that optimal nonuniform price schedules are optimal within any class of selling schemes if buyers are risk neutral.

A second environment studied more recently is one in which the monopoly does not know the empirical distribution of tastes, but instead has only probabilistic assessments. If buyers are risk neutral and the monopoly has constant costs, then again it is optimal to present each buyer with a nonuniform price schedule. But if the cost function is nonlinear, then average profit will depend on the total quantity purchased and hence be uncertain. In this case it may be better to use a selling scheme in which each buyer's allocation depends jointly on the decisions of all buyers.

The polar example of such an environment is one in which there is a binding quantity constraint. It is then infeasible to set a nontrivial price schedule, since there is positive probability that independently made purchase orders will exceed the capacity constraint. Consequently, the class of selling schemes must be enlarged in these environments to include auction-like mechanisms.

Optimal auctions have been studied by Harris and Raviv [4,5], Myerson [18], Riley and Samuelson [19], and Maskin and Riley [9,10]. These authors study an environment in which (a) the good is indivisible; (b) no buyer wants more than one unit; (c) buyers exhibit no income effects, i.e., each buyer's (unobservable) willingness-to-pay for a unit is independent of his income; and (d) buyers are risk neutral. Then, given that buyers' characteristics are independently distributed and that only one unit can be sold, the schemes that maximize expected profit are characterized. It turns out that the set of optimal schemes is quite large, containing, e.g., under a regularity condition, both first-price and second-price auctions with reserve prices.[1,2]

The reason there are so many optimal auctions in this environment is that auctions that differ only in their payment functions, but that have the same expected payment functions, must be equivalent when all agents are risk neutral. This equivalency result cannot hold if buyers are risk averse. The lack of equivalency when buyers are risk averse is suggested by the well-known result that, in this case, first-price auctions generate greater expected profit than second-price auctions [4,8,9,11,12,13,19,25].

In this paper, expected profit maximizing schemes are explicitly characterized for environments in which assumptions (a)-(c) are kept, but (d) is generalized to allow for risk averse buyers. In particular, buyers shall usually be assumed to have a constant measure of absolute risk aversion.

Major consequences follow from this apparently minor generalization. First, when there is no quantity constraint, take-it-or-leave-it offer schemes, which are the optimal "nonuniform" price schedules in this environment, are not optimal in a broader class of selling schemes. Instead, the optimal scheme gives some buyers only a probability of obtaining a unit. The reason for this is that when buyers are risk averse, the expected cost of inducing them to reveal their evaluations can be decreased by allowing them to avoid, and therefore sometimes to bear, risk.[3]

The consequences of risk aversion are more startling when there is a unit quantity constraint. Then, even a small amount of buyer risk aversion results in the class of optimal auctions shrinking essentially to a unique auction. This auction does not resemble any standard auction. It can be described, for the case in which buyers are approximately risk neutral, as a first-price auction in which buyers must pay to submit low bids, but are paid to submit high bids. Buyers with high values are induced to submit bids nearly equal to their values.

Independently, Maskin and Riley [11] have used a similar approach to study optimal auctions with risk averse buyers. They allow more general utility functions, but at the cost of not obtaining conditions that are both necessary and sufficient for an auction to be optimal. Their results are in accord with most of the qualitative properties of the optimal selling schemes found here.

The paper is organized as follows. The environment and a very broad class of selling schemes are defined in Section 2. The assumption of constant absolute risk aversion is introduced in Section 3. The seller's maximization problem is formulated in Section 4. Its solution is presented in Section 5 for the case of no capacity constraint, and in Section 6 for the case of a unit capacity constraint. Summarizing remarks are contained in Section 7, and all proofs are contained in the Appendix.

2. Selling Schemes

The environment contains n buyers and one seller of a good. The following assumptions are made: (i) the good is discrete; (ii) the seller is risk neutral and bears an opportunity cost c_0 for each unit sold; (iii) no buyer wants more than one unit; (iv) the buyers exhibit no income effects, i.e., a buyer with evaluation θ is willing to pay θ dollars for one unit regardless of his income; (v) the buyers' evaluations are realizations of independent random variables whose distribution is common knowledge; (vi) each buyer's evaluation is observed only by himself; and (vii) all buyers have the same, increasing and concave expected utility function u for money. Consequently, a buyer with evaluaton θ, income I, and one unit of the good obtains utility $u(I + \theta)$. After this section, u will be assumed to exhibit constant absolute risk aversion.

A further technical assumption is that the buyers' evaluations are identically distributed according to a cumulative distribution F. It is assumed that F has a continuous, positive density f on the support [0,1]. The seller's unit cost is assumed to satisfy $0 \leq c_0 < 1$.

A <u>selling scheme</u> fundamentally consists of a message set M_i for each buyer, and an outcome function that maps an n-tuple of messages into a (perhaps random) allocation $\tilde{a}$. The behavior of buyers is described by a <u>Nash(Bayesian) equilibrium</u>, which is an n-tuple of functions $(m_1^*,\ldots,m_m^*)$. Each m_i^* maps possible evaluations for buyer i into the message set M_i. The equilibrium property of these functions consists of $m_i^*(\theta_i)$ being the message that maximizes the expected utility of buyer i when he has the evaluation θ_i and every other buyer j uses m_j^* to submit messages.

This notion of a selling scheme is very general, incorporating even iterative schemes when the message sets are defined broadly enough. Such generality is not required here, since any selling strategy, in any equilibrium, results in the same outcomes as a more tractable revelation selling scheme. The argument is the following: any scheme will, in a given equilibrium, result in an outcome $\tilde{a}(\theta_1,\ldots,\theta_n)$ when the buyers' evaluations are $\theta_1,\ldots,\theta_n$. The function $\tilde{a}(\theta_1,\ldots,\theta_n)$ can serve as the outcome function of another selling scheme in which each buyer's message set is the set of possible evaluations. Because the function $\tilde{a}(\theta_1,\ldots,\theta_n)$ is obtained as the composition of a selling scheme with one of its equilibria, the n-tuple of identity functions $(m_i^*(\theta_i) = \theta_i)$ is an equilibrium of the new scheme. We have therefore constructed a revelation scheme in which truth-telling is an equilibrium that results in the same outcomes as did the given equilibrium in the original selling scheme.[4]

Henceforth, attention will be restricted to revelation schemes that are incentive compatible, i.e., schemes in which the n-tuple of identity functions is an equilibrium. The preceeding argument indicates that this involves little loss of generality. However, having the feasible class of selling schemes depend upon utility functions does obscure comparison and implementation issues. Once optimal revelation schemes have been found, some attention will be given to the question of their implementation via equivalent, non-revelation schemes.

A revelation scheme must, most generally, specify a joint probability distribution for the buyers' allocations as a function of reported evaluations. Denoting $\tilde{t}_i$ as the (possibly random) payment of buyer i, and $\tilde{x}_i \in \{0,1\}$ as the (possibly random) amount of the good received by buyer i, an outcome is a joint probability distribution $G(x_1,t_1,\ldots,x_n,t_n \mid \theta_1,\ldots,\theta_n)$. Any revelation scheme can be identified with such a function G.

Two properties of a scheme G are of interest. Implementation is simplified and strategic interactions eliminated if each buyer's allocation is independent of the other buyers' actions. This occurs if G is <u>separable</u> in the sense of being of the form $\prod_{i=1}^{n} G_i(x_i, t_i \mid \theta_i)$. For example, a scheme in which each buyer is presented a price-quantity schedule is separable, whereas a first price auction is not.

The second property of a scheme G that is of particular interest here is whether, given an n-tuple of reported evaluations, it places all probability on just one allocation $(x_1, t_1, \ldots, x_n, t_n)$. If it does, then G is <u>deterministic</u>. Offering price-quantity schedules is a deterministic scheme. So is a first price auction, if the coin flip that resolves ties is ignored.

Buyer i is only concerned with a few marginal distributions of a scheme G. Suppose that the scheme is incentive compatible, so that the expectations of buyer i over other buyers' reports can be taken via the distribution F. Then buyer i's expectations over the random elements in his environment are well-defined by the two distributions F and G, and his report $\hat{\theta}_i$. When buyer i has the true evaluation θ_i but reports $\hat{\theta}_i$, his expected utility is

(1) $$E[u(\theta_i \tilde{x}_i - \tilde{t}_i) \mid \hat{\theta}_i] = \Pr(\tilde{x}_i = 1 \mid \hat{\theta}_i) E[u(\theta_i - \tilde{t}_i) \mid \hat{\theta}_i, \tilde{x}_i = 1]$$
$$+ \Pr(\tilde{x}_i = 0 \mid \hat{\theta}_i) E[u(-\tilde{t}_i) \mid \hat{\theta}_i, \tilde{x}_i = 0].$$

Using G and F, define

(2) $$Q_i(\hat{\theta}_i) = \Pr(\tilde{x}_i = 1 \mid \hat{\theta}_i),$$

(3) $$H_i(p_{iw} \mid \hat{\theta}_i) = \Pr(\tilde{t}_i \leq p_{iw} \mid \hat{\theta}_i, \tilde{x}_i = 1),$$

(4) $$J_i(p_{i\ell} \mid \hat{\theta}_i) = \Pr(\tilde{t}_i \leq p_{i\ell} \mid \hat{\theta}_i, \tilde{x}_i = 0).$$

The probability $Q_i(\hat{\theta}_i)$ is buyer i's perceived probability of obtaining a unit. He perceives his random payment $\tilde{p}_{iw}$ contingent upon obtaining (winning) a unit to be distributed according to H_i, and his random payment $\tilde{p}_{i\ell}$ contingent upon not obtaining (losing) a unit to be distributed according to J_i. Expression (1) indicates that buyer i is indifferent between any two incentive compatible schemes that result in the same Q_i, H_i and J_i functions.

The seller's expected profits are

$$(5) \qquad E\{\sum_{i=1}^{n} (\tilde{t}_i - c_o\tilde{x}_i)\} = \sum_{i=1}^{n} E\{Q_i(\tilde{\theta}_i)E(\tilde{p}_{iw}-c_o \mid \tilde{\theta}_i) + (1-Q_i(\tilde{\theta}_i))E(\tilde{p}_{i\ell} \mid \tilde{\theta}_i)\} .$$

Therefore the seller too is indifferent between any two incentive compatible schemes that result in the same Q_i, H_i and J_i functions.

If there is unlimited capacity, then the problem will be to find a scheme that maximizes (5) subject to the incentive compatibility constraint that $\hat{\theta}_i = \theta_i$ maximizes (1), and subject to the constraint that participation be voluntary, i.e., that expression (1) be no less than u(0) at $\hat{\theta}_i = \theta_i$. By the above arguments, therefore, attention can be restricted to finding the marginal distributions Q_i, H_i and J_i. Once these functions are found, then the simplest scheme consistent with them is the separable one that has buyer i's allocation independent of the reports of other buyers.

This scheme has $Q_i(\theta_i)$ as the actual probability of buyer i obtaining a unit, $H_i(p_{iw} \mid \theta_i)$ as the actual distribution of his payment when he obtains a unit, and $J_i(p_{i\ell} \mid \theta_i)$ as the actual distribution when he does not obtain a unit.

A separable scheme will not suffice if fewer than n units can be sold. This is because in any (nontrivial) separable scheme, there is positive probability that each buyer will obtain a unit. Letting

$$q_i(\theta_1,\ldots,\theta_n) = \Pr(\tilde{x}_i = 1 \mid \theta_1,\ldots,\theta_n),$$

then

$$(6) \qquad 0 \le \sum_{i=1}^{n} q_i(\theta_1,\ldots,\theta_n) \le 1$$

expresses the capacity constraint of having no more than one unit to sell. With this additional constraint, the problem is to find the functions q_i, J_i and H_i that maximize (5), where Q_i and q_i are related by

$$(7) \qquad Q_i(\theta_i) = \int_0^1 \cdots \int_0^1 q_i(\theta_1,\ldots,\theta_n) \prod_{j\neq i} f(\theta_j)d\theta_j \ .$$

Once these functions are found, the simplest scheme consistent with them has the winning buyer determined as a function of all reports via the probabilities $q_i(\theta_1,\ldots,\theta_n)$, but the payments conditional upon who obtains the unit still determined independently via the distributions J_i and H_i.

