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MAXIMIZING EXPECTED UTILITY
AND
THE RULE OF LONG RUN SUCCESS

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1. INTRODUCTION

In the utility theory developed by Von Neuman and Morgenstern [7] (see also Marschak [5]), the preference ordering of probability distributions by an individual is used to derive his utility indices of a finite set of prizes $x_1, \ldots, x_n$ to which the probabilities are assigned. It is shown there that if the ordering of probability distributions satisfies certain, so called, rationality conditions or rules (rules that any rational man, according to Marschak [5] should follow in ordering the probability distributions), then utility indices, u_i^v, constant up to positive linear transformations, can be assigned to the prizes x_i in such a way that the individual orders the probability distributions as if he were maximizing the expected value of the utilities u_i^v.

Professor Marschak in [6], pp. 504-5, proposes the alternative approach of starting by defining what he calls the rule or aim of long run success and then trying to study whether this aim can be satisfied by applying the rule of maximizing expected utility. In other words, what Professor Marschak proposes is to start out with the intuitively very appealing common sense definition that "the best policy or rule is the one that succeeds in the long run" and then try to determine whether the rule of maximizing expected utility is a best rule according to this common sense definition.

In this paper we plan to follow a similar approach. To this end we will proceed as follows: First, I will define how the individual orders sequences of prizes $x_{h_1}, \ldots, x_{h_i}, \ldots, x_{h_k}$, for k = 1,2,.... Then, from the ordering of these sequences of prizes, utility indices $u_i^c = u^c(x_i)$ (i =1,..., n) will be obtained, which are constant up to positive linear transformations, and possess the property that the sequence $x_{h_1}, x_{h_2}, \ldots, x_{h_k}$ is at least as good as the sequence $x'_{h_1}, x'_{h_2}, \ldots, x'_{h_k}$ if and only if $\sum_{i=1}^{k} u^c(x_{h_i}) \geq \sum_{i=1}^{k} u^c(x'_{h_i})$.

By using these utility indices u_i^c, utility functions for finite sequences of prizes $U_k^c\,(x_{h_1}, \ldots, x_{h_k})$ are then defined in the obvious way. The rule of long run success is then formally stated. Finally, we will show that given two lotteries L and L' assigning the prizes x_i $(i = 1, \ldots, n)$ with probabilities P_i and P'_i respectively $(0 \leq P_i \leq 1,\ 0 \leq P'_i \leq 1,\ \sum_{i=1}^{n} P_i = \sum_{i=1}^{n} P'_i = 1)$, the rule that prescribes choosing L over L' whenever $\sum_{i=1}^{n} u_i^c P_i$ is greater than $\sum_{i=1}^{n} u_i^c P'_i$ implies that the rule of long run success, as stated by us, will be satisfied. We will also discuss the Van Neuman and Morgenstern rationality axioms and the utility indices u_i^v derived from them in relation to the utility indices u_i^c derived from our model and the rule of long run success.

I would like to point out here that we, like Marschak, do not assume the successive random variables $U_1^c, U_2^c, \ldots, U_k^c$, to be statistically independent. Neither do we assume, unlike Marschak, that their corresponding variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2$ converge to zero as $k \to \infty$.

Given the extremely subtle character of the problem of rationalizing decision making under uncertainty, we will rely for the developing of our model mainly on a very simple example. We will do this, even at the expense of sacrificing generality, with the hope of reducing the probability of misunderstandings to acceptable levels.

2. THE ORDERING OF SEQUENCES OF PRIZES

The theory of utility to be presented here is a simplified version of the one developed by this author in [2] (See also [3]).

Consider the following simple situation. An individual, say Mr. A, will be offered every day after lunch either x_1, a cup of coffee, or x_2, a cup of tea, or x_3, a cup of camomile. Let us assume that Mr. A prefers to have, after lunch, a cup of coffee to a cup of tea, and a cup of tea to a cup of camomile. Thus he prefers x_1 over x_2 and x_2 over x_3. Suppose that our hostess for some reason, can serve each day only coffee or only tea or only camomile, but that she can decide every day which of the three different drinks will be served. Suppose further that besides Mr. A, there are other guests with tastes different from his. And that our hostess, who is a fair minded person, wants to know more about the preferences of Mr. A (and the other guests of course) in order to find out what could be a fair proportion of days serving coffee, days serving tea, and days serving camomile. Thus, she asks Mr. A the following question: Suppose that one day I serve tea which, I know, is not your favorite, instead of coffee that you like the best. But other days, I can serve tea instead of camomile, that you like the least, and thus compensate you for the loss of pleasure suffered the day that I give you tea instead of coffee. How many days do I have to serve tea instead of camomile in order for you to feel compensated? We will assume that Mr. A can answer this question precisely. We will also assume that if during a period of a <u>finite</u> number of days, say k, coffee is served during k_1 days, tea during k_2 days and camomile during k_3 days (of course $k_1 + k_2 + k_3 = k$), Mr. A would not care regarding the order in which the different drinks are served whenever the number of times that he has coffee, tea and camomile remain k_1, k_2 and k_3.

These and other less crucial conditions will be presented below in a formal way in the form of axioms characterizing the preferences of an individual. And from these axioms utility indices $u_i^c = u^c(x_i)$ will be derived which are constant up to positive linear transformations, and such that, for any finite k, the sequence $x_{h_1}, x_{h_2}, \ldots, x_{h_i}, \ldots, x_{h_k}$ is at least as good as the sequence $x'_{h_1}, x'_{h_2}, \ldots, x'_{h_i}, \ldots, x'_{h_k}$ if and only if

$$\sum_{i=1}^{k} u^c(x_{h_i}) \geq \sum_{i=1}^{k} u^c(x'_{h_i}).$$

We will proceed now to this formalization.

Let $X = \{x_1, \ldots, x_i, \ldots, x_n\}$ represent the set of the n alternative prizes or consumption incomes. Thus in our previous example, n was set equal to 3 and x_1 represented a cup of coffee, x_2 a cup of tea and x_3 a cup of camomile. Let $N = \{1, \ldots, i, \ldots, n\}$ be the set of the n natural numbers 1, 2, ..., i, ..., n.

