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THE DETERMINATION OF MARGINAL-COST PRICES UNDER A SET OF AXIOMS

by

Dov Samet and Yair Tauman

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ABSTRACT

This paper presents a set of axioms which characterizes a family of price mechanisms for consumption goods. Among these prices are marginal-cost prices and Aumann-Shapley prices. Using this characterization one can uniquely determine marginal cost prices (as well as Aumann-Shapley prices) under certain axioms. A discussion of the economic interpretation of the axioms is also provided.

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I. INTRODUCTION

The main purpose of this paper is to provide an axiomatic approach to marginal cost (MC) pricing and to point out the similarity between MC pricing and Aumann-Shapley (A-S) pricing. The latter is a cost-sharing price mechanism that is axiomatically derived by a set of five natural axioms discussed in [3] and [6].

In this paper we consider models in which there is one producer with a given technology and fixed input prices who produces a finite number of consumption goods. Thus, we can uniquely derive the cost function that describes the minimal cost of producing a given vector of consumption goods.

By a price mechanism $P(\cdot,\cdot)$ we mean a rule or a function that associates with each cost function F and vector α of quantities that are actually produced, a vector of prices:

$$P(F,\alpha) = \left(P_1(F,\alpha),\ P_2(F,\alpha),\ldots,P_m(F,\alpha)\right),$$

where m is the dimension of α and $P_i(F,\alpha)$ is the price of a unit of the i-th commodity. The interesting cost functions are those which do not consist only of costs which are directly cuased by each commodity, i.e., cost functions which are not of the form

$$F(x_1,x_2,\ldots,x_m) = F_1(x_1) + \ldots + F_m(x_m).$$

According to our definition a price mechanism does not depend on the utilities of consumers and it can be applied to the case in which information on the private tastes is not available.

Let us point out that although price mechanism are independed of demand, it may lead to prices which are compatible with demand, i.e., given the cost function F one may set prices p and a vector α such that α is demanded at p and moreover, p is determined by the price mechanism at α, i.e., $p = P(F,\alpha)$. For example, Mirman and Tauman [6] showed that the Aumann-Shapley price mechanism is compatible with demand as explained above. In the case that the price mechanism does not generate cost sharing prices rules to share the profit or cover the loss not through prices must be specified.

We shall consider in the sequel price mechanisms which obey the following four axioms. First we require that prices should be independent of the units of measurement (Axiom 1 below). This is a preliminary requirment of any pricing system. We also require that the price of a commodity for which the cost is positive is non negative (Axiom 4, below). This axiom reflects fairness toward the producer. Axiom 2 requires that two commodities having the same effect on the cost have the same price. This emphasizes the fact that a price of a commodity measures its "real value" in production.

Finally axiom 3 below enables us to calculate the prices via its factor of production: If the cost is broken into two factors, e.g., the cost of labor and the cost of raw material then, the prices can be obtained by adding the pries attributable to the two factors separately. (In section IV we will show that Axiom 3 can be replaced by other natural axioms).

In this paper we prove that strengthening slightly the positivity axiom (axiom 4* below) the set of four axioms (1,2,3, and 4*) uniquely characterize MC prices (this is theorem B below). On the way to proving this result we state a theorem (theorem A below) that is interesting in its own right, which characterizes the set of all price-mechanisms satisfying the four "basic" axioms 1,2,3 and 4. Among them is the marginal cost prices and the Aumann-Shapley price mechanism. The latter as mentioned, can be uniquely characterized by an additional requirement that cost equal revenue, i.e., cost is shared by the prices. This price mechanism was first proposed by Billera, Heath and Raanan [2] to set telephone billing rates which share the cost arising in serving the consumers; and it has been adopted for interal telephone billing at Cornell University. Later it was characterized axiomaticaly (independently) by Billera-Heath [3] and Mirman-Tauman [6]. Moreover using this characterization one can easily prove (Theorem C below) that the A-S price mechanism is the unique cost-sharing mechanism which obeys axioms 1-4. This provides an alternative proof for the main results in [3] and [6].

The conclusion is that axioms 1-4 are the key axioms in our study. Both MC prices and A-S prices obey these four axioms. Strengthening axiom 4 yields MC prices while imposing the cost-sharing requirement in addition to the four axioms (1-4) yields A-S prices.

Finally, we should mention that our work stems from ideas that were already developed in game theory. In [2] it is shown that for a given cost function F and vector α of quantities consumed, one can associate a non-atomic game $v(F,\alpha)$ in a way that its Shapley value will measure the effect of each unit of each commodity on the cost. If this magnitude is chosen to be the price of the commodity we get exactly the Aumann-Shapley price mechanism. However, from the same gave $v(F,\alpha)$ one can derive a price mechanism using, instead of the Shapley value, a wider concept of solution called the semi-value. Using the characterization of Dubey, Neyman and Weber [5] for all semi-values of a large space of non-atomic games, it turns out that the corresponding set of all price mechanisms derived by the set of all semi-values is exactly the set of all price mechanisms obeying Axiom 1 - Axiom 4. Thus our Theorem A should be considered as the parallel result of Dubey, Neyman and Weber formulated in purely economic terms.