Because all buyers are <u>ex ante</u> identical, only schemes that treat them symmetrically need be considered. Henceforth, the subscript "i" is deleted from the functions Q_i, H_i and J_i, and the three-tuple $\langle Q, H, J \rangle$ will be referred to as a "scheme." If there is no capacity constraint, then $\langle Q, H, J \rangle$ is feasible if it satisfies the incentive compatibility and voluntary participation constraints. If there is a unit capacity constraint, then feasibility also requires that Q be implementable, i.e., that functions q_i exist such that Q and each q_i are related via (7).

The following lemma, proved in Maskin and Riley [9,11], reduces the capacity constraint to a more tractable form. In words, it states that a given Q is implementable only if it implies that the probability of selling the unit to a buyer with an evaluation greater than θ is no greater than the probability that a buyer exists with an evaluation greater than θ.

Lemma 1: Suppose that $Q:[0,1] \to [0,1]$ is a function for which there exist functions $q_i[0,1]^n \to [0,1]$ $(i = 1,\ldots,n)$ satisfying (6) and (7). Then, for every θ,

$$(8) \qquad Y(\theta) = \int_\theta^1 [F^{n-1}(z) - Q(z)]f(z)dz \geq 0.$$

To incorporate the capacity constraint, (8) will be included as a constraint in the problem of finding a scheme $\langle Q, H, J \rangle$ to maximize expected profits. Once this scheme is found, it will be verified that its Q is indeed implementable, i.e., that the necessary q_i functions exist.

Regardless of the presence of a capacity constraint, one property of an optimal scheme can be obtained at the current level of generality. The following lemma states that attention can be restricted to schemes in which the payment contingent upon not obtaining a unit, $\tilde{p}_\ell$, is deterministic.

<u>Lemma 2</u>: Suppose $\langle Q, H, J \rangle$ is a feasible scheme. Then a scheme $\langle Q, H, \bar{J} \rangle$, where $\bar{J}(p_\ell \mid \theta)$ puts all probability on a point $p_\ell(\theta)$, can be constructed that is both feasible and yields no less expected profit than $\langle Q, H, J \rangle$.

The proof of Lemma 2 relies upon replacing $\tilde{p}_\ell$ by its certainty equivalent. But the proof is not standard, since incentive compatibility must be shown to still hold. It is generally not possible to replace $\tilde{p}_w$ by a deterministic $p_w(\theta)$ that both increases expected profit and maintains incentive compatibility.

Because of Lemma 2, a three-tuple $\langle Q, H, p_\ell \rangle$, where p_ℓ is a real-valued function of evaluations, can be referred to as a scheme.

3. Constant Absolute Risk Aversion

Henceforth it is assumed that u exhibits constant absolute risk aversion R. In particular, the symbol "u" shall denote the function

$$u(x) = (1 - e^{-Rx})/R.$$

There are two reasons why this functional form makes the problem tractable. The first is that the multiplicative separability of u implies that $\tilde{p}_w$ can be made deterministic without destroying incentive compatibility.

Lemma 3: Suppose $\langle Q, H, p_\ell \rangle$ is feasible. Then a scheme $\langle Q, \bar{H}, p_\ell \rangle$, where $\bar{H}(p_w \mid \theta)$ puts all probability on a point $p_w(\theta)$, can be constructed that is both feasible and yields no less expected profit than $\langle Q, H, p_\ell \rangle$.

Attention can now be restricted to schemes with deterministic payment functions p_w and p_ℓ. However, rather than using the function p_w, it will be more convenient to use the bid function b defined by

$$b(\theta) = p_w(\theta) - p_\ell(\theta) .$$

The function p_ℓ will be denoted p and called the bid submission or entry fee function. A scheme shall be denoted $\langle Q, b, p \rangle$. The interpretation now is that a buyer must pay the submission fee $p(\hat{\theta})$ in order to submit a bid $b(\hat{\theta})$. Submitting this bid gives the buyer, from his point of view, a probability $Q(\hat{\theta})$ of obtaining a unit. In the event that he obtains a unit, he must additionally pay the amount that he bid, $b(\hat{\theta})$.

Given a scheme $\langle Q, b, p \rangle$, a buyer with evaluation θ who reports $\hat{\theta}$ is paying $p(\hat{\theta})$ to obtain the lottery that gives him $\theta - b(\hat{\theta})$ dollars with probability $Q(\hat{\theta})$, and zero dollars with probability $1 - Q(\hat{\theta})$. The second simplification due to assuming constant absolute risk aversion is that the buyer's evaluation of the lottery is independent of the entry fee. Specifically, the buyer's expected utility function,

$$\hat{U}(\hat{\theta},\theta) = Q(\hat{\theta})u(\theta - p(\hat{\theta}) - b(\hat{\theta})) + (1 - Q(\hat{\theta}))u(-p(\hat{\theta})),$$

is equivalent to the additively separable function

$$U(\hat{\theta},\theta) = \psi(\theta - b(\hat{\theta}), Q(\hat{\theta})) - p(\hat{\theta}), \tag{9}$$

where ψ is defined by

$$\psi(y,Q) = -\left(\frac{1}{R}\right) \log [1 - Q + Qu'(y)]. \tag{10}$$

The amount $\psi(y,Q)$ is just the certainty equivalent of a lottery offering y dollars with probability Q and zero dollars with probability $1 - Q$; it is equal to the expected value Qy minus a risk premium $\pi(y,Q)$.

The incentive compatibility constraint can now be made tractable by a technique based roughly upon that of Mirrlees [15]. Given a revelation scheme $\langle Q, b, p \rangle$, which then defines $U(\hat{\theta},\theta)$ by (9), define

$$V(\theta) = U(\theta,\theta). \tag{11}$$

If the scheme is incentive compatible, then $V(\theta)$ is the indirect utility function of a buyer with evaluation θ. Incentive compatibility is satisfied if and only if $V(\theta) \geq U(\hat{\theta},\theta)$ for all possible θ and $\hat{\theta}$.

Lemma 4: A scheme $\langle Q, b, p \rangle$ is incentive compatible if and only if

(12) $\dfrac{Q(\theta)e^{Rb(\theta)}}{1 - Q(\theta)}$ is nondecreasing in θ, and

(13) $V(\theta) = V(0) + \int_0^\theta \psi_1(z - b(z), Q(z))dz.$

4. The Seller's Problem

If the seller uses a feasible scheme $\langle Q, b, p \rangle$, then the expected profit obtained from a buyer with evaluation θ is

$$(14) \qquad p(\theta) + Q(\theta)(b(\theta) - c_o)$$

$$= \psi(\theta - b(\theta), Q(\theta)) - V(\theta) + Q(\theta)(b(\theta) - c_o).$$

Therefore the seller's expected profit is

$$(15) \qquad n \int_0^1 [\psi(\theta - b(\theta), Q(\theta)) - V(\theta) + Q(\theta)(b(\theta) - c_o)] f(\theta) d\theta .$$

The problem of the seller is to maximize (15) by choosing control functions Q and b. The constraints are those imposed by feasibility. The incentive compatibility constraint is embodied by (12) and by the equation of motion for the state variable V,

$$(16) \qquad V'(\theta) = \psi_1(\theta - b(\theta), Q(\theta)) \quad \text{with} \quad V(0) \geq 0 .$$

Expression (16), which is derived from (13), also incorporates the voluntary participation constraint. Implementability requires at least the constraint

$$(17) \qquad 0 \leq Q(\theta) \leq 1.$$

Implementability further requires, in the case of a unit capacity constraint, that nonnegative functions q_i exist satisfying (6) and (7).

The problem is solved in the following way. Expression (15) shall be maximized subject to the constraints (16), (17), and, in the case of a unit capacity constraint,

(18) $\quad Y(\theta) \geq 0,$

where Y is a state variable with an equation of motion

(19) $\quad Y'(\theta) = f(\theta)[Q(\theta) - F(\theta)^{n-1}]$ with $Y(1) = 0.$

The constraints (18) and (19) are necessary, by Lemma 1, for Q to be implementable in the case of a unit capacity constraint. The solution $\langle Q,b \rangle$, when adjoined to an entry fee function defined by

(20) $\quad p(\theta) = \psi(\theta - b(\theta), Q(\theta)) - V(\theta),$

constitutes a selling scheme $\langle Q,b,p \rangle$. This scheme must then be shown to be feasible by, first, showing that it satisfies the ignored constraint (12). Secondly, in the case of a unit capacity constraint, functions q_i that implement Q must be shown to exist.

Both tasks are facilitated by imposing the following regularity condition, which is satisfied if the density does not decrease rapidly. It is equivalent to the one used in Myerson [18] and in Maskin and Riley [9].

(RC) For any R in an interval $[0,\overline{R}]$, the function

$$\alpha(\theta,c) = u(\theta - c) + \frac{F(\theta) - 1}{f(\theta)}$$

is increasing in θ at any $(\theta,c) \in [0,1]^2$.

Henceforth, (RC) is imposed and $0 < R \leq \overline{R}$ is assumed. For later use, we remark now that (RC) implies that the function

$$\beta(\theta,c) = \frac{u(\theta - c)}{u'(\theta - c)} + \frac{F(\theta) - 1}{f(\theta)}$$

is also increasing in θ whenever $\theta > c$. Both α and β are negative if $\theta < c$, and $\alpha(\theta,c) \leq \beta(\theta,c)$ holds for all (θ,c), strictly if $\theta \neq c$. Consequently, two increasing functions on the unit interval are defined by $\alpha(\overline{\theta}(c),c) = 0$ and $\beta(\underline{\theta}(c),c) = 0$, with $c < \underline{\theta}(c) < \overline{\theta}(c) < 1$ if $0 \leq c < 1$.

The first step in maximizing (15) subject to (16) - (19) is to define the Hamiltonian

$$\mathcal{H}(\theta,V,Y,Q,b,\lambda,\mu)$$

$$= [\psi(\theta - b,Q) - V + (b - c_o)Q]f(\theta) + \lambda\psi_1(\theta - b,Q) + \mu f(\theta)[Q - F(\theta)^{n-1}],$$

and the Lagrangian $L = \mathcal{H} + \eta Y$. Notice that when Q and b are chosen to maximize $\mathcal{H}$ for fixed (θ,V,Y,λ,μ), the maximized Hamiltonian is concave in V and Y. Therefore the following conditions, in conjunction with (16) - (19), are sufficient as well as necessary for a solution.[5]

(21) $\quad \lambda'(\theta) = -\dfrac{\partial L}{\partial V} = f(\theta), \quad \lambda(1) = 0, \quad \lambda$ continuous;

(22) $\quad \mu'(\theta) = -\dfrac{\partial L}{\partial Y} = -\eta(\theta). \quad \mu(0) \leq 0, \quad \mu(0)Y(0) = 0;$

(23) if μ is discontinuous at θ, then θ is an entry or an exit point of an interval upon which $Y = 0$, and $\mu(\theta^-) > \mu(\theta^+)$;

(24) $\quad \eta(\theta) \geq 0$ and $\eta(\theta)Y(\theta) = 0;$

(25) $\quad \mathcal{H}(\theta,V(\theta),Y(\theta),Q(\theta),b(\theta),\lambda(\theta),\mu(\theta))$ is continuous;

$$(26') \qquad (Q(\theta),b(\theta)) \in \underset{\substack{-\infty<b<\infty \\ 0\le Q\le 1}}{\operatorname{argmax}} \mathscr{H}(\theta,V(\theta),Y(\theta),Q,b,\lambda(\theta),\mu(\theta)).$$

Condition (21) immediately implies $\lambda(\theta) = F(\theta) - 1$. This makes sense. The amount $-\lambda(\theta)$ is the implicit cost, due to the incentive compatibility constraint (16), of increasing by one dollar the dollar-equivalent utility level $V(\theta)$ of a buyer with evaluation θ. The incentive compatibility constraint requires that if one more dollar is to be given to a buyer with evaluation θ, then an extra dollar must also be given to the buyer if his evaluation is greater than θ. Hence, the additional expected cost of increasing $V(\theta)$ by one dollar is $1 - F(\theta) = -\lambda(\theta)$.