Write $X^\infty = X \times X \times \cdots$: the countable infinite Cartesian product of X times itself; and let x^∞ be a generic element of X^∞. Thus in a more explicit way x^∞ represents an infinite sequence $x_{h_1}, \ldots, x_{h_i}, \ldots,$ with $h_i \in N$ for all i.

Our first axiom now states:

<u>Total Ordering Axiom</u>. It is assumed that the individual possesses a

preference ordering Q (also written $\geq$) [1] of the elements of X^{∞}, i.e. Q is a transitive, reflexive, connected relation defined on X^{∞}. In economic terms, Q is the relation "is as good as".

In terms of our example what the total ordering axiom means is that Mr. A imagines himself having an after lunch drink day after day, after day... forever (he has not learned yet that he is going to die some day), and that he is able to rank, according to his tastes, all the possible alternative infinite sequences.

Remark. It should be emphasized that the preference ordering among infinite sequences of prizes postulated in the previous axiom is to be considered here in a timeless context. It is true that it is difficult to think of a person receiving a sequence of prizes without time being involved. We used a time reference in our example regarding Mr. A's after lunch drinks and we will use a time reference again when presenting further examples in order to facilitate their description and understanding. But we will assume that our individual makes abstraction of the time intervals elapsing between successive consumptions (because, say, these time intervals were previously fixed by custom or otherwise and cannot be changed) and cares only about the sequences of consumptions that he can receive.

Let us turn at this point to the permutation axiom. This axiom roughly states that the preferences of the individual with regard to the infinite sequences $x^{\infty} = x_{h_1}, \ldots, x_{h_i}, \ldots;\ h_i \in N$ for $i = 1, 2, \ldots$, do not depend on the way in which the prizes of any finite part of the sequence are arranged. In other words, the individual does not care if the order of any finite number of prizes of a sequence is altered. This is certainly a very strong restriction on the preferences of the individual. To begin with, habit forming consumptions have to be excluded from our set of prizes. Also, even if we exclude habit forming consumptions as prizes, we might still have the case where the order in which the prizes are received and consumed really matters. Consider for instance, the situation where a consumer is offered dinners for two years, half of them fish food and the other half meat food. There is no reason, of course, to assume that our consumer should be indifferent between, say, dining on fish during the whole first year and meat during the whole second year, or dining on meat one night and fish the following night, etc., throughout the two years. This last difficulty can be overcome to a certain extent by assuming that each prize represents not a concrete consumption as a cup of coffee or a steak, but a kind of "opportunity set." as it is the case when each prize represents a certain amount of money. Thus, using a gastronomic example again, we can consider prize x_1 as a menu of fish and meat dishes of high quality from which the consumer can choose, x_2, as a similar menu but with lower quality, etc. --- In this case the order in which the individual receives the prizes seems to us less important and the axiom can be accepted as a good approximation of reality. In any event, the case where the individual can alter his future preferences by means of his present consumptions or through moral persuasion or through advertising, etc., although extremely interesting, is beyond the scope of this paper.

We now present in a formal way the

Permutation Axiom. Let k be any natural number greater than 0. Let π_K

and π'_k be arbitrary one to one functions from the set of natural numbers $\{1, 2, \cdots\}$ onto itself such that $\pi_k(m) = \pi'_k(m) = m$ for all $m > k$. Then we have: for any finite natural number $k > 0$ and any π_k and π'_k,

$$(x_{h_1},\ldots,x_{h_i},\ldots)\ Q\ (x_{h^*_1},\ldots,\ x_{h^*_i},\ldots) \Leftrightarrow$$

$$\Leftrightarrow (x_{h_{\pi_k(1)}},\ldots,\ x_{h_{\pi_k(i)}},\ldots)\ Q\ (x_{h^*_{\pi'_k(1)}},\ldots,\ x_{h^*_{\pi'_k(i)}},\ \ldots),$$

where h_i and h^*_i both belong to N for $i = 1, 2,\ldots$.

The Independence Axiom. Using the example of Mr. A's after lunch drinks, this axiom simply states that if two sequences of drinks

$x_{h_1},\ldots,\ x_{h_i},\ldots$ and $x_{h^*_1},\ldots,\ x_{h^*_i},\ \cdots$ coincide from $(\nu + 1)$-th term on, and Mr. A prefers the sequence $x_{h_1},\ldots,\ x_{h_i},\ \ldots$ to the sequence $x_{h^*_1},\ldots,\ x_{h^*_i},\ldots$, he will also prefer drinking $x_{h_1},\ldots,\ x_{h_\nu}$ to drinking $x_{h^*_1},\ldots,\ x_{h^*_\nu}$ the first ν days no matter what he is offered to drink the rest of the days whenever these drinks are the same in both cases.

Before we present in a formal way the independence axiom, it might be convenient to develop some notation. Consider an infinite sequence

$$[1]\quad x_{h_1},\ldots,\ x_{h_\nu},\ldots\ =\ (x^{\infty}).$$

We will also represent the sequence [1] by $(x^\nu;\ x^{\infty-\nu})$, where x^ν represents the first ν terms of the sequence and $x^{\infty-\nu}$ the remaining terms. Suppose we obtain a new sequence from the sequence [1] by changing some, all or none of the first ν terms. The resulting sequence will be represented by

$(x^{*\nu}; x^{\infty-\nu})$. Similar changes on the remaining part of sequence [1] will be represented accordingly.

We can now formally present the Independence Axiom. For any natural number $\nu > 0$,

$$(x^{\nu}; x^{\infty-\nu}) \gtrless (x^{*\nu}; x^{\infty-\nu}) \Leftrightarrow (x^{\nu}; x^{*\,\infty-\nu}) \gtrless (x^{*\nu}; x^{*\,\infty-\nu}).$$

Remark. A preference ordering Q defined on x^{∞}, satisfying the Permutation and Independence axioms, induces a preference ordering Q_{ν}, which also satisfies the corresponding Permutation and Independence axioms, on the Cartesian product $X^{\nu} = X \times \cdots \times X$ (ν times; ν being any finite natural number greater than 0) as follows:

Write $x^{\nu} = (x_{h_1}, \ldots, x_{h_\nu})$; $x^{*\nu} = (x^{*}_{h_1}, \ldots, x^{*}_{h_\nu})$. Then,

$(x_{h_1}, \ldots, x_{h_\nu})$ Q_{ν} $(x^{*}_{h_1}, \ldots, x^{*}_{h_\nu})$ if and only if

there is a $(x^{\infty-\nu})$ such that $(x^{\nu}; x^{\infty-\nu})$ Q $(x^{*\nu}; x^{\infty-\nu})$. In particular we have the preference ordering Q_1 defined on the set of prizes X. We may write $\gtrsim$ instead of Q_{ν} when it is clear from the context on which Cartesian product X^{ν} the preference ordering is defined.