II. THE AXIOMATIC APPROACH

We define here the notion of a price mechanism and we present four axioms by which we describe desirable mechanisms, then we characterize the set of all price mechanisms that satisfy these axioms. A price mechanism can lead to a profit as well as to a loss for the producer. However, how the profit is shared or how the loss is covered is out of our discussion.

We denote by E^m the m dimensional euclidean space and by E^m_+ the non-negative orthant of E^m.

Let $\mathcal{F}^m$ be the set of all real-valued functions F which are defined on E^m_+, satisfying $F(0) = 0$ and are continuously differentiable on E^m_+.

For any dimension m of the commodity space it is assumed that a producer who produces m commodities has a cost function [2] $F \in \mathcal{F}^m$ defined on E^m_+.

DEFINITION 1 A price mechanism is a function P which associates with each m, each $F \in \mathcal{F}^m$ and each vector of quantities α in E^m_+ a vectof of prices $P(F,\alpha)$ in E^m

$$P(F,\alpha) = \left(P_1(F,\alpha),\ldots,P_m(F,\alpha)\right).$$

We will characterize those price mechanisms that satisfy the following four axioms. The first axiom requires that the prices should be independent of the units of measurment. To illustrate it, suppose that F is a cost function of a producer who produces one commodity only. $F(x)$ is the cost to produce x units of this commodity. Assume that x is measured in kilogram. Let $G(y)$ be the cost function of the same producer where y is measured now in tons. Clearly

$$G(y) = F(1000y).$$

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2 It is worth mentioning that for the results we obtain in this section it is enough to consider only non decreasing cost functions.

According to our notations, if α tons are produced the price per one ton is $P(G,\alpha)$. Since α tons are 1000α kg the price per one kg is $P(F,1000\alpha)$. Therefore, a price mechanism $P(\cdot,\cdot)$ which is coherent with rescaling should obey:

$$P(G,\alpha) = 1000 \cdot P(F,1000\alpha).$$

and in general:

Axiom 1 (**Rescaling**). Let F be in $\mathcal{F}^m$. Let $\lambda_1,\lambda_2,\ldots,\lambda_m$ be m positive real numbers. Let G be a function in $\mathcal{F}^m$ defined by

$$G(x_1,x_2,\ldots,x_m) = F(\lambda_1 x_1,\lambda_2 x_2,\ldots,\lambda_m x_m).$$

Then, for each $\alpha \in E_+^m$ and each i, $1 \leq i \leq m$

$$P_i(G,\alpha) = \lambda_i P_i\big(F,(\lambda_1\alpha_1,\ldots,\lambda_m\alpha_m)\big).$$

The next axiom reflects the requirment that two commodities that are the "same" will have the same price. Since by definition price mechanism yields prices that depend on the cost function and not on demand functions it is clear that being the "same commodity" means playing the same role in the cost function. As an illustration, suppose that a car's producer, produces red and blue cars. He can represent his cost function as a two-variable function $F(x_1, x_2)$ where x_1 and x_2 are the quantities of red and blue cars respectively. But in fact, the cost to produce a red car is the same as the cost to produce a blue car. This can be formulated in this way: There is one-variable function G for which G(x) is the cost to produce a total of x cars (red ones, blue ones or both) and

$$F(x_1,x_2) = G(x_1+x_2).$$

In this case the axiom asserts that the price of a blue car is the same as the price of a red car, which is the price of a car, i.e.,

$$P_1\big(F,(\alpha_1,\alpha_2)\big) = P_2\big(F,(\alpha_1,\alpha_2)\big) = P(G,\alpha_1+\alpha_2).$$

In general:

<u>Axiom 2</u> (**Consistency**). Let F be in F^m and leg G be in F^1. If for every $x \in E_+^m$

$$F(x_1,x_2,\ldots,x_m) = G\left(\sum_{i=1}^{m} x_i\right)$$

then, for each i, $1 \leqslant i \leqslant m$, and for each $\alpha \in E_+^m$

$$P_i(F,\alpha) = P\left(G, \sum_{i=1}^{m} \alpha_i\right).$$

Suppose now that a given cost function F can be broken into two components say G - the cost of raw-materials and H - the cost of labor. In that case it is reasonable to require that the prices born of the cost F will be the sum of the prices born of G and H. (In section IV we show that this axiom can be replaced by other two natural axioms).

<u>Axiom 3</u> (**Additivity**). Let F,G and H be in F^m. If for each $x \in E_+^m$

$$F(x_1,\ldots,x_m) = G(x_1,\ldots,x_m) + H(x_1,\ldots,x_m),$$

then for each $\alpha \in E_+^m$

$$P(F,\alpha) = P(G,\alpha) + P(H,\alpha).$$

The last axiom asserts that prices of commodities for which the producer should invest money to produce are non negative. In a formal way;

<u>Axiom 4</u> (**Positivity**) Let $F \in F^m$ and let $\alpha \in E_+^m$, $\alpha \neq 0$. If F is non decreasing at each $x \leqq \alpha$, then[3]

$$P(F,\alpha) \geqq 0.$$

i.e., for each i, $1 \leqq i \leqq m$, $\quad P_i(F,\alpha) \geqq 0$.