Substituting $1-F$ for λ, and letting $c = c_o - \mu$, and dividing by $f(\theta)$, the important part of the Hamiltonian becomes

$$(27) \qquad \overline{\mathscr{H}}(\theta,Q,b,c) = \psi(\theta-b,Q) + (b-c)Q - [\frac{1-F(\theta)}{f(\theta)}]\psi_1(\theta-b,Q).$$

The cost $c(\theta) = c_o - \mu(\theta)$ is the true opportunity cost of selling a unit to a buyer with evaluation θ; it is the sum of the seller's personal opportunity cost c_o and the implicit cost $-\mu(\theta)$ of not being able to also sell a unit to another buyer.

Condition (26') can be replaced now by

$$(26) \qquad (Q(\theta),b(\theta)) \in \underset{\substack{-\infty<b<\infty \\ 0\le Q\le 1}}{\operatorname{argmax}} \overline{\mathscr{H}}(\theta,Q,b,c(\theta)).$$

The following lemma states that this maximization is virtually unique if buyers are strictly risk averse.

<u>Lemma 5</u>: If $0 \le c \le 1$, then $\bar{\mathcal{V}}(\theta,Q,b,c)$ is maximized for $-\infty < b < \infty$ and $0 \le Q \le 1$ by $\bar{b}(\theta,c) = c$ and

$$(28) \qquad \bar{Q}(\theta,c) = \begin{cases} 0 & \theta < \underline{\theta}(c) \\ \dfrac{\beta(\theta,c)}{\beta(\theta,c) - \alpha(\theta,c)} & \underline{\theta}(c) \le \theta < \bar{\theta}(c) \\ 1 & \theta \ge \bar{\theta}(c). \end{cases}$$

$Q = \bar{Q}(\cdot, c)$ is the unique maximizing Q and, on $(\underline{\theta}(c), \bar{\theta}(c))$, $b = c$ is the unique maximizing b.[6] The function $\bar{Q}$ increases in θ and decreases in c if $\theta \varepsilon (\underline{\theta}(c), \bar{\theta}(c))$.

5. Solution With Unlimited Capacity

By Lemma 5, the optimal scheme is determined once $\mu(\theta)$ is determined. But $\mu(\theta)$, the implicit cost due to a binding capacity constraint, is zero when there is no quantity constraint. Consequently, the seller's optimal scheme in this case has been found.

Theorem 1: With no quantity constraint, an optimal scheme is $Q^*(\theta) = \overline{Q}(\theta, c_o)$, $b^*(\theta) = c_o$, and

$$(29) \qquad p^*(\theta) = \psi(\theta - c_o, Q^*(\theta)) - \int_0^\theta \psi_1(z - c_o, Q^*(z))dz \;.$$

If $\langle Q, b, p \rangle$ is also optimal, then almost everywhere $Q = Q^*$, $p(\theta) = p^*(\theta)$ for $\theta \leq \overline{\theta}(c_o)$, and $b(\theta) = b^*(\theta)$ for $\theta \in (\underline{\theta}(c_o), \overline{\theta}(c_o))$. If $\theta \geq \overline{\theta}(c_o)$, then $b(\theta) + p(\theta) = c_o + p^*(\overline{\theta}(c_o))$.

The scheme described in Theorem 1 could be implemented directly as a separable, nondeterministic revelation scheme. Alternatively, each buyer could be offered the increasing payment-probability schedule

$$(30) \qquad P^*(Q) = p^*(Q^{*-1}(Q))$$

from which to choose a probability Q of obtaining a unit. A buyer thus purchases a probability Q for an amount $P^*(Q)$. Q can be viewed as "reliability," or as the likelihood that the seller will have the good in stock. Upon paying for a Q, a buyer enters the store and finds a unit with probability Q. If he finds a unit, he purchases it at cost c_o.

Theorem 1 implies that an optimal scheme for risk averse buyers must subject them to risk. Hence, it must yield greater expected profit than the particular deterministic take-it-or-leave-it-offer scheme that is well-known (and will be shown) to be optimal when buyers are risk neutral. Consequently, since the deterministic scheme yields the same expected profit regardless of how risk averse buyers are, greater expected profit can be made when the buyers are risk averse.

The reason for this is that risk aviodance weakens the incentive compatability constraint. In particular, as the following argument indicates, greater payments can be extracted from buyers with high values if buyers with low values are subjected to risk. Suppose there are only two buyers, with evaluations $\theta_1 < \theta_2$. Two schemes are to be constructed in which the expected payment of buyer 1 is the same $x > 0$. Scheme A imposes no risk on buyer 1, so that it simply involves selling a unit for the price x to any buyer claiming to have value θ_1. Since buyer 2 can obtain this allocation as well, x is the largest amount that buyer 2 can be induced to pay in Scheme A. In Scheme B, on the other hand, buyer 1 receives the risky allocation (Q_1,b_1,p_1), with $0 < Q_1 < 1$ and $x = p_1 + Q_1b_1$. By mimicking buyer 1 in Scheme B, buyer 2 receives the certainty equivalent $\psi(\theta_2-b_1,Q_1) - p_1$, which is less than the expected value $Q_1\theta_2 - x$ because of risk aversion. Therefore, if y is the largest price that buyer 2 can be induced to pay in Scheme B, then y satisfies $\theta_2-y = \psi(\theta_2-b_1,Q_1) - p_1 < Q_1\theta_2 - x$. This implies $y > x$, so that the risky Scheme B extracts a greater payment from buyer 2 than does the deterministic Scheme A.

The next result emphasizes the usefulness of risk aversion: as buyers become extremely risk averse, expected profit approaches that which could be obtained by perfect price discrimination.

Corollary 1: Assume $\bar{R} = \infty$. Then as $R \to \infty$, the optimal scheme in the case of unlimited capacity converges to the scheme $\langle Q^{\infty}, b^{\infty}, p^{\infty} \rangle$ defined by $b^{\infty}(\theta) = c_o$,

$$p^{\infty}(t) = \begin{cases} 0 & \theta \le c_o \\ \theta - c_o & \theta > c_o \end{cases} \qquad Q^{\infty}(\theta) = \begin{cases} 0 & \theta \le c_o \\ 1 & \theta > c_o \end{cases} .$$

Furthermore, the expected profit converges to the maximum that is possible when information is perfect, $n\int_{c_o}^{1}(\theta - c_o) f(\theta)d\theta$.

Returning to Theorem 1, there is a more direct and perhaps more insightful way of deriving it, based on methods used in some of the nonuniform pricing literature [3,14,16,23]. The first step is to show that if Q^*, b^* and p^* are numbers that maximize profit $p + Q(b - c_o)$ subject to constraints

$$\psi(\theta - b, Q) - p = V \quad \text{and} \quad \psi_1(\theta - b, Q) = V',$$

where V and V' are arbitrary numbers, then $b^* = c_o$. Therefore, given any indirect utility $V(\theta)$ that the seller must give to the buyers with an incentive compatible scheme, the optimal such scheme will have $b^*(\theta) = c_o$. Therefore an optimal feasible scheme will have $b^*(\theta) = c_o$, at least when $0 < Q^*(\theta) < 1$. Then a series of change-of-variable and reversing-of-integration operations show that the expected profit from a scheme with $b = c_o$,

$$n\int_0^1 [\psi(\theta - c_o, Q(\theta)) - V(\theta)] f(\theta) d\theta$$

$$= n\int_0^1 [\psi(\theta - c_o, Q(\theta) - \int_0^\theta \psi_1(z - c_o, Q(z)) dz] f(\theta) d\theta ,$$

can be written as

(31) $$n\int_0^{Q(1)} \psi_2(Q^{-1}(q) - c_o, q)\ [1 - F(Q^{-1}(q))] dq .$$

This expression can be maximized pointwise. Doing so yields a function Q^{-1} that has as its inverse $Q^*(\theta) = \bar{Q}(\theta, c_o)$, which is increasing in θ by (RC) and hence justifies the use of the inverse Q^{-1} in deriving (31).

Expression (31) is interpretable. A buyer with evaluation $\theta = Q^{-1}(q)$ purchases the probability q of obtaining a unit. All buyers with evaluations greater than $Q^{-1}(q)$ purchase probabilities greater than q. Thus $1 - F(Q^{-1}(q))$ is the probability of selling at least an amount q of probability to an arbitrary buyer -- the "demand" for the "qth increment of probability" is $n(1 - F(Q^{-1}(q))$. The maximum price that the marginal buyer $\theta = Q^{-1}(q)$ would pay for an additional increment of probability above the qth is $\psi_2(Q^{-1}(q) - c_o, q)$, since his profit in the event of obtaining a unit is $Q^{-1}(q) - c_o$. This price is what must be charged for the qth increment to all buyers who purchase it. Thus (31) is the sum of the profits made on each increment of probability sold.

Expression (31) is valid even for $R = 0$, in which case $\psi_2(Q^{-1}(q) - c_o, q)$ is simply $Q^{-1}(q) - c_o$. Hence (31) is maximized by setting $Q^{-1}(q)$ equal to whatever constant θ_o maximizes $(\theta - c_o)(1 - F(\theta))$. This results in a particular take-it-or-leave-it scheme. This scheme is equivalent to the take-it-or-leave-it offering of the price θ_o, which is a scheme found by Riley and Zeckhauser [20] and implicitly by Maskin and Riley [9] and by Myerson [18], to be an optimal auction when there is only one risk neutral bidder. The following corollary formally states these results and also a continuity property.

Corollary 2: If there is no quantity constraint, then as $R \to 0$, the optimal scheme converges to the scheme $\langle Q^o, b^o, p^o \rangle$ defined by $b^o(\theta) = c_o$,

$$p^o(\theta) = \begin{cases} 0 & \theta < \theta_o \\ \theta_o - c_o & \theta \geq \theta_o \end{cases} \qquad Q^o(\theta) = \begin{cases} 0 & \theta < \theta_o \\ 1 & \theta \geq \theta_o \end{cases},$$

where

$$\theta_o \in \underset{\theta}{\operatorname{argmax}}\ (\theta - c_o)(1 - F(\theta)). \tag{32}$$

This scheme is equivalent to the take-it-or-leave-it scheme consisting of posting a price θ_o, and it maximizes expected profits when buyers are risk neutral, regardless of (RC) holding.

6. Solution With a Unit Capacity Constraint

The solution is far more complicated if no more than one unit can be sold. The implicit cost μ is nontrivial and interactions between buyers must be allowed. A simple case is presented before the general result.

One more function must first be defined. Let θ^o be the minimal $\theta > 0$ satisfying $\overline{Q}(\theta, c_o) = F(\theta)^{n-1}$, and define $\overline{c}: [\theta^o, 1] \to [c_o, 1]$ by

$$\overline{Q}(\theta, \overline{c}(\theta)) = F(\theta)^{n-1} . \tag{33}$$

Since $\overline{Q}$ decreases in c and $\overline{Q}(\theta, 1) = 0$, $\overline{c}$ is well-defined. By Theorem 1, a seller that had cost $\overline{c}(\theta)$ would, without a quantity constraint, assign a buyer with evaluation θ the probability $F(\theta)^{n-1}$ of obtaining a unit.

It will be shown that if R is small then $\overline{c}$ is an increasing function. In this case, the opportunity cost $c_o - \mu(\theta)$ can be determined by

$$c_o - \mu(\theta) = \begin{cases} c_o & \theta < \theta^o \\ \overline{c}(\theta) & \theta \geq \theta^o \end{cases}$$

without violating the requirement that $c_o - \mu(\theta)$ be nondecreasing (see (22) and (24)). Depicted in Figures 1 and 2, the optimal scheme for this case has

$$b^*(\theta) = \begin{cases} c_o & \theta < \theta^o \\ \overline{c}(\theta) & \theta \geq \theta^o \end{cases}, \qquad Q^*(\theta) = \begin{cases} \overline{Q}(\theta, c_o) & \theta < \theta^o \\ F(\theta)^{n-1} & \theta \geq \theta^o \end{cases},$$

and $p^*(\theta)$ determined from (20). This scheme is virtually the only optimal scheme when $\bar{c}$ is increasing.

