ON THE COMPLEXITY OF POINT COVERING AND LINE COVERING

by

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Abstract The problems of minimum covering of points by lines and minimum covering of lines by points are proved to be *strongly* NP-hard.

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The following two problems arise in the context of designing control systems which involve the utilization of photoelectric cells, laser beams, etc.

1. **POINT COVERING (PC):** A set of points $(x_1,y_1),\ldots,(x_p,y_p)$ (x_i,y_i rationals $i=1,\ldots,p$) is given. Find a collection of straight lines $\{\ell_1,\ldots,\ell_r\}$ of minimum cardinality, such that (x_i,y_i) lies on at least one ℓ_j.

2. **LINE COVERING (LC):** A set of straight lines $L_1,\ldots,L_r$ is given. Find a set of points $\{(x_1,y_1),\ldots,(x_p,y_p)\}$ of minimum cardinality such that each L_j contains at least one (x_i, y_i).

In view of the renowned duality between lines and points (see [3], for example) which is discussed later, the two problems are obviously very close in nature.

We first mention briefly the trivial cases. First, PC is trivial when no three points are colinear, in which case $\lceil p/2 \rceil$ lines are necessary to cover all points. Analogously, LC is trivial when every subset of three lines has an empty intersection and there are no parallel lines.

If there may be parallel lines, but still no intersection points of three lines or more, then an optimal solution for LC can be easily computed as follows. Partition the set of lines into classes $R_1,\ldots,R_s$ such that two lines are parallel if and only if they belong to the same class R_i. Let $r_i = |R_i|$, $i=1,\ldots,s$, and assume $r_1 \geqq r_2 \geqq ,\ldots, r_s \geqq 1$. Now, select an arbitrary line from R_1 and an arbitrary line from R_2. The point of intersection of two selected lines will belong to the final solution. Next, drop the two selected lines from R_1 and R_2, rename the classes so as to conform with the requirement

$r_i \geq r_j$ for $i < j$ and continue in the same manner. We observe that this is in fact a particular case of a well-known scheduling problem, namely minimal-length scheduling of unit-execution-time tasks with tree-structured precedence constraints (see [1, p.54], which is solvable by the "level strategy". The embedding of our problem in the scheduling problem is by viewing each line as a task where members of the same R_i form a chain and the different chains are disjoint. The number of machines is two and the interpretation is that at each time unit at most two lines can be processed and this is feasible if they are not parallel. Furtheremore, the value of the optimal solution is simply $\max(r_1, \lceil r/2 \rceil)$. This is easily proved by induction, distinguishing between the case $r_1 > r_3$ (where both r_1 and $\lceil r/2 \rceil$ decrease by one after the first time unit) and the case $r_1 = r_3$ (where only $\lceil r/2 \rceil$ decreases but $\lceil r/2 \rceil > r_1$, so that $\max(r_1, \lceil r/2 \rceil)$ decreases in any case).

We now turn to the NP-hardness of the problems in the general case. First it is easily verified that both problems are in NP.

We now reduce 3-satisfiability to PC. Let $E_1 \wedge \ldots \wedge E_m$ be an instance of 3-satisfiability, where $E_j = x_j \vee y_j \vee z_j$, $\{x_j, y_j, z_j\} \subset \{v_1, \bar{v}_1, \ldots, v_n, \bar{v}_n\}$, $j=1,\ldots,m$. Assume $|\{v_i, \bar{v}_i\} \cap \{x_j, y_j, z_j\}| \leq 1$. The general idea of the reduction is as follows. We shall construct a set of $m + nm^2$ points, m corresponding to the clauses $E_1, \ldots, E_m$ and m^2 ones corresponding to each pair of variables $(v_i, \bar{v}_i)$. Also, a set of $2nm$ lines will be constructed with the following properties.

