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VALUES OF MARKET WITH A MAJORITY RULE

by

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In this paper we extend the space DIFF of non atomic games to a space NADIFF consisting of games with non-additive derivatives. We use the properties of NADIFF to answer questions like when a value on a subspace Q can be extended to a diagonal value on $(Q \circledast J) \vee$ DIFF (the minimal space contains Q, DIFF and $Q \cdot J$ where J is the set of all majority games).

In this paper we introduce the space NADIFF of nonatomic games which is an extension of the space DIFF defined by Mertens [M]. NADIFF contains, in addition to DIFF, games with non additive derivatives. For example it contains the market game $v = \min(\mu_1,\ldots,\mu_n)$ where $\mu_i \in NA$ (NA is the space of all non atomic bounded measures) and all market games that have an extension i.e. market games in EXT (the space EXT was defined first in [M] and is defined below). The main purpose of this paper is to deal with the existence of a value on the space $Q \circledast J$, where Q is a supsapce of NADIFF, J is the set of all weighted majority games of the form $f_\alpha o \mu$ ($0 < \alpha < 1$ is the quota and μ is the majority measure) and $Q \circledast J$, is the minimal linear and symmetric space that contains Q as well as all games of the form $v \cdot f_\alpha o \mu$ in $Q \cdot J$. If Q is a space of market games which have an extension then the games $v \cdot f_\alpha o \mu$ in $Q \cdot J$ are used to describe economies in which taxation and redistribution are performed according to majority rule. Such games play a central rule in Aumann-Kurz [A-K]. In their model the market games μ are differentiable and therefore are in DIFF. In this paper we develop tools that will enable us to deal with nondifferentiable market games on which a majority rule is imposed. To that end we first prove several properties of NADIFF and then provide conditions that guarantee the existence of an extension of a value ϕ on a subspace Q to a (diagonal) value on the space $(Q \circledast J) \vee$ DIFF which is the minimal linear space containing $Q \circledast J$ and DIFF. Tauman [T] proved the existence of a value on the space Q^n generated by all n handed glove games, i.e., games of the form $v_n = \min(\mu_1,\ldots,\mu_n)$ where μ_i and μ_j, for $i \neq j$, are mutually singular. The results below will enable us to extend this value to a value on the space $(Q^n \circledast J) \vee$ DIFF and moreover to provide a formula for this value. We close the paper by showing the existence of a value on the space generated by all games of the form $v \cdot f_\alpha o \mu$ where v is <u>any</u>

market game and $f_\alpha \circ \mu$ is in J. This value distributes the amount v(I) to the players in the game $v \cdot f_\alpha \circ \mu$ according to their political power only.

Notations In this paper we shall basically follow the notation of Aumann and Shapley [A-S]. Let $(I, \mathcal{C})$ be a measurable space which is isomorphic to $([0,1], \mathcal{B})$ where $\mathcal{B}$ is the set of all Borel subsets of [0,1]. Let J be the set of all weighted majority games. i.e. J is the set of all games of the form $f_\alpha \circ \mu$ where $0 < \alpha < 1$, $\mu \in NA^1$ and where f_α is the jump function defined by

$$f_\alpha(x) = \begin{cases} 0 & 0 \leq x \leq \alpha \\ 1 & \alpha < x \leq 1 \end{cases} \qquad \text{or} \qquad f_\alpha(x) = \begin{cases} 0 & 0 < x < \alpha \\ 1 & \alpha \leq x \leq 1 \end{cases}$$

From now on whenever we will write f_α we will refer to the above definition. Moreover denote $f_0(x) = 1$.

Let Q be a set of games. $Q \circledast J$ is the linear and symmetric space generated by Q and by the set $Q \cdot J$ of all games of the for $v \cdot f_\alpha \circ \mu$ where $v \in Q$ and $f_\alpha \circ \mu \in J$. Any game in $Q \circledast J$ is of the form $\sum_{i=1}^{m} v_i \cdot f_{\alpha_i} \circ \mu_i$ where $v_i \in Q$, $f_{\alpha_i} \circ \mu_i \in J$, $0 \leq \alpha_i < 1$ and $1 \leq i \leq m$. Let Q_1 and Q_2 be two sets of games. Denote by $Q_1 \quad Q_2$ the minimal linear and symmetric supspace that contains Q_1 and Q_2. For every $v \in BV$ define the game v^+ by

$$v^+(S) = \sup \sum_i \max \{v(S_i) - v(S_{i-1}), 0\}$$

where the sup is taken over all chains of coalitions of the form $\emptyset = S_0 \subseteq S_1 \subseteq \ldots \subseteq S_n = S$. The game v^- is defined by $v^+ - v$. v^+ and v^- are both non-decreasing and

$$\|v\|_{BV} = v^+(I) + v^-(I).$$

Let $B_1(I, \mathcal{C})$ be the set of real valued measurable functions on $(I, \mathcal{C})$ with values in [0,1]. Any function w on $B_1(I, \mathcal{C})$ which is of bounded variation can be represented as $w = w^+ - w^-$ where w^+ and w^- are defined similarily to v^+ and v^- respectively. Moreover we have

$$\|w\|_{IBV} = w^+(1) + w^-(1),$$

where $\|w\|_{IBV}$ is the variation norm of w over $B_1(I, C)$. Denote;

$$|w| = w^+ + w^-.$$

Notice that by writing w(t) we consider the argument t as the constant function f(x) = t.

Let DNA (discrete NA topology) be the coarsest topology on the set B(I,C) of bounded real valued measurable functions on (I, C) such that for any $\mu \varepsilon$ NA the mapping $f \to \int f d\mu$ is continuous from B(I, C) to the real line with the discrete topology. Denote by EXT the set of all games $v \varepsilon$ BV that have a DNA continuous extension v* to $B_1(I, C)$ such that $|v^*|(t)$ is continuous at t=0 and t=1.