This scheme is feasible, since functions q_i that implement Q^* can be found explicitly. Letting $\theta_{max} = \max(\theta_1, \ldots, \theta_n)$ and $m(\theta_1, \ldots, \theta_n)$ be the number of buyers with evaluation θ_{max}, define

$$q_i^*(\theta_1, \ldots, \theta_n) = \begin{cases} 0 & \theta_i < \theta_{max} \\ \dfrac{Q^*(\theta_i)}{m(\theta_1, \ldots, \theta_n) F(\theta_i)^{n-1}} & \theta_i = \theta_{max} \end{cases}$$

Because $Q^*(\theta) < F(\theta)^{n-1}$ for $\theta < \theta^o$, q_i^* is a well-defined probability. It is easy to show that $0 \le \Sigma q_i^* \le 1$, and that

$$Q_i^*(\theta_i) = \int_0^1 \ldots \int_0^1 q_i^*(\theta_1, \ldots, \theta_n) \prod_{j \neq i} f(\theta_j) d\theta_j \, .$$

These q_i^* functions imply that a buyer obtains the unit only if his evaluation is the largest, and that even in this event, if his evaluation is less than θ^o he has only a probability $Q^*(\theta_i)$ of obtaining the unit. This scheme first identifies the buyer with the maximum evaluation, and then either awards the unit to him if his evaluation is greater than θ^o, or, if $\theta < \theta^o$, treats him exactly as all buyers are treated by the scheme that is optimal when no quantity constraint exists.

Before further discussion, let me present the general result.

<u>Theorem 2:</u> Suppose $I_0 = [0,\theta^1)$, $I_1 = [\theta^1,\theta^2]$, $I_2 = (\theta^2,\theta^3),\ldots,I_{2K+1} = [\theta^{2K+1},1]$ is a partition with $\theta^1 \geq \theta^o$ that satisfies the following criteria:

$$\bar{c}(\theta^{2k}) = \bar{c}(\theta^{2k+1}) \quad (k = 0,\ldots,K) \tag{34}$$

$$\bar{c} \text{ is increasing on } I_{2k+1} \quad (k = 0,\ldots,K) \tag{35}$$

$$\int_{\theta}^{\theta^{2k+1}} [F(z)^{n-1} - \bar{Q}(z,\bar{c}(\theta^{2k}))]f(z)dz \geq 0 \quad \forall\ \theta \in I_{2k} \quad (k = 0,\ldots,K) \tag{36}$$

with equality holding in (36) for $\theta = \theta^{2k}$ $(k = 1,\ldots,K)$. Define $\langle Q^*,b^*,p^* \rangle$ by

$$Q^*(\theta) = \begin{cases} \bar{Q}(\theta,\bar{c}(\theta^{2k})) & \theta \in I_{2k} \ (k = 0,\ldots,K) \\ F(\theta)^{n-1} & \theta \in I_{2k+1} \ (k = 0,\ldots,K) \end{cases} \tag{37}$$

$$b^*(\theta) = \begin{cases} \bar{c}(\theta^{2k}) & \theta \in I_{2k} \ (k = 0,\ldots,K) \\ \bar{c}(\theta) & \theta \in I_{2k+1} \ (k = 0,\ldots,K) \end{cases} \tag{38}$$

$$p^*(\theta) = \psi(\theta - b^*(\theta),\ Q^*(\theta)) - \int_0^{\theta} \psi_1(z - b^*(z),Q^*(z))dz. \tag{39}$$

Then, $\langle Q^*,b^*,p^* \rangle$ maximizes expected profit and is feasible. Furthermore, if $\langle Q,b,p \rangle$ is any feasible piecewise continuous scheme maximizing expected profit, then $Q = Q^*$, $p = p^*$, and, for all $\theta \in (\underline{\theta}(c_o),1)$, $b(\theta) = b^*(\theta)$.

Remark 1: In the original version of this paper, it was only conjectured that the Q^* in Theorem 2 is implementable. Its implementability now follows from a new theorem of Maskin and Riley [11, Theorem 7], which states that any nondecreasing function Q that satisfies constraint (8), such as Q^*, can be implemented.

Remark 2: Because the partition satisfying (34) - (36) is assumed to exist, Theorem 2 is a characterization rather than an existence theorem. However, F must be pathological for the partition not to exist.

A more complicated example is shown in Figures 3 and 4. Here there is a region in which $\bar{c}$ decreases, so that setting the cost $c_o - \mu$ equal to $\bar{c}$ on this region would violate the restriction that $c_o - \mu$ be nondecreasing. The points θ^2 and θ^3 are found, loosely speaking, by letting $b^*(\theta) = \bar{c}(\theta)$ hold for $\theta > \theta^o$ until the function $\bar{Q}(\cdot, \bar{c}(\theta))$ has fallen enough to make the two shaded areas (weighted by the density) equal. After this point, θ^2, b^* is held equal to the constant $\bar{c}(\theta^2)$ and $Q^*(\cdot)$ is set equal to $\bar{Q}(\cdot, \bar{c}(\theta^2))$ until the endpoint θ^3 of this "irregular" segment is reached.

There are two qualitative features of the general scheme in Theorem 2 that are due to the buyers' risk aversion. The first is that buyers with high evaluations are induced to bid nearly their true evaluation, i.e., $\theta - b^*(\theta) \to 0$ as $\theta \to 1$. This result follows from Theorem 2 because $b^* = \bar{c} < 1$ on $[\theta^{2K+1}, 1]$, and $\bar{c} \to 1$ as $\theta \to 1$. But it can also be shown more generally whenever buyers are risk averse. No analagous result is obtained when buyers are risk neutral, as then only the expected payment

$x(\theta) = p(\theta) + Q(\theta)b(\theta)$, as well as the probability $Q(\theta)$, are determined in an optimal scheme.

The second distinctive feature casts some light upon the first. The bid submission fee p^*, while positive for buyers with modest evaluations, becomes negative for buyers with high evaluations. This can be seen from (39), since $\theta - b^*(\theta) \to 0$ implies $\psi(\theta - b^*(\theta)) \to 0$. But again the result appears to hold generally when buyers are risk averse.

What is an intuition for these features? Potentially, the largest payments can be extracted from the buyers with the greatest evaluations. But a buyer must be rewarded to induce him to reveal that his evaluation is high. One way of rewarding is to let the probability of obtaining a unit increase rapidly with the reported evaluation. This is the strategy utilized when a capacity constraint is absent. With this rewarding strategy the seller's profit is due to the increasing bid submission fee. The success of this strategy depends upon the perceived probability-of-winning function increasing rapidly enough that buyers with high evaluations do not bear too much risk and hence pay only small bid submission fees. A capacity constraint therefore limits the use of this strategy, since then a buyer's perceived probability of obtaining a unit cannot be made too large. When the constraint becomes binding, buyers with high evaluations are rewarded by lowering their entry fees, with the seller making the extra profit instead from an increasing bid function. By making the bid submission fee negative for buyers with high evaluations, the seller is selling them insurance against the event of not obtaining a unit. By increasing the insurance coverage with the reported evaluation, the expected value of the reward necessary to induce buyers with high evaluations to reveal themselves is minimized.

Further light is shed upon the role of risk aversion by examining the scheme when buyers are nearly risk neutral. To derive the limiting scheme, observe first that the function $\bar{c}$ is defined by

$$\bar{Q}(\theta,\bar{c}(\theta)) = \frac{\beta(\theta,\bar{c}(\theta))}{\beta(\theta,\bar{c}(\theta)) - \alpha(\theta,\bar{c}(\theta))} = F(\theta)^{n-1} .$$

Rearranging and substituting in for α and β yields

$$\frac{u(\theta-\bar{c}(\theta))}{u'(\theta-\bar{c}(\theta))} + \frac{F(\theta)-1}{f(\theta)} = \left[\frac{u(\theta-\bar{c}(\theta))}{u'(\theta-\bar{c}(\theta))} - u(\theta-\bar{c}(\theta))\right] F(\theta)^{n-1} .$$

Since $u(y) = R^{-1}(1-e^{-Ry})$ and $u(y)/u'(y) = R^{-1}(e^{Ry}-1)$ both converge to y as $R \to 0$, the function $\bar{c}$ converges to the function

$$c^{o}(\theta) = \theta + \frac{F(\theta)-1}{f(\theta)} .$$

The regularity condition (RC) implies c^{o} is an increasing function. Also, as $R \to 0$ the points $\underline{\theta}(c_0)$ and θ^{o} both converge to a point θ_o defined by

$$c_o = \theta_o + \frac{F(\theta_o)-1}{f(\theta_o)} = c^{o}(\theta_o) ,$$

which is also the $\theta_o = \underset{\theta}{\operatorname{argmax}}\,(\theta-c_o)(1-F(\theta))$ of Corollary 2. Theorem 2 therefore implies that the optimal scheme converges to the one presented in Corollary 3. This limiting scheme is optimal for risk neutral buyers, given (RC).

<u>Corollary 3</u>: As $R \to 0$, the optimal scheme with a unit capacity constraint converges to one defined by $b^o(\theta) = c^o(\theta)$ and

$$Q^o(\theta) = \begin{cases} 0 & \theta < \theta_o \\ F(\theta)^{n-1} & \theta \geq \theta_o \end{cases}$$

$$p^o(\theta) = \begin{cases} 0 & \theta < \theta_o \\ \int_0^\theta \max(z,\theta_o)\,dF(z)^{n-1} - c^o(\theta)F(\theta)^{n-1} & \theta \geq \theta_o \ . \end{cases}$$

In the limiting scheme of Corollary 3, a buyer with value θ obtains the unit if and only if θ exceeds both θ_o and the other buyers' values. Buyers with high values still submit bids nearly equal to their values, since $b^o(\theta) = c^o(\theta) \to 1$ as $\theta \to 1$. Buyers with high values are still paid to participate, since $p^o(\theta) \to \int_0^1 \max(z,\theta_o)\,dF(z)^{n-1} - 1 < 0$ as $\theta \to 1$. Buyers with values only slightly larger than θ_o pay positive entry fees, since $p^o(\theta_o) = (\theta_o - c_o)F(\theta_o)^{n-1} > 0$. Any buyer with value greater than θ_o has the total expected payment $p^o(\theta) + b^o(\theta)Q^o(\theta) = \int_0^\theta \max(z,\theta_o)\,dF(z)^{n-1}$, which, if we call θ_o the seller's (reported) "value," is the expectation of the largest of the other agents' values conditional upon θ being the largest value. This is the same expected payment made by risk neutral buyers in a first or second-price auction with reserve price θ_o (see, e.g., [19]).

There is a simple nonrevelation scheme that is equivalent to the one of Corollary 3. This scheme is a first price auction in which the entry fee depends upon the size of the bid. If the entry fee for a buyer who bids b is

$$P^o(b) = p^o(b^{o^{-1}}(b)),$$

then no buyer with evaluation less than θ_o will enter the auction, and buyers with evaluations $\theta \geq \theta_o$ will bid $b = b^o(\theta)$ and pay the bid submission fee $P^o(b) = p^o(\theta)$.

The final corollary identifies the limiting scheme as $R \to \infty$. It can be shown that as $R \to \infty$, $\bar{c}(\theta)$ converges to θ and θ^o converges to c_o (see the proof of Corollary 2). Hence Theorem 2 implies

Corollary 4: Assume $\bar{R} = \infty$. Then as $R \to \infty$, the seller's optimal scheme with a unit capacity constraint converges to $\langle Q^\infty, b^\infty, p^\infty \rangle$, where

$$Q^\infty(\theta) = \begin{cases} 0 & \theta < c_o \\ F(\theta)^{n-1} & \theta \geq c_o \end{cases},$$

$b^\infty(\theta) = \theta$, and $p^\infty(\theta) = 0$. Consequently, the seller's maximum expected profit approaches that obtainable by perfect price discrimination.

7. Summary and Discussion

Initially, an environment was considered in which buyers have the same no-income-effects, risk averse utility function, but different private evaluations for the good. For a selling scheme to maximize expected profit in this environment, the payments made by buyers who do not obtain a unit must be nonstochastic. On the other hand, the payments made by buyers who do obtain units cannot generally be assumed to be nonstochastic. It was shown, however, that attention could be restricted to schemes with deterministic payment functions if buyers exhibit constant absolute risk aversion, which was subsequently a maintained assumption.