Price mechanisms which satisfy these four axioms are of special form.

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3 By $x \leqslant \alpha$ we mean $x_i \leqq \alpha_i$ for every i, $1 \leqq i \leqq m$.

THEOREM A. **$P(\cdot,\cdot)$ is a price mechanism which obeys A1 – A4 if and only if there is a non-negative measure μ on $([0,1], \mathcal{B})$ ($\mathcal{B}$ is the set of all Borel subsets of $[0,1]$) such that for each m, for each $F \in \mathcal{F}^m$ and for each $\alpha \in E_+^m$, $\alpha \neq 0$**

$$(*)\quad P_i(F,\alpha) = \int_0^1 \frac{\partial F}{\partial x_i}(t\alpha)d\mu(t), \qquad i = 1,\ldots, m.$$

Moreover, for a given price mechanism $P(\cdot,\cdot)$ there is a unique measure μ which satisfies (*). In other words, (*) defines an one-to one mapping from the set of all non-negative measures on $([0,1], \mathcal{B})$ onto the set of all price mechanisms obeying A1-A4.

For an intuitive interpretation of the formula (*), assume that the vector α is produced in an homogenous way, starting from 0 and ending at α. Suppose also that along this production process each time a "small" proportion (an infinitesimal one) of α is produced, the m-th commodity is then charged by its current marginal production cost. The price of the m-th commodity once α has been produced, will be the average of these marginal costs weighted by the measure μ which corresponds to the given price mechanism. If this measure happens to be the atomic probability measure whose whole mass is concentrated at the point, $t = 1$, i.e., if $\mu(\{1\}) = 1$, the associated price mechanism $P(\cdot,\cdot)$ is the well-known marginal cost price mechanism. For any m, for any $F \in \mathcal{F}^m$ and for any $\alpha \in E_+^m$

$$P_i(F,\alpha) = \frac{\partial F}{\partial x_i}(\alpha), \qquad i = 1,\ldots,m.$$

If μ is chosen to be the Lebesgue measure on $[0,1]$ the associated price mechanism $P(\cdot,\cdot)$ is the Aumann-Shapley price mechanism (see [3] and [6])

$$P_i(F,\alpha) = \int_0^1 \frac{\partial F}{\partial x_i}(t\alpha)dt, \qquad i=1,\dots,m$$

These prices are the uniform average of the marginal cost along the diagonal $[0,\alpha]$.

We shall prove theorem A through Proposition 1 to Proposition 4, below. Let $P(\cdot,\cdot)$ be a price mechanism obeying A1 - A4.

PROPOSITION 1. **Let m be a positive integer. Let F and G be in $\mathcal{F}^m$ and let $\alpha \in E^m_+$. If $F(x) = G(x)$ for each $x \leqq \alpha$ then**

$$\mathbf{P(F,\alpha) = P(G,\alpha).}$$

Proof. It follows (e.g. by the additivity axiom) that $P(0,\alpha) = 0$. Therfore $P(-F,\alpha) = -P(F,\alpha)$ for each $F \in \mathcal{F}^m$. Let $H = F-G$. For each $x \leqq \alpha$ $H(x) = 0$ and therefore H and -H are non-decreasing at each $x \leqq \alpha$. Hence, by the positivity axiom $P(H,\alpha) \geqq 0$ and $P(-H,\alpha) \geqq 0$. Therefore $P(H,\alpha) = 0$ and hence $P(F,\alpha) = P(G,\alpha)$.

Let $\alpha \in E^m_+$, $\alpha \neq 0$, and let C_α be the box $\{x \in E^m_+ | x \leqq \alpha\}$. Let $\mathcal{F}^m(C_\alpha)$ be the set of all continuously differentiable functions on C_α with $F(0) = 0$. Each $F \in \mathcal{F}^m(C_\alpha)$ can be extended to a function on E^m_+ which is continuously differentiable (for a proof see for example Whitney [8]). If $\hat{F}$ and $\bar{F}$ are two such extension of F we have by Proposition 1

$$P(\hat{F},\alpha) = P(\bar{F},\alpha).$$

Therefore the function $P(\cdot,\alpha)$ on $\mathcal{F}^m$ can be considered also as a function on $\mathcal{F}^m(C_\alpha)$ which is positive and additive, i.e.,

$$P(F+G,\alpha) = P(F,\alpha) + P(G,\alpha),$$

for each F and G in $F^m(C_\alpha)$, and

$$P(F,\alpha) \geqq 0,$$

for each F which is non-decreasing on C_α.

From now on we will refer to $P(\cdot,\alpha)$ as a function on F^m as well as a function on $F^m(C_\alpha)$.

PROPOSITION 2. **There exists a non-negative measure μ on $([0,1], \mathcal{B})$ such that**

$$P(F,\alpha) = \int_0^1 \frac{\partial F}{\partial x}(t\alpha)d\mu(t)$$

for each $F \varepsilon F^1$ and $\alpha > 0$. Moreover, the measure μ is uniquely determined by the above equation.