The rate of substitution axiom tries to formalize the intuitive notion that if an individual is disappointed because in one instance he is given prize x_q instead of prize x_p that he likes better, he can be compensated by giving him in a sufficiently large number of instances a prize x_{p^*} instead of a prize x_{q^*} that he likes less. Formally, we have

Rate of Substitution Axiom. If $x_{h_p} \gtrsim x_{h_q}$ and $x_{h_{p^*}} > x_{h_{q^*}}$ then there exists a real and non-negative number (that depends on h_p, h_q, h_{p*}, h_{q*}), $R(h_p, h_q, h_{p*}, h_{q*})$ such that the following is true:

(a) If in a sequence $x_{h_1}, \ldots, x_{h_i}, \ldots,$ we substitute x_{h_q} for x_{h_p} r times ($r > 0$) and $x_{h_{p*}}$ for $x_{h_{q*}}$ s times ($s \geq 0$), then the resulting sequence is $>$, $<$, or $\sim$ with regard to the original one if and only if $\frac{s}{r} > R(h_p, h_q, h_{p*}, h_{q*})$, $\frac{s}{r} < R(h_p, h_q, h_{p*}, h_{q*})$, or $\frac{s}{r} = R(h_p, h_q, h_{p*}, h_{q*})$, respectively.

(b) If in a sequence $x_{h_1}, \ldots, x_{h_i}, \ldots,$ we substitute x_{h_p} for x_{h_q} r times ($r > 0$) and $x_{h_{q*}}$ for $x_{h_{p*}}$ s times ($s \geq 0$), then the resulting sequence is $>$, $<$, or $\sim$ with regard to the original one if and only if $\frac{s}{r} < R(h_p, h_q, h_{p*}, h_{q*})$, $\frac{s}{r} > R(h_p, h_q, h_{p*}, h_{q*})$ or $\frac{s}{r} = R(h_p, h_q, h_{p*}, h_{q*})$, respectively.

We will sometimes represent a finite sequence containing c_1 times the term $x_{h_1}, \ldots,$ c_k times the term x_{h_k}, where the c_i's ($i = 1, \ldots, k$) are integer and positive numbers, by $c_1 \times x_{h_1}, \ldots, c_k \times x_{h_k}$. This representation is legitimate in view of the Permutation axiom.

We now turn to our last axiom, the

Repetition Axiom. For any integer and positive numbers k and c,

$$x_{h_1},\ldots, x_{h_k} \gtrsim x_{h_1^*},\ldots, x_{h_k^*} \Leftrightarrow c \times x_{h_1},\ldots, c \times x_{h_k} \gtrsim c \times x_{h_1^*},\ldots, c \times x_{h^*_k}$$

The following theorem, whose proof we will postpone until the appendix, enunciates some of the properties of the function $R\ (h_p, h_q, h_{p*}, h_{q*})$.

Theroem 1. The function $R\ (h_p, h_q, h_{p*}, h_{q*})$, that clearly is unique, satisfies the following properties:

(i) If $R\ (h_p, h_q, h_{p*}, h_{q*}) > 0$, then

$$R\ (h_{p*}, h_{q*}, h_p, h_q) = \frac{1}{R(h_p, h_q, h_{p*}, h_{q*})}$$

(ii) $$R\ (h_p, h_q, h_{p*}, h_{q*} = \frac{R\ (h_p, h_q, h_g, h_k)}{R\ (h_{p*}, h_{q*}, h_g, h_k)}\ .$$

(iii) If $x_{h_1} > x_{h_2} > \ldots > x_{h_p}$ $(2 \leq p \leq n)$,

and $x_g > x_k$, then

$$R\ (h_1,h_p,g,k) = R(h_1,h_2,g,k) + R(h_2,h_3,g,k) + \ldots + R(h_{p-1},h_p,g,k).$$

Utility indices. We try to determine now if they exist, utility indices for the different prizes x_i, $u^c\ (x_i) = u_i^c$, $i = 1,\ldots, n$, that satisfy the following condition:

$[\alpha]$ $x_{h_1},\ldots, x_{h_k} \gtrsim x_{h_1^*},\ldots, x_{h^*_k} \Leftrightarrow$

$\Leftrightarrow \sum_{i=1}^{k} u^c_{h_i} \gtrsim \sum_{i=1}^{k} u^c_{h_i^*}$, where k is any finite natural number greater than 0.

Theorem 2. (Proof in the Appendix.) There exist utility indices u_i^c satisfying condition [α] if and only if:

(i) $u_r^c > u_s^c \Leftrightarrow x_r > x_s$; and

(ii) if $x_r > x_s$ and $x_{r^*} > x_{s^*}$, then

$$\frac{u_r^c - u_s^c}{u_{r^*}^c - u_{s^*}^c} = R\,(r,\ s,\ r^*,\ s^*).$$

Write $u^c = (u_1^c, \cdots, u_n^c)$. We will call the vectors u^c, utility vectors.

Theorem 3. (Proof is in the Appendix)

(i) There exists a class $\mathcal{U}^c$ of utility vectors u^c that satisfy conditions (i) and (ii) of Theorem 2 and therefore condition [α]. Two vectors $\bar{u}^c = (\bar{u}_1^c, \ldots, \bar{u}_n^c)$ and $\bar{\bar{u}}^c - (\bar{\bar{u}}_1^c, \ldots, \bar{\bar{u}}_n^c)$ belong to the class $\mathcal{U}^c$ if and only if $\bar{u}^c \in \mathcal{U}^c$ and $\bar{\bar{u}}_i^c = a\, u_i^c + b$ for $i = 1, \ldots, n$, where a and b are constants, $a > 0$.

(ii) If a utility vector u^c satisfies conditions (i) and (ii) of Theorem 2, then $u^c \in \mathcal{U}^c$.