One fundamental conclusion is that maximum expected profits are greater if the buyers are risk averse rather than risk neutral. If buyers are extremely risk averse, then expected profits from an optimal scheme approximate those achievable by perfect price discrimination. This result can be explained from the revelation point of view: the desire of buyers with high values to avoid risk can be used to decrease the reward that is necessary to prevent them from mimicking buyers with lower values.

In the case of unlimited capacity, an expected profit maximizing scheme presents the buyers with a nontrivial, nonuniform price schedule for the probability of acquiring a unit. Additional payment is made by buyers who obtain a unit. Given that buyers exhibit constant absolute risk aversion, this conditional payment is equal to the seller's opportunity cost.

In the case of a unit capacity constraint, the expected profit maximizing scheme was found in "reduced form." In other words, only an individual's perceived probability of obtaining the unit ($Q^*(\theta_i)$), rather than his actual probability as a function of all buyers' values ($q_i(\theta_1,\ldots,\theta_n)$), was constructed. The functions q_i were, however, constructed for the case which necessarily occurs when buyers are only mildly risk averse. Then, the optimal scheme approximates a first price auction with both a reserve price and a bid submission fee, where the submission fee is positive for low bids and negative for high bids. As long as the buyers are not risk neutral, the optimal scheme is virtually unique, and it induces buyers with high values to submit bids approximately equal to their values. In the sense that their marginal utility of income does not depend upon obtaining a unit, buyers with high values are nearly perfectly insured.

Schemes that perfectly insure all buyers do exist. When there is no capacity constraint, a take-it-or-leave-it offer scheme subjects no buyers to risk. Similarly, to sell a single unit, a first price auction found in [19] involves paying buyers to submit bids according to a schedule that induces each one, as a dominant strategy, to bid his true value.[7] Setting the reserve price at c_o in these schemes results in both ex ante and ex post efficiency, since they subject only the risk neutral agent, the seller, to risk, and they award units only to those who value them the most.

The schemes that have been found here are also ex ante efficient or, more specifically, incentive-efficient in the sense of Myerson [12].[8] This is obvious, since switching to another feasible scheme must decrease the expected profit of the seller. It is somewhat paradoxical, therefore, that they impose risk on the risk averse buyers. It is, of course, the incentive compatibility requirement that prevents Pareto dominating schemes from being feasible.

Risk is imposed on a buyer only to decrease the amount that must be paid to keep him from being mimicked by buyers with greater values. It is therefore unnecessary to impose much risk on a buyer with a high value, since it is unlikely that another buyer with a greater value exists. This explains why, in the optimal scheme with a unit capacity constraint, the difference between a buyer's marginal utilities in the two states "win" and "lose" is approximately zero for buyers with high values. Thus the marginal conditions for *ex ante* optimality under complete information, which requires buyers to be perfectly insured, are satisfied for buyers with the highest values. This result is analogous to ones found in the optimal taxation (individuals with the highest ability have zero marginal tax rates) and nonuniform pricing (individuals with the highest demand curves pay a marginal price equal to marginal cost) literatures.

Proof of Lemma 2:

Let $-\bar{p}_\ell(\theta)$ be a buyer's certainty equivalent for $-\tilde{p}_\ell$:

$$u(-p_\ell(\theta)) = \int_{-\infty}^{\infty} u(-t)\,dJ(t \mid \theta).$$

Let $\bar{J}(p_\ell \mid \theta)$ then be the distribution putting all probability on $p_\ell(\theta)$. Since u is increasing and concave, $p_\ell(\theta)$ is greater than $E(\tilde{p}_\ell \mid \theta)$. Therefore, replacing J with $\bar{J}$ yields greater expected profit if feasibility is maintained. A capacity constraint only involves Q, so that any existing capacity constraint is still satisfied. The expected utility of a buyer with evaluation θ who reports $\hat{\theta}$ in the original scheme is

$$Q(\hat{\theta})\int u(\theta - t)\,dH(t \mid \hat{\theta}) + (1 - Q(\hat{\theta}))\int u(-t)\,dJ(t \mid \hat{\theta}).$$

By the construction of $p_\ell(\theta)$, this expression is also equal to the expected utility of a buyer with evaluation θ who reports $\hat{\theta}$ in the new scheme. Hence, since they are satisfied by the original scheme, the constraints of incentive compatibility and voluntary participation are also satisfied by the new scheme. It is therefore feasible. □

Proof of Lemma 3:

Let $\bar{H}(p_w \mid \theta)$ be the distribution that puts all probability on the $p_w(\theta)$ defined by

$$e^{Rp_w(\theta)} = \int_{-\infty}^{\infty} e^{Rt}\,dH(t \mid \theta) .$$

Since $p_w(\theta) \geq E(\tilde{p}_w \mid \theta)$, $\langle Q, \bar{H}, p_\ell \rangle$, if it is feasible, yields greater expected profits than $\langle Q, H, p_\ell \rangle$. Because Q is unchanged, the new scheme still satisfies any capacity constraint. In either scheme, the expected utility of a buyer with evaluation θ who reports $\hat{\theta}$ is

$$1 - Q(\hat{\theta})\, e^{-R\theta} \int_{-\infty}^{\infty} e^{Rt}\, dH(t|\hat{\theta}) - (1 - Q(\hat{\theta}))\, e^{Rp_{\ell}(\theta)} .$$

Hence the new scheme satisfies the incentive compatibility and voluntary participation constraints because the original scheme satisfies them. So $\langle Q, \overline{H}, p_{\ell} \rangle$ is feasible. □

Proof of Lemma 4:

Suppose first that $\langle Q, b, p \rangle$ is incentive compatible. Let $\theta_2 > \theta_1$, and let $Q_i = Q(\theta_i)$ and $b_i = b(\theta_i)$. Incentive compatibility implies

(A1) $$V(\theta_i) \geq U(\theta_j,\theta_i) = V(\theta_j) + \psi(\theta_i - b_j, Q_j) - \psi(\theta_j - b_j, Q_j).$$

Combining these two expressions results in

$$\psi(\theta_1 - b_2, Q_2) - \psi(\theta_1 - b_1, Q_1) \leq \psi(\theta_2 - b_2, Q_2) - \psi(\theta_2 - b_1, Q_1).$$

This inequality proves (12), since it can be rearranged, utilizing (10), to yield

$$\frac{Q_1 e^{Rb_1}}{1 - Q_1} \leq \frac{Q_2 e^{Rb_2}}{1 - Q_2} .$$

Because $\psi(y,Q)$ increases in y, expression (A1) with $(i,j) = (2,1)$ implies that $V(\theta_2) > V(\theta_1)$. Therefore, using (A1) again with $(i,j) = (1,2)$,

$$0 \leq V(\theta_2) - V(\theta_1) \leq \psi(\theta_2 - b_2, Q_2) - \psi(\theta_1 - b_2, Q_2) \quad .$$

Consequently, since

$$\text{(A2)} \qquad \psi_1(y,Q) = \frac{1}{1 + \dfrac{1-Q}{Qu'(y)}} \leq 1 \quad ,$$

$0 \leq V(\theta_2) - V(\theta_1) \leq \theta_2 - \theta_1$. This shows that V is absolutely continuous and hence equal to the integral of its derivative. Since $V(\theta) - U(\hat{\theta},\theta) \geq 0$, with equality when $\theta = \hat{\theta}$, it is true that

$$\hat{\theta} \in \underset{\theta}{\operatorname{argmin}}\ V(\theta) - U(\hat{\theta},\theta).$$

Hence, whenever $V'(\hat{\theta})$ exists,

$$V'(\hat{\theta}) = U_2(\hat{\theta},\hat{\theta}) = \psi_1(\hat{\theta} - b(\hat{\theta}), Q(\hat{\theta})).$$

Therefore expression (13) holds.

Now we must show that (12) and (13) imply incentive compatibility. Expression (13) implies

$$V(\theta) = V(\hat{\theta}) + \int_{\hat{\theta}}^{\theta} \psi_1(z - b(z), Q(z))dz.$$

Subtracting (A1) with $(\theta_j,\theta_i) = (\hat{\theta},\theta)$ from this expression yields

$$V(\theta) - U(\hat{\theta},\theta) = \int_{\hat{\theta}}^{\theta} \{\psi_1(z - b(z),Q(z)) - \psi_1(z - b(\hat{\theta}),Q(\hat{\theta}))\}dz.$$

Hence (A2) implies

$$V(\theta) - U(\hat{\theta},\theta) = \int_{\hat{\theta}}^{\theta} \left\{ \frac{1}{1 + \frac{1 - Q(z)}{Q(z)\exp(b(z) - z)}} - \frac{1}{1 + \frac{1 - Q(\hat{\theta})}{Q(\hat{\theta})\exp(b(\hat{\theta}) - z)}} \right\} dz.$$

Because of (12), the expression in brackets is positive (negative) only if $z > \hat{\theta}$ ($z < \hat{\theta}$). Therefore $V(\theta) - U(\hat{\theta},\theta) \geq 0$, which proves incentive compatibility. □

Proof of Lemma 5:

The derivatives of $\bar{\mathcal{V}}$ are

$$\bar{\mathcal{V}}_2(\theta,Q,b,c) = b-c + \psi_{12}(\theta-b,Q) \left\{ \frac{\psi_2(\theta-b,Q)}{\psi_{12}(\theta-b,Q)} + \frac{F(\theta)-1}{f(\theta)} \right\}$$

$$\bar{\mathcal{V}}_3(\theta,Q,b,c) = -\psi_{11}(\theta-b,Q) \left\{ \frac{\psi_1(\theta-b,Q)-Q}{\psi_{11}(\theta-b,Q)} + \frac{F(\theta)-1}{f(\theta)} \right\}$$

Calculation yields

$$\psi_1(y,Q) = \frac{Qu'(y)}{1-Q + Qu'(y)}, \qquad \psi_2(y,Q) = \frac{u(y)}{1-Q + Qu'(y)},$$

$$\psi_{11}(y,Q) = \frac{-RQ(1-Q)u'(y)}{[1-Q + Qu'(y)]^2}, \qquad \psi_{12}(y,Q) = \frac{u'(y)}{[1-Q + Qu'(y)]^2}.$$

Therefore $\psi_2/\psi_{12} = (\psi_1 - Q)/\psi_{11} = Qu(y) + (1 - Q)u(y)/u'(y)$. Hence, from the definitions of α and β,

(A3) $$\bar{\mathcal{V}}_2(\theta,Q,b,c) = b-c + \{Q\alpha(\theta,b) + (1-Q)\beta(\theta,b)\}\psi_{12}(\theta-b,Q)$$

(A4) $$\bar{\mathcal{V}}_3(\theta,Q,b,c) = - \{Q\alpha(\theta,b) + (1-Q)\beta(\theta,b)\}\psi_{11}(\theta-b,Q).$$

Now, fix θ and c, and let $g(Q,b) = Q\alpha(\theta,b) + (1-Q)\beta(\theta,b)$. Then, letting $h(Q,b) = V(\theta,Q,b,c)$, (A3) and (A4) imply $h_1(Q,b) = b - c + g(Q,b)\psi_{12}(\theta-b,Q)$ and $h_2(Q,b) = -g(Q,b)\psi_{11}(\theta-b,Q)$. Inspection of α and β indicates that $g_2 < 0$ everywhere, and $g_1 < 0$ unless $b = \theta$. Also, if $Q < 1$ then $g(b,Q) = 0$ has a unique solution $b = b(Q)$ that is decreasing. Let $b(1) \in [-\infty,\infty)$, the limiting value of $b(Q)$. If $0 < Q < 1$, $(b-b(Q))h_2(Q,b) < 0$ for $b \neq b(Q)$, since $\psi_{11} < 0$ and $b(Q)$ decreases. Hence, $b(Q)$ uniquely maximizes h for fixed $0 < Q < 1$. Since $\psi_{11} = 0$ if $Q = 0$ or $Q = 1$, any b maximizes $h(0,\cdot)$ and $h(1,\cdot)$. Now, $dh(Q,b(Q))/dQ = h_1(Q,b(Q)) = b(Q) - c$, and $d^2h(Q,b(Q))/dQ^2 = b'(Q) \leq 0$, with strict inequality if $b(Q) \neq \theta$. Hence $h(Q,b(Q))$ is strictly concave on $[0,1]$. Thus $h(Q,b(Q))$ has a unique maximizer Q^*, and any (Q,b) maximizing h has $Q = Q^*$. Also, if $0 < Q^* < 1$ then $(Q^*,b(Q^*)) = (Q^*,c)$ uniquely maximizes h. That $Q^* = \overline{Q}(\theta,c)$ now follows by verifying that $b(Q) < c$ for all Q if $\theta < \underline{\theta}(c)$, $b(\overline{Q},(\theta,c)) = c$ if $\underline{\theta}(c) \leq \theta \leq \overline{\theta}(c)$, and $b(Q) > c$ for all Q if $\theta > \overline{\theta}(c)$. Verifying these is straightforward.