Proof. We will first prove the proposition in case $\alpha = 1$. By the last remark we will consider here $P(\cdot, 1)$ as a function on $F^1([0,1])$ (the set of all functions in F^1 restricted to [0,1]) and we will prove this proposition for functions F in $F^1([0,1])$.

There is 1-1 linear mapping τ from C[0,1] (the class of continuous real functions on [0,1]) onto $F^1([0,1])$, defined by

$$\tau f(x) = \int_0^x f(t)dt, \qquad f \varepsilon C[0,1].$$

$P(\cdot,1)$ defines a functional ψ on C[0,1] by

(1) $\psi f = P(\tau f,1), \qquad f \varepsilon C[0,1].$

By the additivity and the positivity of $P(\cdot,1)$ we get the additivity and the positivity of ψ (positivity means here that $\psi f \geqq 0$ whenever $f \geqq 0$). By the additivity of ψ, $\psi(rf) = r\psi(f)$ for any rational number r. Using the positivity axiom it is easy to verify that the last equation holds for any real number. Thus ψ is linear and positive functional on C [0,1]. Apply the Riesz Representation Theorem for ψ (see for example [7,p. 40]) to get the existence of a unique non-negative measure μ on $([0,1], \mathcal{B})$ such that

$$\psi(f) = \int_0^1 f(t)d\mu(t).$$

Together with (1) we thus have

$$(2) \qquad P(F,1) = \int_0^1 \frac{\partial F}{\partial x}(t)d\mu(t).$$

Now let $F \varepsilon \mathcal{F}^1$ and let $\alpha > 0$. Define a function G in $\mathcal{F}^1$ by $G(x) = F(\alpha x)$. By (2) we have

$$P(G,1) = \int_0^1 \frac{\partial G}{\partial x}(t)d\mu(t).$$

By the rescaling axiom

$$P(G,1) = \alpha \cdot P(F,\alpha).$$

Therefore

$$P(F,\alpha) = \frac{1}{\alpha} \int_0^1 \frac{\partial G}{\partial x}(t)d\mu(t) = \int_0^1 \frac{\partial F}{\partial x}(t\alpha)d\mu(t).$$

<u>DEFINITION 2</u>. Let $C = C_\beta$ for $\beta >> 0$. The norm $\| \ \|_1$ on $\mathcal{F}^m(C)$ (the set of all continuously differentiable function F on C with $F(0) = 0$) is defined by

$$\| F \|_1 = \sum_{i=1}^{m} \| \frac{\partial F}{\partial x_i} \|_{\sup}$$

where the sup is taken over C.

It is easy to check that $\| \|_1$ indeed defines a norm on $\mathcal{F}^m(C)$. The property $\|F\|_1 = 0 \Longrightarrow F = 0$, follows from $F(0) = 0$.

PROPOSITION 3. **Let $C = C_\beta$ for $\beta \gg 0$. For each $\alpha \in C$ the function $P(\cdot,\alpha)$ is continuous in the norm $\| \|_1$ on $\mathcal{F}^m(C)$.**

Proof. Since $P(\cdot,\alpha)$ is additive it is sufficient to prove that if $(F_n)_{n=1}^{\infty}$ is a sequence of functions in $\mathcal{F}^m(C)$ satisfying $\|F_n\|_1 \to 0$ as $n \to \infty$ then,

$$P_i(F_n,\alpha) \to 0, \text{ as } n \to \infty,$$

for each i, $1 \leqq i \leqq m$.

From the additivity axiom one can easily verify that for each rational number λ and for each $F \in \mathcal{F}^m$

$$P(\lambda F,\alpha) = \lambda P(F,\alpha).$$

For each integer n let us choose a positive rational number ε_n such that

$$(3) \qquad \varepsilon_n \to 0, \text{ as } n \to \infty \text{ and } \|F_n\|_1 < \varepsilon_n.$$

Let R be the function in $\mathcal{F}^m$ defined by

$$R(x_1,x_2,\ldots,x_m) = \sum_{j=1}^{m} x_j.$$

By (3) for each i, $1 \leqq i \leqq m$, and for each $x \in C$

$$\frac{\partial(\varepsilon_n R - F_n)}{\partial x_i}(x) = \varepsilon_n - \frac{\partial F_n}{\partial x_i}(x) > 0$$

and

$$\frac{\partial(\varepsilon_n R + F_n)}{\partial x_i}(x) = \frac{\partial F_n}{\partial x_i}(x) + \varepsilon_n > 0.$$

Therefore, $\varepsilon_n R - F_n$ and $\varepsilon_n R + F_n$ are non decreasing functions on C. By the positivity and the additivity axioms we have

$$P_i(F_n,\alpha) \leqq P_i(\varepsilon_n R,\alpha) = \varepsilon_n P_i(R,\alpha)$$

and

$$P_i(F_n,\alpha) \geqq -P_i(\varepsilon_n R,\alpha) = -\varepsilon_n P_i(R,\alpha).$$

We thus have

$$P_i(F_n,\alpha) \to 0 \text{ as } n \to \infty,$$

and the proof of Proposition 3 is completed.