Now that we have developed a theory of utility for sequences of prizes, we can attack the problem suggested by Marschak in [6] of relating the rule of "long run success" to the rule of maximizing expected utility. To this task we will turn in the next sections.

3. MAXIMIZATION OF EXPECTED UTILITY AND THE RULE OF LONG RUN SUCCESS

Let us go back to our example regarding Mr. A and his after lunch drinks. Suppose that our hostess decides to determine each day by means of a random device, say a roulette, whether coffee, tea or camomile will be served. Thus, the set of all possible outcomes of playing once the roulette is partitioned into three events E_1, E_2 and E_3 and the commitment is made that those days in which the outcome of playing the roulette belongs to E_1 coffee will be served, those in which the outcome belongs to E_2 tea will be served, and those in which E_3 obtains camomile will be served.

Let P_i designate the probability of the event E_i $(i = 1, 2, 3)$. The word probability here will be used in the objective or statistical sense. That is, P_i will be taken here as the limit to which the frequency of occurrence of the event E_i "converges" when the random experiment is repeated infinite many times.

Let $\mathcal{E}$ be the random experiment that corresponds to a partition of all the possible outcomes of playing our roulette into three events E_1, E_2 and E_3 with probabilities P_1, P_2 and P_3, respectively. Let $\mathcal{E}'$ be the random experiment that corresponds to the partition into the events E_1', E_2', E_3' with probabilities P_1', P_2', P_3'. Suppose now that our hostess presents Mr. A with the two random experiments and asks him which of the two he prefers that be used every day in order to determine whether coffee, tea, or camomile will be served. What random experiment should he choose $\mathcal{E}$ or $\mathcal{E}'$? Or equivalently, what probability distribution should he choose (P_1, P_2, P_3) or (P_1', P_2', P_3')?

Remark. We want to stress here that for Mr. A to choose the random experiment, say $\mathcal{E}$, or equivalently its corresponding probability distribution (P_1, P_2, P_3), means that each and every day the roulette is played; and that coffee will be served when E_1 obtains, tea when E_2 and camomile when E_3. Also given the objective or statistical interpretation that we have adopted for the probabilities P_1, P_2 and P_3, this means that by choosing the probability distribution (P_1, P_2, P_3), Mr. A will obtain a sequence of drinks $x_{h_1}, x_{h_2}, \ldots, x_{h_k}$, such that the frequency with which coffee, tea and camomile will be served "converges", respectively, to the probabilities P_1, P_2 and P_3 when $k \to \infty$.

The rule of long run success. Let $x_{h_1}, \ldots, x_{h_k}$ and $x'_{h_1}, \ldots, x'_{h_k}$ designate, respectively, the sequences of the first k prizes generated by the random devices $\mathcal{E}$ and $\mathcal{E}'$, or equivalently by their corresponding probability distributions (P_1, P_2, P_3) and (P'_1, P'_2, P'_3). Clearly, the sequences $x_{h_1}, \ldots, x_{h_k}$ and $x'_{h_1}, \ldots, x'_{h_k}$ are random and therefore we can calculate the probability that, say, the sequence $x_{h_1}, \ldots, x_{h_k}$ be preferred by Mr A to the sequence $x'_{h_1}, \ldots, x'_{h_k}$. In symbols $P_r\,[(x_{h_1}, \ldots, x_{h_k}) > (x'_{h_1}, \ldots, x'_{h_k})]$. Now, we say that Mr. A satisfies or follows the rule of long run success if he chooses the probability distribution (P_1, P_2, P_3) over the probability distribution (P'_1, P'_2, P'_3) whenever,

$$\lim_{k \to \infty} P_r\,[(x_{h_1}, \ldots, x_{h_k}) > (x'_{h_1}, \ldots, x'_{h_k})] = 1.$$

It will be easy to prove now the proposition that if Mr. A orders the possible sequences of prizes $x_{h_1}, \ldots, x_{h_k}, \ldots$ in a way that satisfies our permutation, independence, rate of substitution and repetition axioms, then the rule of maximizing the expected value of the utilities u_i^c implies that the rule of long run success, as stated above, is satisfied.

Proof. As it was shown in section 2, if the preference ordering, by Mr. A, of the possible sequences $x_{h_1}, \ldots, x_{h_k}, \ldots$ satisfies our axioms, then utility indices $u^c(x_1)$, $u^c(x_2)$, $u^c(x_3)$, constant up to positive linear transformations, can be assigned to the prizes x_1, x_2, x_3 with the property that the sequence $x_{h_1}, \ldots, x_{h_k}$ is at least as good as the sequence $x'_{h_1}, \ldots, x'_{h_k}$ if and only if $\sum_{i=1}^{k} u^c(x_{h_i}) \geq \sum_{i=1}^{k} u^c(x'_{h_i})$. If the sequences $x_{h_1}, \ldots, x_{h_k}$ and $x'_{h_1}, \ldots, x'_{h_k}$ are generated, respectively, by the probability distributions (P_1, P_2, P_3) and (P'_1, P'_2, P'_3), then

$$Z_k = \frac{1}{k} \sum_{i=1}^{k} u^c(x_{h_i}) \quad \text{and} \quad Z'_k = \frac{1}{k} \sum_{i=1}^{k} u^c(x'_{h_i}) \quad \text{are random variables}$$

and by the law of large numbers we know that Z_k and Z'_k converge in probability respectively to $\sum_{i=1}^{3} P_i\, u^c(x_i)$ and $\sum_{i=1}^{3} P'_i\, u^c(x_i)$ as $k \to \infty$.