If $\underline{\theta}(c) < \theta < \overline{\theta}(c)$, then $\beta(\theta,c) > \alpha(\theta,c)$. Therefore (RC) implies

$$\overline{Q}_1(\theta,c) = \frac{\overline{Q}(\theta,c)\alpha_1(\theta,c) + (1-\overline{Q}(\theta,c))\beta_1(\theta,c)}{\beta(\theta,c) - \alpha(\theta,c)} > 0.$$

Hence $\overline{Q}$ increases in θ. Similarly, $\overline{Q}$ decreases in c for $\theta \in (\underline{\theta}(c),\overline{\theta}(c))$ because $\alpha_2(\theta,c) < 0$ and $\beta_2(\theta,c) < 0$. □

Proof of Theorem 1:

Without a capacity constraint, the constraints (18) and (19) can be dropped and both μ and η set to zero. Therefore $c(\theta) = c_0 - \mu(\theta) = c_0$. Hence (26) and lemma 5 imply that $Q^*(\theta) = \overline{Q}(\theta, c_0)$ is the unique optimizing Q, and that $b^*(\theta) = \overline{b}(\theta, c_0) = c_0$ is an optimizing b and is the only optimizing b on $(\underline{\theta}(c_0), \overline{\theta}(c_0))$. Letting

$$V^*(\theta) = \int_0^\theta \psi_1(z - c_0, \overline{Q}(z, c_0))dz,$$

satisfaction of (16), (17), (25) and (26) by $\langle Q^*, b^*\rangle$ is immediate. Expression (29) is obtained from (20) by substituting $V^*(\theta)$ for $V(\theta)$, and is required for $\langle Q^*, b^*, p^*\rangle$ to be incentive compatible. Since Q^* is nondecreasing and b^* is constant, the neglected constraint (12) is satisfied. Hence $\langle Q^*, b^*, p^*\rangle$ is feasible. The uniqueness of Q^* and b^* on $(\underline{\theta}(c_0), \overline{\theta}(c_0))$ imply the uniqueness of p^* on the same interval. For $\theta < \underline{\theta}(c_0)$, $Q^*(\theta) = 0$ is unique -- hence $p^*(\theta) \leq 0$ is necessary by the voluntary participation constraint. Hence $p^*(\theta) = 0$ for $\theta \leq \underline{\theta}(c_0)$ is the unique optimizing p. For any $\theta \geq \overline{\theta}(c_0)$, $p^*(\theta) = p^*(\overline{\theta}(c_0))$ follows from (29). Also, incentive compatibility requires the total payment $p(\theta) + b(\theta)$ to be constant on the set of θ for which $Q(\theta) = 1$. Hence $b^*(\theta) + p^*(\theta) = c_0 + p^*(\overline{\theta}(c_0))$ is unique on $[\overline{\theta}(c_0), 1]$ $\square$

Proof of Corollary 1:

For $y > 0$,

$$\lim_{R\to\infty} u(y) = \lim_{R\to\infty} \frac{1 - e^{-Ry}}{R} = 0$$

$$\lim_{R\to\infty} \frac{u(y)}{u'(y)} = \lim \frac{e^{Ry} - 1}{R} = \infty$$

Hence $\alpha(\theta,c_0) \to (F(\theta) - 1)/f(\theta)$ and $\beta(\theta,c_0) \to \infty$ as $R \to \infty$ for any $\theta > c_0$. Therefore $\underline{\theta}(c_0) \to c_0$ and $\overline{\theta}(c_0) \to 1$ as $R \to \infty$. Thus if $\theta \le c_0$, $Q*(\theta) = 0 = Q^{\infty}(\infty)$ for any R. If $\theta > c_0$, then

$$\lim_{R\to\infty} Q^*(\theta) = \lim_{R\to\infty} \frac{\beta(\theta,c_0)}{\beta(\theta,c_0) - \alpha(\theta,c_0)} = 1.$$

Thus $Q^* \to Q^{\infty}$ as $R \to \infty$. Because $Q^*(\theta) = \overline{Q}(\theta,c_0)$,

$$1 - Q^*(\theta) + Q^*(\theta)\ddot{u}'(\theta-c_0) = \left[\frac{u'(\theta-c_0)}{u(\theta-c_0)}\right]\left[\frac{1-F(\theta)}{f(\theta)}\right]$$

if $\theta \in (\underline{\theta}(c_0), \overline{\theta}(c_0))$. Consequently, if $c_0 < \theta < 1$ then

$$\lim_{R\to\infty} \psi(\theta-c_0,Q^*(\theta)) = \lim -(\tfrac{1}{R}) \log\left\{ 1 - Q^*(\theta) + Q^*(\theta)u'(\theta-c_0)\right\}$$

$$= \lim_{R\to\infty} -(\tfrac{1}{R}) \log\left\{\left[\frac{1-F(\theta)}{f(\theta)}\right]\left[\frac{u'(\theta-c_0)}{u(\theta-c_0)}\right]\right\}$$

$$= \lim_{R\to\infty} -(\tfrac{1}{R}) \log\left\{\frac{Re^{-R(\theta-c_0)}}{1-e^{-R(\theta-c_0)}}\right\} = \theta - c_0.$$

If $\theta = 1$, then $\psi(\theta - c_0, Q^*(\theta)) = \psi(\theta - c_0, 1) = \theta - c_0$ for every R. Also if $c_0 < \theta < 1$ then

$$\lim_{R\to\infty} \psi_1(\theta - c_0, Q^*(\theta)) = \lim_{R\to\infty} \frac{Q^*(\theta)u'(\theta - c_0)}{1 - Q^*(\theta) + Q^*(\theta)u'(\theta - c_0)}$$

$$= \lim_{R\to\infty} \left[\frac{f(\theta)}{1-F(\theta)}\right] Q^*(\theta)u(\theta - c_0) = 0.$$

Hence, as $0 \le \psi_1 \le 1$, the Dominated Convergence Theorem implies that

$$\lim_{R\to\infty} p^*(\theta) = \lim_{R\to\infty} \left\{ \psi(\theta - c_0, Q^*(\theta)) - \int_0^\theta \psi_1(z - c_0, Q^*(z))dz \right\}$$

$$= \theta - c_0.$$

Hence $p^* \to p^\infty$ as $R \to \infty$. Finally, the convergence of expected profits

$$\lim_{R\to\infty} \int_0^1 p^*(\theta)f(\theta)d\theta = \int_{c_0}^1 (\theta - c_0)f(\theta)d\theta$$

also follows from the Dominated Convergence Theorem, since $0 \le p^* \le p^\infty$ for all R. □

Proof of Corollary 2:

Because of (RC) there is a unique θ_0 maximizing $(\theta - c_0)(1-F(\theta))$. Both $\underline{\theta}(c_0)$ and $\overline{\theta}(c_0)$ converge to θ_0 as $R \to 0$. Hence $\overline{Q}(\theta, c_0) \to Q^o(\theta)$

as $R \to 0$. Since $\psi(y,Q) \to yQ$ and $\psi_1(y,Q) \to Q$ as $R \to 0$, taking the limit in (29) yields

$$p^o(\theta) = (\theta - c_0)Q^o(\theta) - \int_0^\theta Q^o(z)\,dz\,.$$

Hence $p^o(\theta) = 0$ if $\theta < \theta_0$. If $\theta \geq \theta_0$, $p^o(\theta) = (\theta - c_0) - (\theta - \theta_0) = \theta_0 - c_0$. Therefore $\langle Q^*, b^*, p^* \rangle \to \langle Q^o, b^o, p^o \rangle$ as $R \to 0$. The scheme $\langle Q^o, b^o, p^o \rangle$ is obviously incentive compatible and equivalent to posting a take-it-or-leave-it price θ_0.

Now assume buyers are risk neutral. Holding Q fixed, they are indifferent among all schemes $\langle Q, b, p \rangle$ having the same sum $Qb + p$. Hence, in finding an optiaml scheme, $b^* = c_0 = b^0$ can be set. The derivation of (31) is valid for $R = 0$. But now $\psi_2(y,Q) = y$, so that (31) is maximized by a constant function $Q^{-1}(q) = \theta_0$, where θ_0 satisfies (32). This Q^{-1} is obviously nondecreasing, regardless of (RC). Hence the optimal scheme has any buyer with $\theta \geq \theta_0$ purchasing a unit with probability one for the amount $c_0 + (\theta - c_0) = \theta_0$. This is equivalent to posting the price θ_0. $\square$

<u>Proof of Theorem 2</u>:

Let $V^*(\theta) = \psi(\theta - b^*(\theta), Q^*(\theta)) - p^*(\theta)$, $Y^*(\theta) = \int_\theta^1 [F(z)^{n-1} - Q^*(z)]f(z)\,dz$, and $\mu^*(\theta) = c_o - b^*(\theta)$. Then (39) implies that V^* satisfies (16), and Y^* satisfies (19) by construction. Also by construction, (17) $0 \leq Q^* \leq 1$ holds. Expressions (36) and (37) imply (18) $Y^* \geq 0$. Since b^* is continuous, μ^* is continuous and (23) vacuously holds.

Letting $\lambda^* = F - 1$, (21) is satisfied. The Hamiltonian composed with the functions $V^*, Y^*, Q^*, b^*, \lambda^*$ and μ^* is continuous in θ because each of these functions is continuous. Hence (25) holds. Notice that if c^* is defined by $c^*(\theta) = c_o - \mu^*(\theta)$, then $b^*(\theta) = c^*(\theta) = \overline{b}(\theta, c^*(\theta))$ and $Q^*(\theta) = \overline{Q}(\theta, c^*(\theta))$. Hence lemma 5 implies that (26) Q^* and b^* maximize the Hamiltonian.

It remains to show (22) and (24) hold for an appropriately defined η^*. If $\theta \varepsilon I_{2k}$, then $b^{*\prime}(\theta) = 0$. If $\theta \varepsilon I_{2k+1}$, then $b^{*\prime}(\theta) = \overline{c}'(\theta) \geq 0$. Hence $\mu^{*\prime} = - b^{*\prime}$ exists (except at endpoints of the intervals) and is nonpositive on I_{2k+1} and zero on I_{2k}. Let $-\eta^*$ be the right (left at $\theta = 1$) derivative of μ^*. Then $\eta^* \geq 0$. If $\eta^*(\theta) > 0$, then $\theta \varepsilon I_{2k+1}$ and hence $Y^*(\theta) = 0$. Thus (24) holds. Expression (22) holds because $\mu^{*\prime} = -\eta^*$ by construction, and $\mu^*(0) = c_o - b^*(0) = c_o - \overline{c}(\theta^o) = 0$.

We have shown that $\langle Q^*, b^*, p^* \rangle$ satisfies the sufficient conditions for the problem in Section 4. Thus it maximizes expected profit, provided it is feasible. It has been shown that b^* is nondecreasing. From (37), $Q^*(\theta)$ is also nondecreasing, since $\overline{Q}$ and F are both nondecreasing in θ. Hence the neglected constraint (12) that $Q^* e^{Rb^*}/(1 - Q^*)$ be non-decreasing is satisfied. Lemma 3 therefore implies $\langle Q^*, b^*, p^* \rangle$ is incentive compatible. As mentioned in Remark 1, Q^* is implementable by [11, Theorem 7], since it is nondecreasing and satisfies (8). Hence $\langle Q^*, b^*, p^* \rangle$ is feasible.