PROPOSITION 4. **For any polynomial p in $\mathcal{F}^m$ and for any $\alpha \in E^m_+$, $\alpha \neq 0$**

$$P(p,\alpha) = \int_0^1 \frac{\partial p}{\partial x_i}(t\alpha)d\mu(t).$$

Proof. Any polynomial p in $\mathcal{F}^m$ is a linear combination of monomials (i.e., polynomials of the form $x_1^{k_1},\ldots,x_m^{k_m}$). By formula 7.3 of [1,p.41] any monomial in $\mathcal{F}^m$ is a linear combination of polynomials of the form

(4) $\quad F(x_1,\ldots,x_m) = (n_1x_1+\ldots+n_mx_m)^\ell$

where the n_i's are non-negative integers and ℓ is a positive integer. Thus, any polynomial in $\mathcal{F}^m$ is a linear combination of polynomials of the form (4). The additivity of the three: the price mechanism, the differentiation operator and the integration operator makes it sufficient to prove the proposition for the function $a\cdot F$ where $a \in E^1$ and F is given by (4). Let us assume first, that for each i, $1 \leqslant i \leqslant m$, $n_i > 0$.

Let L be the function in $\mathcal{F}^1$ defined by

$$L(x) = x^\ell,$$

and let T be the function in $\mathcal{F}^m$ defined by

$$T(x_1,\dots,x_m) = L(\sum_{j=1}^{m} x_j).$$

Since

$$F(x_1,\dots,x_m) = T(n_1x_1,\dots,n_mx_m),$$

it follows by the rescaling and the consistency, that for each $\alpha \in E_+^m$ and each i, $1 \leqq i \leqq m$,

$$(5) \qquad P_i\big(F,(\alpha_1,\dots,\alpha_m)\big) = n_iP_i\big(T,(n_1\alpha_1,\dots,n_m\alpha_m)\big)$$

$$= n_iP(L, \sum_{j=1}^{m} n_j\alpha_j).$$

The positivity of the n_i's implies that for $\alpha \neq 0$ in E_+^m, $\sum_{j=1}^{m} n_j\alpha_j > 0$ and therefore by Proposition 2

$$(6) \qquad P(L, \sum_{j=1}^{m} n_j\alpha_j) = \int_0^1 \frac{dL}{dx}\,(t\cdot\sum_{j=1}^{m} n_j\alpha_j)d\mu(t).$$

Using the equality

$$F(x_1,\dots,x_m) = L(\sum_{j=1}^{m} n_jx_j).$$

we have

$$\frac{\partial F}{\partial x_i}(x_1,\dots,x_m) = n_i\,\frac{dL}{dx}\,(\sum_{j=1}^{m} n_jx_j).$$

This, together with (5) and (6), imply

$$(7)\qquad P_i\big(F,\ (\alpha_1,\ldots,\alpha_m)\big) = \int_0^1 \frac{\partial F}{\partial x_i}(t\alpha)d\mu(t). \qquad i=1,\ldots,m$$

In the general case however some of the n_i's might be zero. In that case we define for each $\varepsilon > 0$ a function F_ε in $\mathcal{F}^m$ by

$$F_\varepsilon(x) = \big((n_1+\varepsilon)x_1 +,\ldots,+(n_m+\varepsilon)x_m\big)^\ell.$$

Let $C = C_\beta$ for some $\beta \gg \alpha$. Clearly

$$(8)\qquad \|F_\varepsilon - F\|_1 \to 0 \text{ as } \varepsilon \to 0$$

Since the coefficients of the x_i's in F_ε are all positive we have by (7)

$$(9)\qquad P_i(F_\varepsilon,\alpha) = \int_0^1 \frac{\partial F_\varepsilon}{\partial x_i}(t\alpha)d\mu(t), \qquad i=1,\ldots,m.$$

The left hand side of (9) tends, by Proposition 3, to $P_i(F,\alpha)$ when $\varepsilon \to 0$. The right hand side of (9) tends, by (7), to $\int_0^1 \frac{\partial F}{\partial x_i}(t\alpha)d\mu(t)$, and so

$$P_i(F,\alpha) = \int_0^1 \frac{\partial F}{\partial x_i}(t\alpha)d\mu(t)$$

and the proof of Proposition 4 is complete.

Now, we are ready to prove Theorem A.

Proof of Theorem A. Let $P(\cdot,\cdot)$ be a price mechanism obeying the four axioms. Let $F \in \mathcal{F}^m$ and let $\alpha \neq 0$ be in E^m_+. Choose β with $\beta \gg \alpha$ and denote $C = C_\beta$. The polynomials in m variables are dense in $\mathcal{F}^m(C)$ with C^1 norm (for a proof see [4,p.68])(The C^1 norm is defined by

$$\|F\|_{C^1} = \|F\|_{\sup} + \sum_{i=1}^m \left\|\frac{\partial F}{\partial x_i}\right\|_{\sup}).$$

Therefore, there exists a sequence of polynomial $(\hat{p}_n)_{n=1}^\infty$ such that $\|\hat{p}_n - F\|_{C^1} \to 0$ as $n \to \infty$. Thus if

$$p_n = \hat{p}_n - \hat{p}_n(0),$$

then $p_n(0) = 0$ and $\|p_n - F\|_1 \to 0$ as $n \to \infty$.