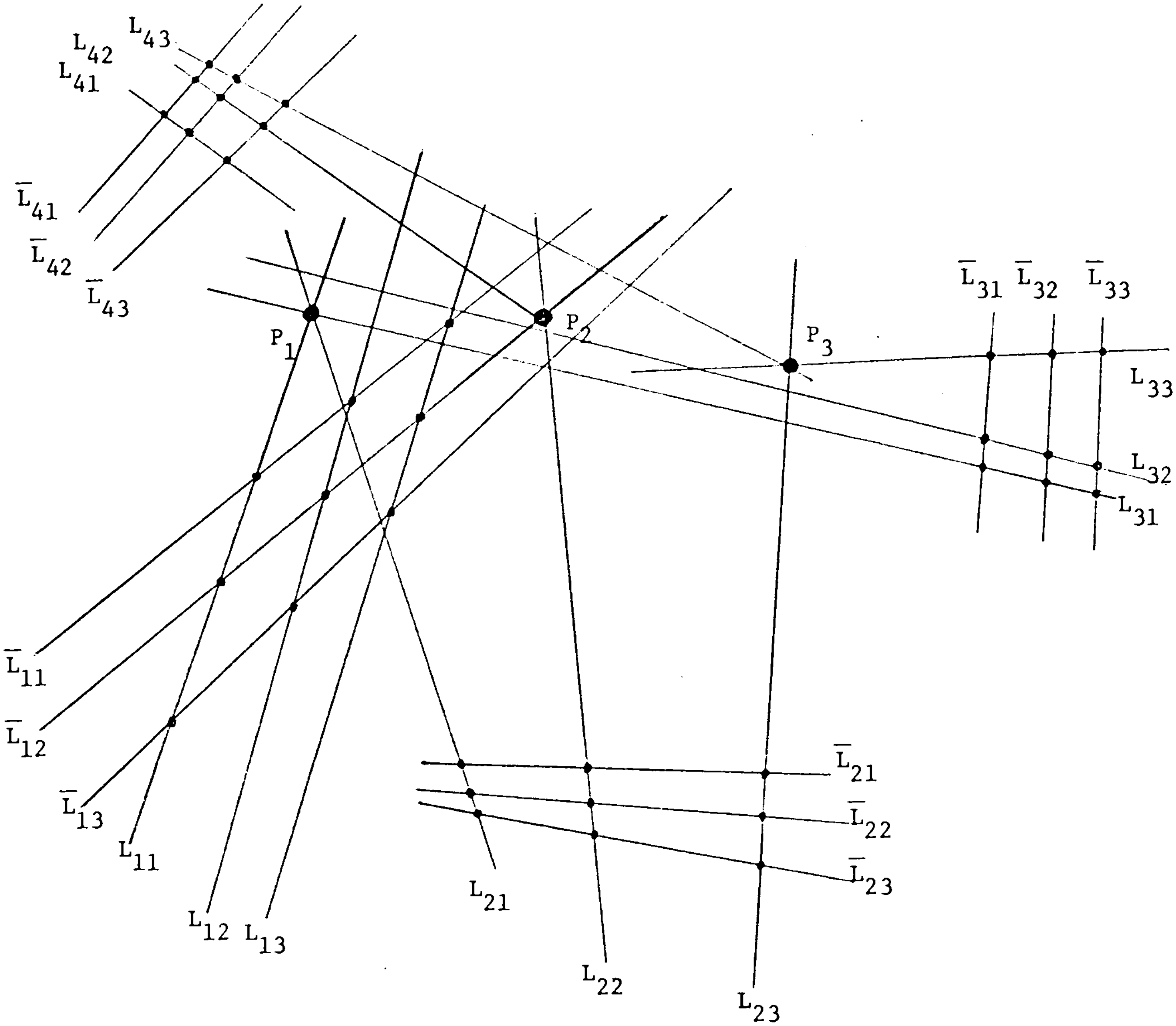
1. Each clause E_j is represented by a point P_j.
2. Each pair of variables $(v_i, \bar{v}_i)$ is represented by a grid of m^2 points $P^i_{k\ell} (1 \leq k, \ell \leq m)$.
3. For each $i (i=1,\ldots,n)$ and $j (j=1,\ldots,m)$, the points $P^i_{1j}, \ldots, P^i_{mj}$ lie

on a straight line denoted by L_{ij} and the points $P^i_{j1},\ldots,P^i_{jm}$ be on a straight line denoted by $\bar{L}_{ij}$.

4. Except for the lines L_{ij}, $\bar{L}_{ij}$ $(i=1,\ldots,n;\ j=1,\ldots,m)$, no other straight line of the plane contains more than two points of the set

$$\{P^i_{k\ell}:\ i=1,\ldots,n;\ k=1,\ldots,m;\ \ell=1,\ldots,m\} \cup \{P_1,\ldots,P_m\}.$$

5. For every $j(j=1,\ldots,m)$ the point P_j lies on the line L_{ik} if and only if $j=k$ and $v_i \varepsilon \{x_j,\ y_j,\ z_j\}$ and P_j lies on $\bar{L}_{ik}$ if and only if $j=k$ and $\bar{v}_i \varepsilon \{x_j, y_j, z_j\}$.