Now, the rule of maximizing the expected value of the utilities $u^c(x_i)$ prescribes that the probability distribution (P_1, P_2, P_3) be preferred over the probability distribution (P'_1, P'_2, P'_3) whenever $\sum_{i=1}^{3} P_i\, u^c(x_i)$ is greater than $\sum_{i=1}^{3} P'_i\, u^c(x_i)$, which implies that

$$\lim_{k \to \infty} P_r [Z_k > Z'_k] = 1.$$

But,

$$P_r [Z_k > Z'_k] = P_r [k Z_k > k Z'_k] =$$

$$P_r [\sum_{i=1}^{k} u^c(x_{h_i}) > \sum_{i=1}^{k} u^c(x'_{h_i})] = P_r [(x_{h_1},\ldots,x_{h_k}) > (x'_{h_1},\ldots,x'_{h_k})].$$

Thus,

$\sum_{i=1}^{3} P_i u^c(x_i) > \sum_{i=1}^{3} P'_i u^c(x_i)$ implies that

$$\lim_{k \to \infty} P_r [(x_{h_1}, \ldots, x_{h_k}) > (x'_{h_1}, \ldots, x'_{h_k})] = 1 \qquad \text{Q. E. D.}$$

Remark. Observe that the rule of long run success, as stated by us, requires, to be satisfied, only that the probability distribution $P = (P_1, P_2, P_3)$ be ranked above the probability distribution $P' = (P'_1, P'_2, P'_3)$ whenever $\sum_{i=1}^{3} P_i u^c(x_i) = E [u_i^c \mid P]$ is greater than $\sum_{i=1}^{3} P'_i u^c(x_i) = E [u_i^c \mid P']$.

But it does not require that the probability distributions P and P' be ranked as indifferent to each other when $E [u_i^c \mid P] = E[u_i^c \mid P']$. Thus, if we designate by $\sigma[u_i^c \mid P]$ the standard deviation of the random utility given P, i.e., the standard deviation of the random variable taking the values u_i^c with probabilities P_i $(i = 1, 2, 3)$, then the individual can order the probability distributions, without violating the rule of long run success, according to the following lexicographic ordering:

$$P > P' \begin{cases} \text{if } E [u_i^c \mid P] > E [u_i^c \mid P'], \\ \text{or if } E [u_i^c \mid P] = E [u_i^c \mid P'], \ \sigma [u_i^c \mid P] > \sigma [u_i^c \mid P']. \end{cases}$$

4. SOME FINAL REMARKS

Remark 1. In this paper we have presented a utility theory for sequences of prizes that allows us to discuss rational behavior in terms of the intuitively very appealing rule of long run success. Whether or not this utility theory is a "good" one to deal with sequences of prizes remains to be seen. We are aware of the limitations of its applicability and we indicated this fact in section 2 when we presented and discussed our axioms. But in cases similar to the example regarding Mr. A and his after lunch drinks we believe that both our axioms and the conclusions obtained from our model by defining rational behavior as that behavior that does not violate the rule of long run success are indeed reasonable. For those situations only we will claim that our model and its conclusions are valid.

Remark 2. It should be noted that the utility indices u_i^c that we have used in this paper are derived from axioms which do not involve the ordering of probability distributions or uncertain prospects. Thus, in the derivation of the u_i^c no uncertainty is involved. In the Von Neuman and Morgenstern expected utility theory, on the contrary, the utility indices u_i^v are derived from axioms regarding the ordering of probability distributions or uncertain prospects.

Remark 3. An interesting question then arises: What is the relationship between the u_i^v and the u_i^c? More concretely, suppose that Mr. A satisfies our axioms and, consequently, by asking him the different rates of substitution postulated by the Rate of Substitution Axiom, we can calculate his utility indices u_1^c, u_2^c, u_3^c. Suppose also that Mr. A satisfies the Von Neuman and Morgenstern, so called, rationality axioms and, consequently, by observing how he orders the different probability distributions (P_1, P_2, P_3) we can obtain his utility indices u_1^v, u_2^v, u_3^v. What is the relationship between these two sets of indices? Are the u_i^v equal to the u_i^c, up to a positive linear transformation? Or, are the u_i^v increasing concave or convex transformations[2] of the u_1^c?

Remark 4. In situations where our model applies, it appears as reasonable to adopt as the only rationality criterion the rule of long run success. But then, the Von Neuman and Morgenstern rationality axioms imply rationality in the sense of the rule of long run success only if the utility indices u_i^v derived from these axioms are equal to the utility indices u_i^c up to a positive linear transformation. Thus, suppose that after learning from Mr. A what are his rate of substitution values, we make the necessary calculations and obtain a set of utility indices u_i^c, say, $u_1^c = 2$ for the utility of coffee, $u_2^c = 1$ for the utility of tea and $u_3^c = 0$ for the utility of camomile. To calculate the utility indices u_i^v, assuming that Mr. A satisfies the Von Neuman and Morgenstern rationality axioms, we can put $u_1^v = 2$, $u_3^v = 0$ and then calculate u_2^v by asking Mr. A to reveal the probability distribution $(P_1, 0, 1 - P_1)$ that is indifferent from his point of view to having tea with certainty. In order for Mr. A to satisfy the rule of long run success, as stated by us, he must choose $P_1 = \frac{1}{2}$, that will give $u_2^v = 1$. But he can choose, without violating the rationality axioms, say, $P_1 = \frac{3}{4}$, that will give $u_2^v = \frac{3}{2}$; or $P_1 = \frac{1}{4}$, in which case

$u_2^v = \frac{1}{2}$, etc. And in all these cases he will be violating the rule of long run success although he behaves in accordance with the Von Neuman and Morgenstern rationality axioms.

Remark 5. If we define rational behavior, as we have proposed before for those situations where our model applies, as any behavior that does not violate the rule of long run success then it follows that the rationality axioms are neither necessary nor sufficient for rational behavior. They are not necessary because the lexicographic ordering of probability distributions

$$P > P' \text{ if and only if } \begin{cases} E\,[u_i^c \mid P] > E\,[u_i^c \mid P'], \\ \text{or if } E\,[u_i^c \mid P] = E\,[u_i^c \mid P'],\ \sigma\,[u_i^c \mid P] > \sigma\,[u_i^c \mid P'] \end{cases}$$

does satisfy the rule of long run success but not the rationality axioms. They are not sufficient because the utility indices u_i^v derived from them are not necessarily equal to the utility indices u_i^c, up to a positive linear transformation, and when this is the case the rule of maximizing the expected value of the utilities u_i^v does violate the rule of long run success.[3]

Remark 6. Since no uncertainty is involved in obtaining the utility indices u_i^c, and, on the other hand, the u_i^v's are derived from the ordering of probability distributions, it appears attractive to try to characterize the behavior toward risk of Mr. A by the relationship between his utility indices u_i^c and u_i^v : $u_i^v = F(u_i^c)$. Thus, we propose on a tentative basis the following definitions:

(i) Mr. A is risk neutral if F is a positive linear transformation.