By Proposition 3

(10) $P_i(p_n,\alpha) \to P_i(F,\alpha)$, as $n \to \infty$.

By Proposition 4

$$P_i(p_n,\alpha) = \int_0^1 \frac{\partial p_n}{\partial x_i}(t\alpha)d\mu(t).$$

Since $\|p_n - F\|_1 \to 0$

$$\frac{\partial p_n}{\partial x_i} \xrightarrow[\text{on } C]{\text{sup}} \frac{\partial F}{\partial x_i} \text{ as } n \to \infty.$$

Therefore,

$$\int_0^1 \frac{\partial p_n}{\partial x_i}(t\alpha)d\mu(t) \to \int_0^1 \frac{\partial F}{\partial x_i}(t\alpha)d\mu(t), \text{ as } n \to \infty.$$

This together with (9) imply

$$P_i(F,\alpha) = \int_0^1 \frac{\partial F}{\partial x_i}(t\alpha)d\mu(t).$$

Thus we proved the first direction of Theorem A. To prove the other direction we have to show that for any nonnegative measure μ the formula

(11) $P_i(F,\alpha) = \int_0^1 \frac{\partial F}{\partial x_i}(t\alpha)d\mu(t)$

which is defined for each $F \in \mathcal{F}^m$ and each $\alpha \in E^m_+$, yields a price mechanism that obeys the four axioms. For the rescaling axiom, assume that F and G are functions as defined in Axiom 1. Then

$$P_i(G,\alpha) = \int_0^1 \frac{\partial G}{\partial x_i}(t\alpha)d\mu(t) = \lambda_i \int_0^1 \frac{\partial F}{\partial x_i}(t\lambda_1\alpha_1,\ldots,t\lambda_m\alpha_m)d\mu(t)$$

$$= \lambda_i \cdot P_i\big(F,(\lambda_1\alpha_1,\ldots,\lambda_m\alpha_m)\big).$$

For the consistency axiom let F and G be as defined in Axiom 2. Then

$$\frac{\partial F}{\partial x_i}(t\alpha) = \frac{dG}{dx}\left(t\sum_{i=1}^{m}\alpha_i\right)$$

which implies that

$$P_i(F,\alpha) = \int_0^1 \frac{\partial F}{\partial x_i}(t\alpha)d\mu(t) = \int_0^1 \frac{dG}{dx}\left(t\sum_{i=1}^{m}\alpha_i\right)d\mu(t)$$

$$= P\left(G, \sum_{i=1}^{m}\alpha_i\right)$$

The additivity axiom follow immediately by the additivity of both the differentiation and the integration operators.

The positivity axiom follows from the inequality $\frac{\partial F}{\partial x_i}(x) \geqq 0$ which holds at any $x \leqq \alpha$ for any function F in F^m which is nondecreasing at each $x \leqq \alpha$. Thus for any non-negative measure μ on [0,1], (10) defines a price mechanism which satisfies the four axioms.

III. THE DETERMINATION OF THE MARGINAL COST PRICES BY SET OF AXIOMS

Let us strengthen the positivity axiom (Axiom 4).

Axiom 4* Let $F \in F^m$ and let $\alpha \in E_+^m$. If F is nondecreasing in a neighborhood of α then $P(F,\alpha) \geqslant 0$.

i.e. we require that the prices are non-negative at α even if F is non-decreasing in a neighborhood of α only.

It is clear that Axiom 4* implies Axiom 4 and therefore by Theorem A a price mechanism $P(\cdot,\cdot)$ which satisfies A1, A2, A3, and A4* is of the form

$$P_i(F,\alpha) = \int_0^1 \frac{\partial F}{\partial x_i}(t\alpha)d\mu(t).$$

But in fact the available set of mechanisms is now much smaller.

<u>THEOREM B</u>. **A price mechanism $P(\cdot,\cdot)$ satisfies A1, A2, A3, and A4* if and only if there is a constant $c > 0$ such that for each m, for each $F \varepsilon \mathcal{F}^m$ and $\alpha \varepsilon E_+^m$ $(\alpha \neq 0)$**

$$P_i(F,\alpha) = c \cdot \frac{\partial F}{\partial x_i}(\alpha) \quad i = 1,\ldots,m.$$

<u>Proof</u>. It is obvious that a price mechanism $P(\cdot,\cdot)$ defined by

$$P_i(F,\alpha) = c \cdot \frac{\partial F}{\partial x_i}(\alpha) \qquad c \geqslant 0,$$

obeys the four mentioned axioms.

Assume now that a price mechanism $P(\cdot,\cdot)$ satisfies the four axioms. Then, by Theorem A there exists a non-negative measure μ on $([0,1], \mathcal{B})$ such that

$$P_i(F,\alpha) = \int_0^1 \frac{\partial F}{\partial x_i}(t\alpha)d\mu(t)$$

For each m, $F \in \mathcal{F}^m$ and $\alpha \in E_+^m$.