Now assume $\langle Q, b, p \rangle$ is another piecewise continuous scheme maximizing expected profit subject to the feasibility constraints. Then $\langle Q, b \rangle$ satisfies the constraints (16) - (19), where V is derived from p in (20). Since $\langle Q, b, p \rangle$ results in the same expected profit as $\langle Q^*, b^*, p^* \rangle$, $\langle Q, b \rangle$ must also maximize

(15) subject to (16) - (19). Thus λ, μ and η exist such that they and $\langle Q,b\rangle$ satisfy the necessary conditions (21) - (26).

We first show that the multiplier μ is continuous. If not, then (23) implies that θ exists such that $\mu(\theta^-) > \mu(\theta^+)$ and either $Q(\theta^-) = F(\theta)^{n-1}$ or $Q(\theta^+) = F(\theta)^{n-1}$. For $i \in \{+,-\}$, let $\mu^i = \mu(\theta^i)$, $Q^i = Q(\theta^i)$, and $b^i = b(\theta^i)$. Because of (26'), (Q^i,b^i) maximizes

$$\mathscr{H}^i(Q,b) = \psi(\theta-b,Q) + (b-c_o)Q + \left[\frac{F(\theta)-1}{f(\theta)}\right]\psi_1(\theta-b,Q) + \mu^i\,[Q-F(\theta)^{n-1}] .$$

Continuity of the Hamiltonian requires $\mathscr{H}^+(Q^+,b^+) = \mathscr{H}^-(Q^-,b^-)$. If $Q^- = F(\theta)^{n-1}$, then $\mathscr{H}^-(Q^-,b^-) = \mathscr{H}^+(Q^-,b^-)$. However, $0 \le F(\theta)^{n-1} < 1$ and $\mu^- > \mu^+$ imply

$$0 = \mathscr{H}_1^-(Q^-,b^-) > \mathscr{H}_1^+(Q^-,b^-).$$

Thus we have the contradiction that $\mathscr{H}^+(Q^+,b^+) > \mathscr{H}^+(Q^-,b^-) = \mathscr{H}^-(Q^-,b^-)$. A similar argument leads to the contradiction $\mathscr{H}^-(Q^-,b^-) > \mathscr{H}^-(Q^+,b^+) = \mathscr{H}^+(Q^+,b^+)$ if $Q^+ = F(\theta)^{n-1}$. Hence μ must be continuous. Since $Q(\theta) = \overline{Q}(\theta,c_o - \mu(\theta))$ and $b(\theta) = c_0 - \mu(\theta)$, Q and b are also continuous.

Now we show $\mu(0) = 0$. Note that (22) and (24) imply $\mu \le 0$. Hence, because $\overline{Q}(\theta,c)$ decreases in c, $Q(\theta) = \overline{Q}(\theta,c_o - \mu(\theta) \le \overline{Q}(\theta,c_o)$. But $\overline{Q}(\theta,c_o) = 0$ on a nondegenerate interval $[0,\underline{\theta}(c_o)]$. Hence $Q(\theta) = 0$ on $[0,\underline{\theta}(c_o)]$, which in turn implies $Y(\theta) > 0$ on $[0,\underline{\theta}(c_o))$. Hence (22) implies $\mu(0) = 0$.

We show next that $\mu = \mu^*$. First, notice that if $\mu' < 0$ and $\mu^{*\prime} < 0$ on an interval I, then (22) and (24) imply $Y = Y^* = 0$ on I, and hence $Q = Q^* = F^{n-1}$ on I. Therefore, since $Q(\theta) = \overline{Q}(\theta, c_o - \mu(\theta))$ and $Q^*(\theta) = \overline{Q}(\theta, c_o - \mu^*(\theta))$, and $\overline{Q}(\theta, c)$ decreases in c, $\mu = \mu^*$ on I. Consequently, if we assume $\mu \neq \mu^*$ and let

$$t = \sup \{\theta \mid \mu(\theta) \neq \mu^*(\theta)\} ,$$

then there exists an interval $J = [s,t)$ upon which μ is constant and μ^* is strictly decreasing and $\mu < \mu^*$, or vice versa. We may assume $\mu < \mu^*$ on J, as the argument is symmetric in μ and μ^*. We can then let s be minimal, i.e.,

$$s = \inf \{ \theta \mid \mu(\theta) = \mu(t) \} .$$

We know $s > 0$, since μ is continuous, $\mu(0) = 0$, and $\mu(s) < \mu^*(s) \leq 0$. Hence μ is strictly decreasing on a nondegenerate interval (r,s). Observe that if $x > s$, then $Q(x) = \overline{Q}(x, c_o - \mu(x)) \leq \overline{Q}(x, c_o - \mu^*(x)) = Q^*(x)$, with the inequality strict for $x \in J$. Hence

$$\begin{aligned} Y(\theta) &= \int_\theta^1 [F(x)^{n-1} - Q(x)] f(x)dx \\ &> \int_\theta^1 [F(x)^{n-1} - Q^*(x)] f(x)dx = Y^*(\theta) \geq 0 \end{aligned}$$

for any $\theta \varepsilon J$. In particular, $Y(s) > 0$. But (22), (24) and $\mu' < 0$ almost everywhere on (r,s) imply $Y(s) = 0$, a contradiction. Therefore $\mu = \mu^*$.

But $\mu = \mu^*$ and lemma 5 imply $Q = Q^*$. Since $0 < Q^*(\theta) < 1$ for any $\theta > \underline{\theta}(c_o)$, lemma 5 also implies $b(\theta) = b^*(\theta)$ for $\theta > \underline{\theta}(c_o)$. Incentive compatibility now implies $p = p^*$. □

Proof of Corollary 3:

Since $\bar{c} \to c^o$ as $R \to 0$, and c^o is increasing, we know immediately that $Q^* \to Q^o$ and $b^* \to b^o$. If $\theta < \theta_o$, then $p^o(\theta) = 0$ is implied by $Q^o(\theta) = 0$. If $\theta \geq \theta_o$, then integration by parts yields

$$p^o(\theta) = \psi(\theta - b^o(\theta), Q^o(\theta)) - \int_{\theta_o}^{\theta} \psi_1(z - b^o(z), Q^o(z))dz$$

$$= (\theta - c^o(\theta))F(\theta)^{n-1} - \int_{\theta_o}^{\theta} F(z)^{n-1}dz$$

$$= -c^o(\theta)F(\theta)^{n-1} + \int_0^{\theta} \max(\theta_o, z)\,dF(z)^{n-1}. \quad \square$$

FOOTNOTES

1. In a first-price (discriminatory) auction with reserve price, the high bidder obtains the unit and pays a price equal to his bid, provided it exceeds the (known) reserve price. A second-price (competitive, Vickrey) auction with reserve price is the same, except that the price is the maximum of the reserve price and the second highest bid.

2. The fact that first and second-price auctions generate the same expected profit in this environment was first observed by Vickrey [25]. However, Matthews [12] observes that a risk averse seller prefers the first-price auction,whereas Milgrom and Weber [13] show that a risk neutral seller prefers the second-price auction if buyers' types are "affiliated" rather than independent random variables.

3. The desirability of incentive schemes yielding random outcomes has previously been suggested by, e.g., Riley and Zeckhauser [20] and Stiglitz [24].

4. Myerson [18] discusses the restriction to revelation selling strategies more fully, aptly naming the idea the "Revelation Principle." The idea dates back at least to Gibbard [2] (for dominant strategy mechanisms) and is discussed generally in [1, 6, 17].

5. See Theorem 8 in Seierstad and Sydsaeter [22]. As is usual, attention is restricted to piecewise continuous controls (Q and b) and continuous, piecewise differentiable state variables (V and Y).

6. It is an artifact of constant absolute risk aversion that $\bar{b}(\theta,c) = c$. If buyers exhibited decreasing (increasing) absolute risk aversion, then a generalization of Lemma 5 would conclude that $\bar{b}(\theta,c) > c$ $(\bar{b}(\theta,c) < c)$,

where $\bar{b}$ is a component of the optimal scheme within the class of schemes with deterministic payment functions.

7. In this auction a buyer who bids b is paid the amount $\int_r^b F(z)^{n-1}\, dz$, where r is the reserve price.

8. In our context, a scheme is <u>incentive-efficient</u> if any other incentive compatible scheme either decreases the seller's expected profit or, for some evaluation θ, decreases the expected utility of some buyer having the evaluation θ. This efficiency notion incorporates the informational constraints of the environment. See Holmstrom [7], Myerson [17] or Harris and Townsend [6] for original discussions of the concept.

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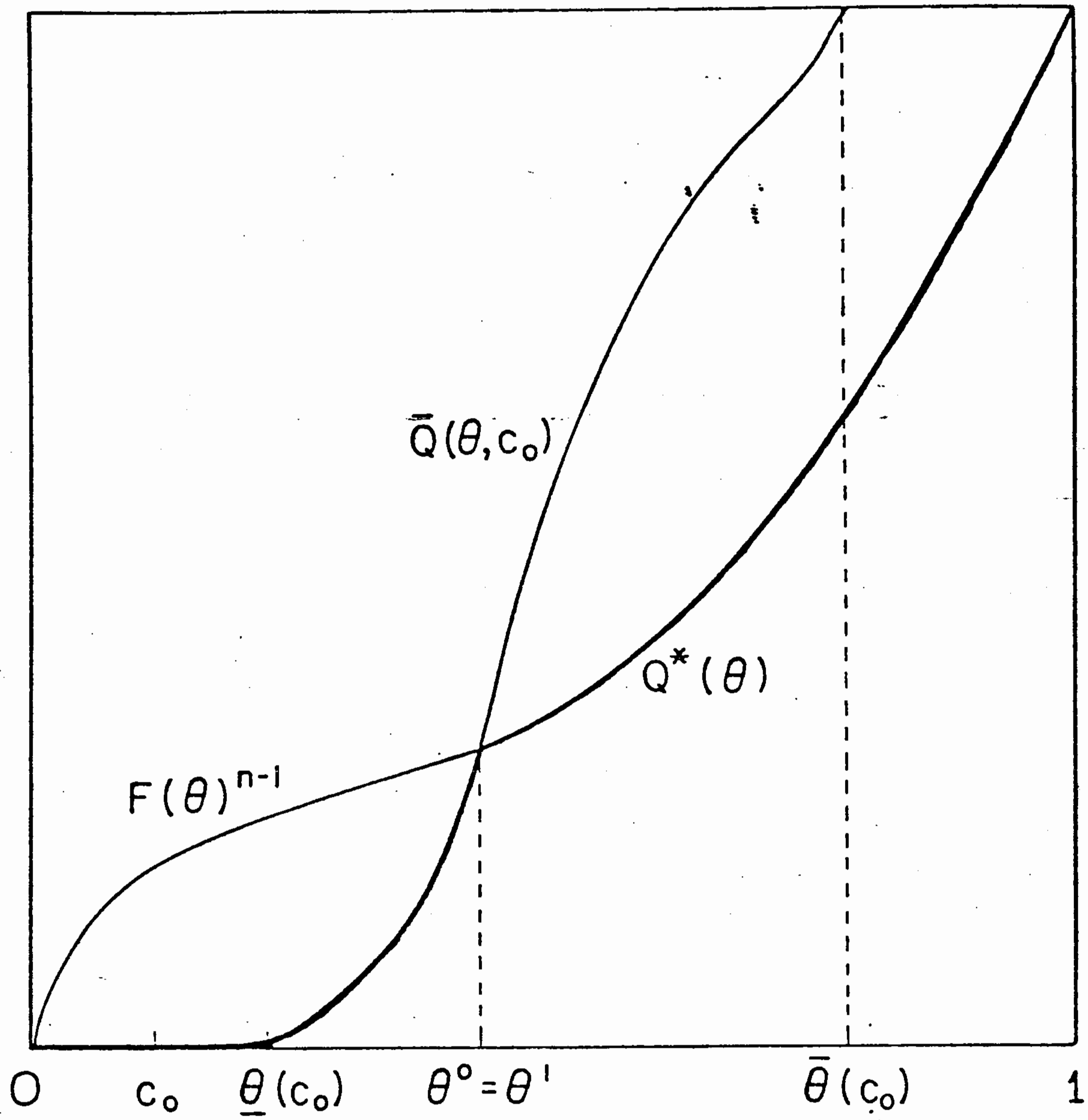