(ii) Mr. A is risk averse if F is a monotonic increasing and strictly concave transformation.

(iii) Mr. A is risk loving if F is a monotonic increasing and strictly convex transformations.

It is worth noting here that both risk averse and risk loving behaviors are not rational in the sense that they do not satisfy the rule of long run success. We do not plan to study here the advantages or disadvantages of these definitions. We want to point out however that they allow us to make comparisons of risk averseness with many commodities without restricting those comparisons, as Kihlstrom and Mirman do [4], to cases where the ordinal preferences are the same.

APPENDIX

Proof of Theorem 1.

Part (i). Assume that $R(h_p, h_q, h_{p*}, h_{q*}) > 0$

and $R(h_{p*}, h_{q*}, h_p, h_q \neq \frac{1}{R(h_p, h_q, h_{p*}, h_{q*})}$.

Then, either (α) $R(h_{p*}, h_{q*}, h_p, h_q) \cdot R(h_p, h_q, h_{p*}, h_{q*}) > 1$,

or (β) $R(h_{p*}, h_{q*}, h_p, h_q) \cdot R(h_p, h_q, h_{p*}, h_{q*}) < 1$.

Assume that (α) holds. Since $R(h_p, h_q, h_{p*}, h_{q*}) > 0$ and $R(h_{p*}, h_{q*}, h_p, h_q) \cdot R(h_p, h_q, h_{p*}, h_{q*}) > 1$, there exist two numbers a and b integer and positive such that,

$$R(h_p, h_q, h_{p*}, h_{q*}) > \frac{a}{b}, \text{ and } R(h_{p*}, h_{q*}, h_p, h_q) > \frac{b}{a}.$$

Consider now a finite sequence containing b times the term x_{h_p}, and a times the term $x_{h_{q*}}$. We can write this finite sequence as follows: $b \times x_{h_p}, a \times x_{h_{q*}}$. By applying the Rate of Substitution Axiom, we have:

(α_1) Since $R(h_p, h_q, h_{p*}, h_q{*}) > \frac{a}{b}$,

$$b \times x_{h_p}, a \times x_{h_{q*}} > b \times x_{h_q}, a \times x_{h_{p*}}.$$

(α_2) Since $R(h_{p*}, h_{q*}, h_p, h_q) > \frac{b}{a}$,

$$b \times x_{h_q}, a \times x_{h_{p*}} > b \times x_{h_p}, a \times x_{h_{q*}}.$$

(α_1) and (α_2) imply that $b \times x_{h_p}, a \times x_{h_{q*}} > b \times x_{h_p}, a \times x_{h_{q*}}$, which is impossible. Therefore (α) cannot hold. In a similar way we can show that (β) cannot hold, which completes the proof of part (i).

Part (ii). Assume that $R(h_p, h_q, h_{p*}, h_{q*}) > 0$ and that

$$R(h_p, h_q, h_{p*}, h_{q*}) \neq \frac{R(h_p, h_q, h_g, h_k)}{R(h_{p*}, h_{q*}, h_g, h_k)} = R(h_p, h_q, h_g, h_k) \cdot R(h_g, h_k, h_{p*}, h_{q*}.$$

Then either

(α) $R(h_p, h_q, h_{p*}, h_{q*}) > R(h_p, h_q, h_g, h_k) \cdot R(h_g, h_k, h_{p*}, h_{q*})$,

or

(β) $R(h_p, h_q, h_{p*}, h_{q*}) < R(h_p, h_q, h_g, h_k) \cdot R(h_g, h_k, h_{p*}, h_{q*})$.

Suppose that (α) holds. Then, there exist numbers b, c and d integer and positive such that:

$$R(h_p, h_q, h_{p*}, h_{q*}) > \frac{b}{c}; \quad R(h_p, h_q, h_g, h_k) < \frac{d}{c}; \ R(h_g, h_k, h_{p*}, h_{q*}) < \frac{b}{d}.$$

Consider the finite sequence $c \times x_{h_p}, b \times x_{h_{q*}}, d \times x_{h_k}$.

By applying the Rate of Substitution Axiom we have:

(i) $c \times x_{h_p}, b \times x_{h_{q*}}, d \times x_{h_k} > c \times x_{h_q}, b \times x_{h_{p*}}, d \times x_{h_k}$, since $R(h_p, h_q, h_{p*}, h_{q*}) > \frac{b}{c}$

(ii) $c \times x_{h_p}, b \times x_{h_{q*}}, d \times x_{h_k} < c \times x_{h_q}, b \times x_{h_{q*}}, d \times x_{h_g}$, since $R(h_p, h_q, h_g, h_k) < \frac{d}{c}$;

(iii) $c \times x_{h_q}, b \times x_{h_{q*}}, d \times x_{h_g} < c \times x_{h_q}, b \times x_{h_{p*}}, d \times x_{h_k},$

since $R(h_g, h_k, h_{p*}, h_{q*}) < \frac{b}{d}$.

(i), (ii) and (iii) cannot be satisfied simultaneously, therefore (α) cannot hold.

In the same way we can see that (β) cannot hold which completes the proof of Part (ii) of Theorem 1.

Part (iii). It suffices to prove that if $x_{h_1} > x_{h_2} > x_{h_3}$ and $x_g > x_k$, then

$$R(h_1, h_3, g, k) = R(h_1, h_2, g, k) + R(h_2, h_3, g, k).$$

Suppose not. Then, either

(α) $R(h_1, h_3, g, k) > R(h_1, h_2, g, k) + R(h_2, h_3, g, k)$, or

(β) $R(h_1, h_3, g, k) < R(h_1, h_2, g, k) + R(h_2, h_3, g, k)$.

Assume that (α) holds. Then, there exist three numbers b_1, b_2, and c integers and positive such that $\frac{b_1}{c} > R(h_1, h_2, g, k)$, $\frac{b_2}{c} > R(h_2, h_3, g, k)$ and $\frac{b_1 + b_2}{c} < R(h_1, h_3, g, k)$. The rest of the proof consists in constructing an appropriate finite sequence as in Part (ii), and show by using the Rate of Substitution Axiom that the assumption that (α) holds leads to an impossible result. Case (β) can be dealt with in a similar way.