Notice now that if F is a constant in a neighborhood of α then $P_i(F,\alpha) = 0$ (apply axiom 4* for F and $-F$). Define now for each ε, $1 > \varepsilon > 0$ a function $f_\varepsilon : E_+^1 \to E_+^1$ by

$$f_\varepsilon(x) = \begin{cases} 1 & 0 \leq x \leq 1-\varepsilon \\ -\frac{2}{\varepsilon} x + \frac{2}{\varepsilon} - 1 & 1-\varepsilon \leq x \leq 1 - \frac{\varepsilon}{2} \\ 0 & 1 - \frac{\varepsilon}{2} \leq x \end{cases}$$

since f_ε is continuous the function F_ε defined by

$$F_\varepsilon(x) = \int_0^x f_\varepsilon(t)dt$$

is in $\mathcal{F}^1$. F_ε is constant in a neighborhood of $\alpha=1$ therefore

$$P(F_\varepsilon,1) = 0.$$

Hence

$$\int_0^1 f_\varepsilon(t)d\mu(t) = 0.$$

On the other hand

$$\int_0^1 f_\varepsilon(t)d\mu(t) \geq \mu\big([0,1-\varepsilon]\big) \geq 0.$$

Therefore for each $0 < \varepsilon < 1$

$$\mu([0,1-\varepsilon]) = 0.$$

Thus, $\mu([0,1)) = 0$ which implies that $\mu([0,1]) = \mu(\{1\})$, and the proof is complete.

Corollary If in addition to axioms 1,2,3, and 4* we require for the identity one variable function $H(x) = x$ that $P(H,1) = 1$ then the MC pricing is the only price mechanism which obeys these requirements (i.e. in this case the constant c of the Theorem B must equal 1).

Proof. According to Theorem B $P(H,1) = c \cdot \frac{dH}{dx}(1) = 1$. Thus $c = 1$.

Finally, let us return back to the four original axioms A1-A4 and add up another axiom that requires cost sharing in the model (total cost equal total revenue).

Axiom 5 **(Cost sharing)** for each m, each $F \varepsilon \mathcal{F}^m$ and each $\alpha \varepsilon E^m_+$

$$\alpha \cdot P(F,\alpha) = F(\alpha).$$

THEOREM C. **There is a unique price mechanism $P(\cdot,\cdot)$ which satisfies A1-A5. $P(\cdot,\cdot)$ is given by**

$$\mathbf{P(F,\alpha) = {}_0\int^1 \frac{\partial F}{\partial x_i}(t\alpha)dt.}$$

$P(\cdot,\cdot)$ is called the Aumann-Shapley price mechanism.

This result was previously stated (independently) by Billera-Health [3] and by Mirman-Tauman [6]. However it is also an immediate corollary of Theorem A above.

Proof. First it is clear that the Aumann-Shapley price mechanism satisfies A1-A5 since the cost sharing axiom is implied by

$$\frac{\partial F}{\partial t}(t\alpha) = \sum_i \alpha_i \frac{\partial F}{\partial x_i}(t\alpha),$$

and therefore

$$\sum_i \alpha_i P_i(F,\alpha) = \sum_i \alpha_i \int_0^1 \frac{\partial F}{\partial x_i}(t\alpha)\,dt = \int_0^1 \frac{\partial F}{\partial t}(t\alpha)\,dt =$$

$$= F(\alpha) - F(0) = F(\alpha).$$

Assume now that $P(\cdot,\cdot)$ is a price mechanism obeys A1–A5. By Theorem A there is a non-negative measure μ on $([0,1]),\mathcal{B})$ s.t.

$$(12) \quad P_i(F,\alpha) = \int_0^1 \frac{\partial F}{\partial x_i}(t\alpha)d\mu(t),$$

for each m, $1 \leqslant i \leqslant m$, $F \in \mathcal{F}^m$ and $\alpha \in E_+^m$, $(\alpha \neq 0)$. Since $P(\cdot,\cdot)$ satisfies A5

$$P(F,1) = \frac{F(1)}{1} = \int_0^1 \frac{dF}{dx}(t)dt,$$

for each $F \in \mathcal{F}^1$. Therefore by (12) we get

$$\int_0^1 \frac{dF}{dx}(t)dt = \int_0^1 \frac{dF}{dx}(t)d\mu(t).$$

It follows then, that the measure μ and the Lebesgue measure coincide on $C[0,1]$ as linear functionals on $C[0,1]$. Therefore by Riesz Representation Theorem these two measures are the same.

IV. SEPARABILITY REPLACING ADDITIVITY

In this section we show that the additivity axiom (A3) can be replaced by two natural axioms. The first is very similar in spirit to the consistency axiom (A2), and the second deals with a set of commodities that can be separated into two sets, which are independently produced.