Figure 1

Steven Matthews

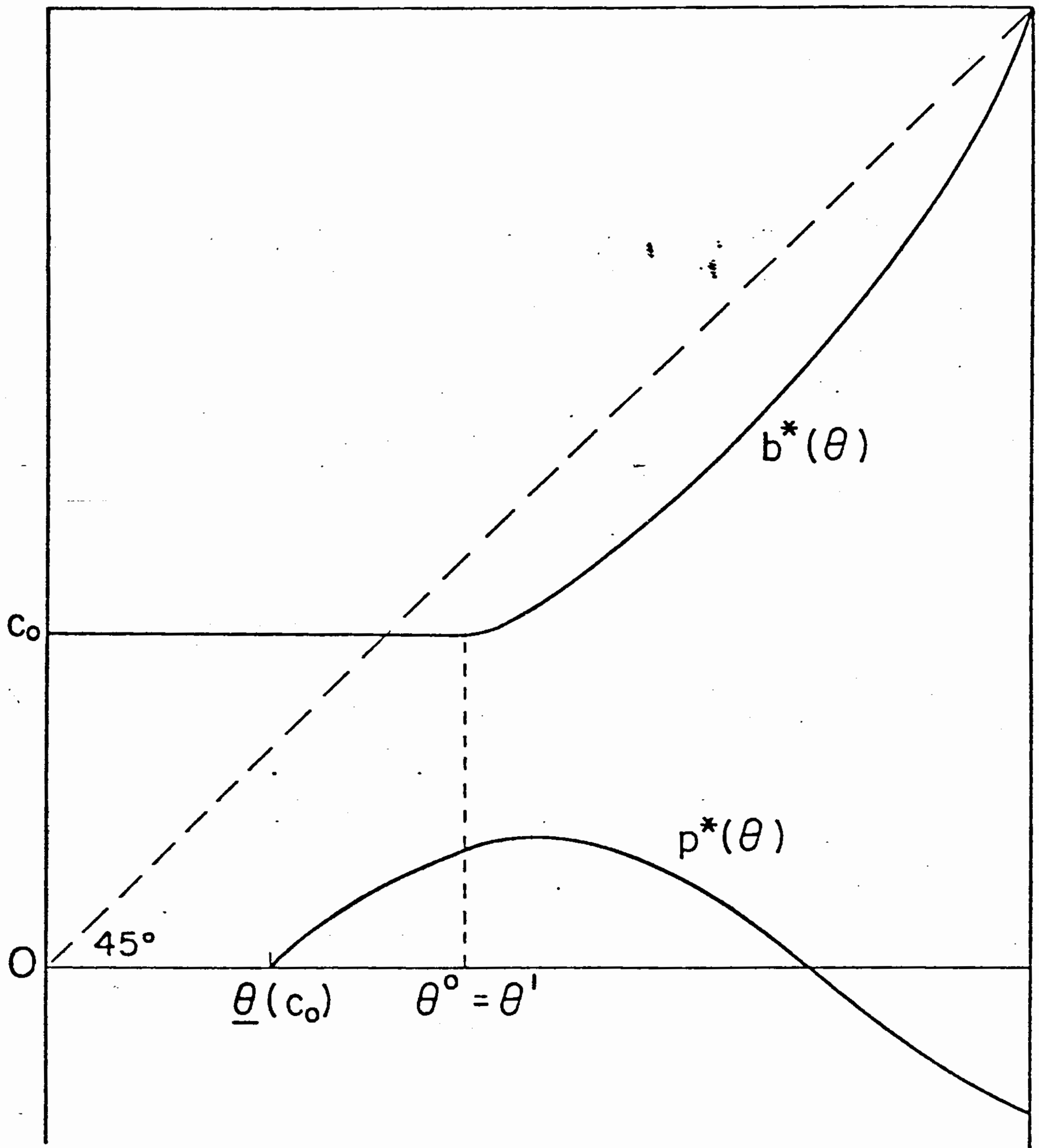

Figure 2

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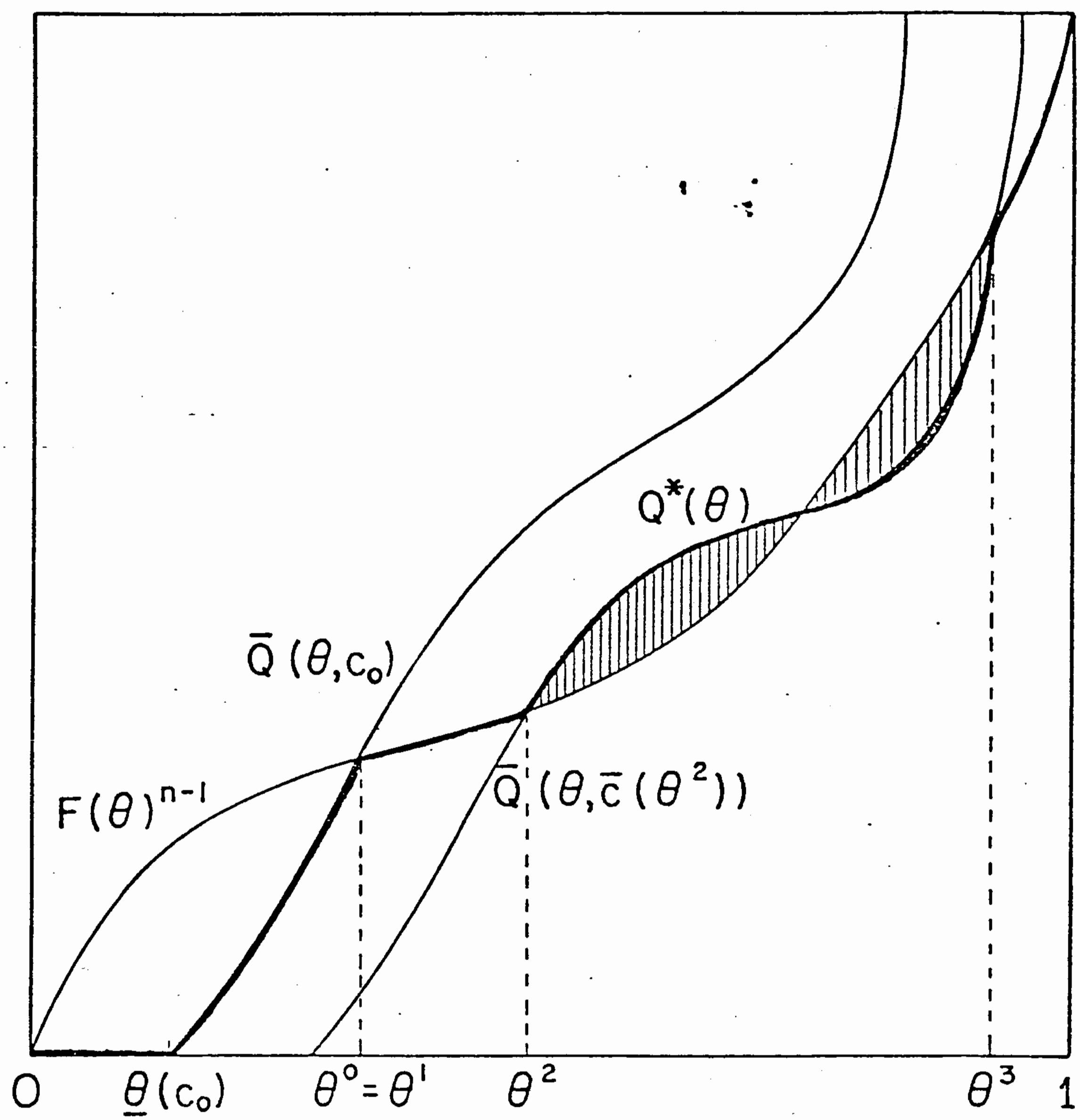

Figure 3

Steven Matthews

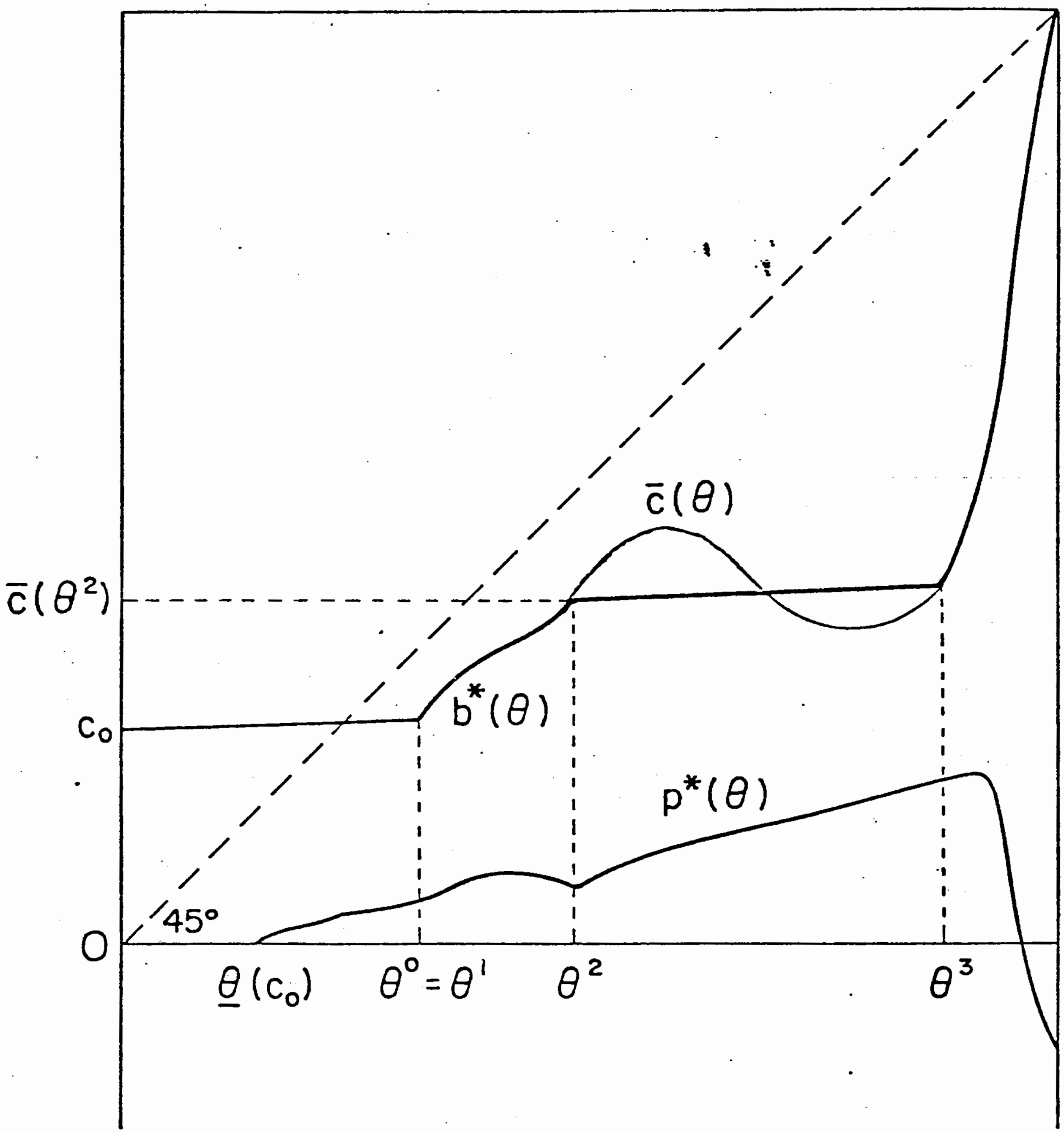

Figure 4

Steven Matthews