Proof of Theorem 2. We will first prove that if there exist utility indices u_i^c satisfying condition [α], then they must satisfy conditions

(i) and (ii) of theorem 2. That they must satisfy condition (i) is obvious. We will then show that they must satisfy condition (ii). Suppose they do not. Suppose, for instance, that

$$x_r > x_s,\ x_{r*} > x_{s*} \quad \text{and} \quad \frac{u_r^c - u_s^c}{u_{r*}^c - u_{s*}^c} > R(r, s, r*, s*).$$

Then, there exist numbers b and c, integer and positive, such that

$$\frac{u_r^c - u_s^c}{u_{r*}^c - u_{s*}^c} > \frac{b}{c} > R(r, s, r*, s*).$$

Now compare the finite sequences $c \times x_r, b \times x_{s*}$ and $c \times x_s, b \times x_{r*}$. Since $c \times u_r^c + b \times u_{s*}^c > c \times u_s^c + b \times u_{r*}^c$, we should have

$c \times x_r, b \times x_{s*} > c \times x_s, b \times x_{r*}$. But, since $\frac{b}{c} > R(r, s, r*, s*)$,

we should also have $c \times x_s, b \times x_{r*} > c \times x_r, b \times x_{s*}$, which is not possible. Thus our assumption that

$\frac{u_r^c - u_s^c}{u_{r*}^c - u_{s*}^c} > R(r, s, r*, s*)$ cannot hold. We can dispose of the case

$\frac{u_r^c - u_s^c}{u_{r*}^c - u_{s*}^c} < R(r, s, r*, s*)$ in a similar way.

Let us now conclude the proof of Theorem 2. Suppose that we have utility indices u_i^c that satisfy conditions (i) and (ii) of Theorem 2, but not condition [α]. Suppose, for instance, that

$$x_{h_1}, \ldots, x_{h_k} > x_{h*_1}, \ldots, x_{h*_k} \quad \text{and} \quad \sum_{i=1}^{k} u_{h_i}^c < \sum_{i=1}^{k} u_{h*_i}^c.$$

Without loss of generality, we can assume that

$$u^c_{h_i} < u^c_{h^*_i} \quad \text{for } i = 1,\dots, j \quad (1 \leq j \leq k); \quad \text{and}$$

$$u^c_{h_i} \geq u^c_{h^*_i} \quad \text{for } i = j+1,\dots, k.$$

By condition (i) of Theorem 2, we have:

$$x_{h^*_i} > x_{h_i}, \text{ for } i = 1,\dots, j;$$

$$x_{h^*_i} \leq x_{h_i}, \quad \text{for } i = j+1,\dots,k.$$

Since $\sum_{i=1}^{k} u^c_{h_i} < \sum_{i=1}^{k} u^c_{h^*_i}$, we have that

$$\sum_{i=1}^{j} [u^c_{h^*_i} - u^c_{h_i}] > \sum_{i=j+1}^{k} [u^c_{h_i} - u^c_{h^*_i}].$$

By now dividing both members of the previous inequality by $u^c_{h^*_1} - u^c_{h_1} > 0$

we obtain, taking into account condition (ii) of Theorem 2,

$$\sum_{i=1}^{j} R(h^*_i, h_i, h^*_1, h_1) > \sum_{i=j+1}^{k} R(h_i, h^*_i, h^*_1, h_1).$$

We can find integer and positive numbers $c, b_1, \dots, b_k$ such that:

$$\sum_{i=j+1}^{k} R(h_i, h^*_i, h^*_1, h_1) < \sum_{i=j+1}^{k} \frac{b_i}{c} < \sum_{i=1}^{j} \frac{b_i}{c} < \sum_{i=1}^{j} R(h^*_i, h_i, h^*_1, h_1);$$

$$\frac{b_i}{c} < R(h^*_i, h_i, h^*_1, h_1), \quad \text{for } i = 1, \dots, j;$$

$$\frac{b_i}{c} > R(h_i, h^*_i, h^*_1, h_1), \quad \text{for } i = j+1, \dots, k.$$

Consider now the sequence,

$$c \times x_{h_1}, \ldots, c \times x_{h_j}, c \times x_{h_{j+1}}, \ldots, c \times x_{h_k}, \left(\sum_{i=1}^{j} b_i\right) \times x_{h^*_1}, \left(\sum_{i=j+1}^{k} b_i\right) \times x_{h_1}.$$

Since $\frac{b_i}{c} < R(h^*_i, h_i, h^*_1, h_1)$ for $i = 1, \ldots, j$, we have by using the Rate of Substitution Axiom that,

$$\text{(i)}\ c \times x_{h_1}, \ldots, c \times x_{h_j}, c \times x_{h_{j+1}}, \ldots, c \times x_{h_k}, \left(\sum_{i=1}^{j} b_i\right) \times x_{h^*_1}, \left(\sum_{i=j+1}^{k} b_i\right) \times x_{h_1} <$$

$$< c \times x_{h^*_1}, \ldots, c \times x_{h^*_j}, c \times x_{h_{j+1}}, \ldots, c \times x_{h_k}, \left(\sum_{i=1}^{j} b_i\right) \times x_{h_1}, \left(\sum_{i=j+1}^{k} b_i\right) \times x_{h_1}.$$

Since $\frac{b_i}{c} > R(h_i, h^*_i, h^*_1, h_1)$ for $i = j + 1, \ldots, k$, we can obtain by applying the Rate of Substitution Axiom,

$$\text{(ii)}\ c \times x_{h^*_1}, \ldots, c \times x_{h^*_j}, c \times x_{h_{j+1}}, \ldots, c \times x_{h_k}, \left(\sum_{i=1}^{j} b_i\right) \times x_{h_1}, \left(\sum_{i=j+1}^{k} b_i\right) \times x_{h_1} <$$

$$< c \times x_{h^*_1}, \ldots, c \times x_{h^*_j}, c \times x_{h^*_{j+1}}, \ldots, c \times x_{h^*_k}, \left(\sum_{i=1}^{j} b_i\right) \times x_{h_1}, \left(\sum_{i=j+1}^{k} b_i\right) \times x_{h^*_1}.$$