For the first axiom consider a producer who produces two commodities with a the cost function $F(x,y)$. The producer can decide to generate a new commodity consists of the other two: Each unit of the new commodity consists of a unit of the first commodity and a unit of the second one. The cost function G for the new commodity satisfies

$$G(x) = F(x,x).$$

It is only natural to ask that the price per unit of the new commodity will be sum of the two prices of the original commodities. In general:

<u>Axiom I</u> <u>**(Aggregation)**</u>. Let $F(x_{11},\ldots,x_{1n_1},x_{21},\ldots,x_{2n_2},\ldots,x_{m1},\ldots,x_{mn_m})$ be in F^{ℓ} where $\ell = \sum_{i=1}^{m} n_i$. Let G be the function in F^m defined by

$$G(x_1,\ldots,x_m) = F(\underbrace{x_1,\ldots,x_1}_{n_1},\ \underbrace{x_2,\ldots,x_2}_{n_2},\ldots,\ \underbrace{x_m,\ldots,x_m}_{n_m}).$$

Then for each i, $1 \leq i \leq m$,

$$P_i\big(G,(\alpha_1,\ldots,\alpha_m)\big) = \sum_{j=1}^{n_i} P_{ij}\big(F,(\alpha_1,\ldots,\alpha_1,\ \alpha_2,\ldots,\alpha_2,\ldots,\alpha_m,\ldots,\alpha_m)\big).$$

For the second axiom assume that a producer produces $n+m$ commodities with cost function F. The first n commodities are n types of cars and the remaining are m types of shoes, which are independently produced. i.e. there are two cost functions $G(x_1,\ldots,x_n)$ and $H(y_1,\ldots,y_m)$ such that

$$F(x_1,\ldots,x_n,x_{n+1},\ldots,x_{n+m}) = G(x_1,\ldots,x_n) + H(x_{n+1},\ldots,x_{n+m})$$

The axiom we state requires that a price of a commodity should be born only from that part of the cost function that it affects. In order to formulate this axiom we shall use the following notation. Let $N = \{i_1, i_2, \ldots, i_n\}$, where $i_1 < i_2 < \ldots < i_n$, be a subset of $\{1,\ldots,m\}$, and let $x \varepsilon E^m$. Denote by x_N the vector in E^n defined by:

$$X_N = (x_{i_1}, \ldots, x_{i_n}).$$

Axiom II (**Separability**) Let N_1 and N_2 be disjoint sets with n_1 and n_2 elements respectively such that $N_1 \cup N_2 = \{1,\ldots,m\}$. Let F,G and H be functions defined on E_+^m, $E_+^{n_1}$ and $E_+^{n_2}$ respectively. If for each $x \varepsilon E_+^m$

$$F(x) = G(x_{N_1}) + H(x_{N_2})$$

then for each $\alpha \varepsilon E_+^m$

$$P_{N_1}(F,\alpha) = P(G,\alpha_{N_1})$$

and

$$P_{N_2}(F,\alpha) = P(H,\alpha_{N_2})$$

PROPOSITION 5 **Axioms I and II imply the additivity axiom A3.**

Proof. Let F,G and H be functions defined on E_+^m such that

$$F(x) = G(x) + H(x)$$

for each $x \varepsilon E_+^m$. Define a function L on E_+^{2m} by

$$L(x_1, x_2, \ldots, x_{2m}) = G(x_1, x_3, \ldots, x_{2m-1}) + H(x_2, x_4, \ldots, x_{2m})$$

Denote by N_1 and N_2 the sets of odd and even numbers, in the set $\{1,\ldots,2m\}$, respectively. By the separability axiom for each $\alpha \varepsilon E_+^m$

$$(13)\begin{cases} P_{N_1}(L,\alpha) = P(G,\alpha_{N_1}), \\ \\ P_{N_2}(L,\alpha) = P(H,\alpha_{N_2}). \end{cases}$$

For $x = (x_1, \ldots, x_m)$ let us denote $\hat{x} = (x_1, x_1, \ldots, x_m, x_m)$. By (13) for each

$\alpha \varepsilon E_{+}^{m}$

$$(14)\quad \begin{cases} P_{N_1}(L,\hat{\alpha}) = P(G,\alpha), \\ \\ P_{N_2}(L,\hat{\alpha}) = P(H,\alpha). \end{cases}$$

By the definition of L it follows that for each $x \varepsilon E_{+}^{m}$

$$L(\hat{x}) = G(x) + H(x) = F(x).$$

From the aggregation axiom we deduce that for each i, $1 \leqslant i \leqslant m$, and for each $\alpha \varepsilon E_{+}^{m}$

$$(15)\quad P_i(F,\alpha) = P_{2i-1}(L,\hat{\alpha}) + P_{2i}(L,\hat{\alpha})$$

but by (14)

$$(16)\quad \begin{cases} P_{2i-1}(L,\hat{\alpha}) = P_i(G,\alpha), \\ \\ P_{2i}(L,\hat{\alpha}) = P_i(H,\alpha). \end{cases}$$

Therefore from (15) and (16)

$$P_i(F,\alpha) = P_i(G,\alpha) + P_i(H,\alpha),$$

for each $\alpha \varepsilon E_{+}^{m}$ and $1 \leqslant i \leqslant m$. Thus the proof of Proposition 5 is complete.

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