Example:

$$E_1 = v_1 \vee v_2 \vee v_3$$

$$E_2 = \bar{v}_1 \vee v_2 \vee v_4$$

$$E_3 = v_2 \vee v_3 \vee v_4$$

The above five properties establish the reduction by the following arguement. The points of the form $P^i_{k\ell}$ cannot be covered by less than nm lines, since no straight line contains more than m of them and altogether they number nm^2. Moreover, to achieve that number, for every i (i=1,...,n), the points $P^i_{k\ell}(1 \leqq k,\ell \leqq m)$ must be covered either by the lines $L_{ij}(j=1,\ldots,m)$ or by the lines $\bar{L}_{ij}(j=1,\ldots,m)$. No other collection of m lines can cover the collection of m^2 points $P^i_{k\ell}$ $(1 \leqq k,\ell \leqq m)$(assuming $m > 2$). We claim that $E_1 \wedge \ldots \wedge E_m$ is satisfiable if and only if the entire collection of points $\{P_1,\ldots,P_m\} \bigcup \{P^i_{k\ell}: i=1,\ldots,n;\ k=1,\ldots,m; \ell=1,\ldots,m\}$ can be covered by nm lines. For, the choice between $\{L_{ij}\}^m_{j=1}$ and $\{\bar{L}_{ij}\}^m_{j=1}$ for a given i simply corresonds to the assignment of a truth-value to $(v_i,\bar{v}_i)$. Specifically, for each i, v_i is true if and only if $\{L_{ij}\}$ is chosen to cover the m^2 points $P^i_{k\ell}$.

Finally, we have to discuss the actual construction of the points P_j and $P^i_{k\ell}$. We will construct points with rational coordinates, maintaining the enumerators and the denominators separately. The numerical values of all the enumerators and denominators will be bounded by a polynomial in m and n. First, let $P_j = (j, j^2)$, j=1,...,m. Thus, no three of the points $P_1,\ldots,P_m$ are colinear. The construction of the points $P^i_{k\ell}$ will be carried out with the aid of the lines L_{ij}, $\bar{L}_{ij}$ as follows. For each i(i=1,...,n), $P^i_{k\ell}$ is the point of intersection of L_{ik} with $\bar{L}_{i\ell}$. The lines L_{ij}, $\bar{L}_{ij}$ are successively constructed in the order $L_{11},\ldots,L_{1m},\bar{L}_{11},\ldots,\bar{L}_{1m}$, $L_{21},\ldots,L_{2m}$, $\bar{L}_{21},\ldots$ so as to satisfy properties 3,4,5. When a specfic line L_{ij} has to be constructed the following conditions should be satisfied: (i) L_{ij} should contain P_i if and only if $v_i \varepsilon \{x_j,y_j,z_j\}$. (ii) L_{ij} should not contain any previously constructed point of the form P_k (except possibly for P_j as explained before) or $P^i_{k\ell}$. (iii) L_{ij} should not coincide with any previously constructed line.

When a specific line $\bar{L}_{ij}$ has to be constructed the following conditions should be satisfied: (i) $\bar{L}_{ij}$ should contain P_j if and only if $\bar{v}_i \varepsilon \{x_j, y_j, z_j\}$. (ii) $\bar{L}_{ij}$ should not contain any previously constructed point (except possibly for P_j) (iii) $\bar{L}_{ij}$ should not contain a point of intersection of two lines of the form L_{ik}, $L_{i\ell}$ (in order for the two points P^i_{kj}, $P^i_{\ell j}$ to be distinct). (iv) $\bar{L}_{ij}$ should not contain a point of intersection between some L_{ik} and another line which contains at least two previously constructed points (in order to satisfy condition 4; the intersection $\bar{L}_{ij}$ with L_{ik} becomes the point P^i_{kj}. (v) $\bar{L}_{ij}$ should not be parallel to any L_{ik}, in order to ensure the existence of the point P^i_{kj}.

Thus, a typical step is that a line has to be constructed so as to (possibly) contain one specified point of the P_j's and not any other point from a finite collection of "forbidden" points, and also so as not to parallel any one of a finitely numbered lines. Suppose that we always construct the line whose slope is the integer closest to zero among the feasible slopes. The number of "forbidden" slopes is obviously bounded by some polynomial in m and n and hence the slope of every constructed line is an integer whose absolute value is bounded by that polynomial. If the constructed line also crosses through one of the P_j's (whose coordinates are of the form (j,j^2)) then the coefficients of its equation will be polynomially bounded integers. Similarly, if the line should not cross through any P_j, then we may construct it so as to cross through an integer point, which is not forbidden, whose distance from the origin is minimal. It follows that the coordinates of such a point are polynomially bounded and hence all our constructed lines will have polynomially bounded integers as their coefficients. This implies that all the points $P^i_{k\ell}$ will have coordinates which are rationals with polynomially bounded enumerators and denominators. This establishes that PC is <u>strongly</u>

NP -hard.

To establish that LC is strongly NP-hard we reduce PC to LC by using the point-line duality argument. Specifically, given the points $(a_1,b_1),\ldots,(a_n,b_n)$, we first find a translation $(a_i,b_i) = (a_i + a,\ b_i + b)$ that will assure that no two points are colinear with the origin. Next, we represent the point (a_i,b_i) by a line $a_i x + b_i y + 1 = 0$. Thus, two lines corresponding to two distinct points are not parallel and the main property is that points are colinear if and only if their corresponding lines all interesect at a single point.

References

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