Since $\sum_{i=1}^{j} b_i > \sum_{i=j+1}^{k} b_i$, by using again the Rate of Substitution Axiom, we obtain,

$$\text{(iii)}\ c \times x_{h^*_1}, \ldots, c \times x_{h^*_j}, c \times x_{h^*_{j+1}}, \ldots, c \times x_{h^*_k}, \left(\sum_{i=1}^{j} b_i\right) \times x_{h_1}, \left(\sum_{i=j+1}^{k} b_i\right) \times x_{h^*_1} <$$

$$< c \times x_{h^*_1}, \ldots, c \times x_{h^*_j}, c \times x_{h^*_{j+1}}, \ldots, c \times x_{h^*_k}, \left(\sum_{i=1}^{j} b_i\right) \times x_{h^*_1}, \left(\sum_{i=j+1}^{k} b_i\right) \times x_{h_1}.$$

(i), (ii) and (iii) imply that,

$$c \times x_{h_1},\dots, c \times x_{h_j}, c \times x_{h_{j+1}},\dots, c \times x_{h_k}, (\sum_{i=1}^{j} b_i) \times x_{h^*_1}, (\sum_{i=j+1}^{k} b_i) \times x_{h_1} <$$

$$< c \times x_{h^*_1},\dots, c \times x_{h^*_j}, c \times x_{h^*_{j+1}},\dots, c \times x_{h^*_k}, (\sum_{i=1}^{j} b_i) \times x_{h^*_1}, (\sum_{i=1}^{j} b_i) \times x_{h_1},$$

which in view of the Independence Axiom implies that

$c \times x_{h_1},\dots, c \times x_{h_k} < c \times x_{h^*_1},\dots, c \times x_{h^*_k}$. And by using now the Repetition Axiom, we finally obtain, $x_{h_1},\dots, x_{h_k} < x_{h^*_1},\dots, x_{h^*_k}$, which contradicts the assumption that $x_{h_1},\dots, x_{h_k} > x_{h^*_1},\dots, x_{h^*_k}$. We can dispose of the other cases in a similar way.

Proof of Theorem 3. Without loss of generality we can assume $x_1 > x_2 > \cdots > x_{n-1} > x_n$.

Consider now the following vector $\bar{u}^c = (\bar{u}_1^c, \bar{u}_2^c,\dots, \bar{u}_i^c,\dots, \bar{u}_{n-1}^c, \bar{u}_n^c)$:

$\bar{u}_n^c = 0$; $\bar{u}_{n-1}^c = 1$; $\bar{u}_i^c = \bar{u}_{i+1}^c + R(i, i+1, n-1, n)$, for $i = 1, \quad, n-2$.

Since $R(i, i+1, n-1, n) > 0$ for $i = 1,\cdots, n-2$, it follows that the $\bar{u}_i^c$'s satisfy condition (i) of Theorem 2. They also satisfy the following condition (ii)':

$$\frac{\bar{u}_i^c - \bar{u}_{i+1}^c}{\bar{u}_{n-1}^c - \bar{u}_n^c} = R(i, i+1, n-1, n), \text{ for } i = 1,\dots,n-1.$$

By taking into account now conditions (i), (ii) and (iii) of Theorem 1, it can be seen easily that the $\bar{u}_i^c$'s also satisfy condition (ii) of Theorem 2.

If $\bar{\bar{u}}^c = (\bar{\bar{u}}_1^c, \cdots, \bar{\bar{u}}_i^c, \ldots, \bar{\bar{u}}_n^c)$ is any vector such that $\bar{\bar{u}}_i^c = a\, \bar{u}_i^c + b$, where a and b are real numbers, $a > 0$, it follows immediately that $\bar{\bar{u}}^c$ also satisfies conditions (i) and (ii) of Theorem 2. This completes the proof of Part (i) of Theorem 3.

Part (ii) of Theorem 3 is easily proved by observing that if a vector u^c satisfies conditions (i) and (ii) of Theorem 2, then it also satisfies condition (ii)$'$ stated above and therefore

$$\frac{u_i^c - u_{i+1}^c}{u_{n-1}^c - u_n^c} = \frac{\bar{u}_i^c - \bar{u}_{i+1}^c}{\bar{u}_{n-1}^c - \bar{u}_n^c}$$

, which implies that

$u_i^c = a\, \bar{u}_i^c + b$, $a > 0$, for $i = 1, \cdots, n$.

Footnotes

[1] We will write $x^{\infty} > x^{*\infty}$ to mean $x^{\infty} \geq x^{*\infty}$ but not $x^{*\infty} \geq x^{\infty}$. $x^{\infty} \sim x^{*\infty}$ means $x^{\infty} \geq x^{*\infty}$ and $x^{*\infty} \geq x^{\infty}$. We will use also "$\geq$" ("$>$) to express that the real number written to the left of $\geq$ is greater or equal (greater) than the real number written to the right.

[2] A transformation F from the u_i^c's to the u_i^v's is said to be:

(i) positive linear if $F(u_i^c) = a\, u_i^c + b$, with $a > 0$;

(ii) monotonic increasing if $u_j^c > u_i^c$ implies that $F(u_j^c) > F(u_i^c)$;

(iii) strictly concave (convex) if $u_j^c > u_i^c$ and

$u_h^c = \alpha\, u_j^c + (1 - \alpha)\, u_i^c$, $0 < \alpha < 1$, implies that

$\alpha\, F(u_j^c) + (1-\alpha)\, F(u_i^c) < (>)\ F[\alpha\, u_j^c + (1 - \alpha)\, u_i^c]$.

[3] The conclusion of Remark 5 is similar to the contention made by M. Allais [See [1] p. 505, (9) and (10)]. He asserts there that for the rational man there does not exist in general utility indices u_i^v (he calls them utility indicators $B(x_i)$) such that the decision maker orders the probability distributions as if he were maximizing the expected value of u_i^v. This would be the case if we define "rational man" as one that does not violate the rule of long run success and he happens to order the probability distributions according to the lexicographic order described in Remark 4. He also asserts that when such utility indactor $B(x_i)$ exists it must coincide, up to a positive linear transformation, with what he calls the psychological value $\bar{s}(x_i)$. If $\bar{s}(x_i)$ is equal to $u^c(x_i)$, up to a positive linear transformation, then the same conclusion is reached in this paper. M. Allais, however, does not justify his assertions by using the rule of long run success and the law of large numbers as we do.