Any $v \varepsilon$ EXT can be extended to v* on B(I, C) by

$$v^*(f) = v^*\left([\max(0,\min(1,f))]\right)$$

<u>Definition (Mertens)</u>. DIFF is the set of all games $v \varepsilon$ EXT s.t. for each continuous function g on [0,1] the limit

$$\lim_{\substack{\tau > 0 \\ \tau \to 0}} \int_0^1 g(t) \cdot \frac{v^*(t + \tau\chi) - v^*(t)}{\tau} \, dt$$

exists (denote it by $m_v^g(\chi)$) for any $\chi \varepsilon B_1(I, C)$ and such that m_v^g is additive in χ. If g=1 we write m_v^1 instead of m_v^g.

The following theorem is due to Mertens [M].

Theorem ([M]) The space DIFF is linear symmetric and closed supspace of BV that contains bv^1NA. A value ϕ_D on DIFF does exist and

(1) $\phi_D v = m_v^1$

(2) $\phi_D v \in NA$.

Definition The set NADIFF is defined as DIFF but without the requirement that the derivative m_v^g is additive. Obviously NADIFF is a linear and summetric subspace of BV that contains DIFF.

Proposition 1 Let $\mu_1,\ldots,\mu_n$ be n measures in NA^1. Then the games

$$v_1 = \min(\mu_1,\ldots,\mu_n)$$

$$v_2 = \max(\mu_1,\ldots,\mu_n)$$

$$v_3 = \prod_{i=1}^{n} f_\alpha o \mu_i$$

are all in NADIFF and moreover

$$m^g_{v_1} = v_1 \int_0^1 g(t)\, dt$$

$$m^g_{v_2} = v_2 \int_0^1 g(t)\, dt$$

$$m^g_{v_3} = v_1 \cdot g(\alpha)$$

(i.e. none of them are in DIFF).

Proof It is easy to check that v_1, v_2, and v_3 are in EXT.

$$m^g_{v_1}(\chi) = \lim_{\substack{\tau>0\\ \tau\to 0}} \int_0^1 g(t)\, \frac{t + \tau \min(\mu_1^*(\chi),\ldots,\mu_n^*(\chi)-t}{\tau}\, dt$$
$$= \lim_{\substack{\tau>0\\ \tau\to 0}} \int_0^1 g(t) v_1^*(\chi) = v_1^*(\chi) \cdot \int_0^1 g(t)dt.$$

The second equality follows in the same manner.

$$m^g_{v_3}(\chi) = \lim_{\substack{\tau>0\\ t\to 0}} \int_0^1 g(t) \cdot \frac{1}{\tau} \left[\prod_{i=1}^{n} (f_\alpha o \mu_i^*)(t + \tau\chi) - \prod_{i=1}^{n} (f_\alpha o \mu_i^*)(t) \right] dt$$

$$= \lim_{\substack{\tau>0 \\ \tau\to 0}} \frac{1}{\tau} \int_{\alpha-\tau\cdot\min(\mu_1^*(\chi),\dots,\mu_n^*(\chi))}^{\alpha} g(t)\, dt.$$

Since g is continuous in $(\alpha-\tau v_1^*(\chi),\alpha)$ there exists $c(\tau)$ in this interval such that

$$m_{v_3}^{g}(\chi) = \lim_{\substack{\tau>0 \\ \tau\to 0}} v_1^*(\chi)\cdot\big(c(\tau)\big)$$
$$= g(\alpha)\cdot v_1^*(\chi).$$

A game of the form v_3 is called n parlaments majority game.

For convenience let us use from now on the notation m_v^1 also as a function on (when identified with the indicator functions) i.e. we will refer to m_v^1 sometimes as a function on $B(I, \mathcal{C})$ and sometimes as a function on $\mathcal{C}$.

Proposition 2 Let $v\in EXT$ and let f be a continuous function on $[0,1]$. If for each $\chi\in B(I,\mathcal{C})$ the limit

$$m_v^f(\chi) = \lim_{\substack{\tau>0 \\ \tau\to 0}} \int_0^1 f(t)\cdot\frac{v^*(t+\tau\chi)-v^*(t)}{\tau}\, dt$$

exists then for each $a,b\in E^1$ and for each $\chi\in B(I,\mathcal{C})$

$$m_v^f(a+b\chi) = a\cdot m_v^f(1) + b\cdot m_v^f(\chi).$$

Proof See [M, p. 527]

Proposition 3 The same conditions as Proposition 2 imply that for each continuous function g on $[0,1]$

$$m_{m_v^f}^{g}(\chi) = m_v^f\cdot\int_0^1 g(t)dt.$$

In particular if $v\in EXT$ and if $m_v^1(\chi)$ is well defined for each χ then,

$$m_{m_v^1}^{1} = m_v^1.$$

Proof Follows immediately from Proposition 2.

Proposition 4 If $v \in NADIFF$ is nondecreasing on $B(I,\mathcal{C})$ then m_v^1 is also nondecreasing.

Proof A restatement of Lyapunov's theorem is that $\mathcal{C}$ (when identified with the indicator functions) is DNA dense in $B_1(I,\mathcal{C})$. Therefore if $v \in EXT$ is nondecreasing then v^* is nondecreasing. Thus for each χ_1 and χ_2 in $B(I,\mathcal{C})$ with $\chi_1 \geqslant \chi_2$

$$\frac{v^*(t+\tau\chi_1)-v^*(t)}{\tau} \geqslant \frac{v(t+\tau\chi_2)-v^*(t)}{\tau}$$

for each $\tau > 0$ and $0 \leqslant t \leqslant 1$. Hence $m_v^1(\chi_1) \geqslant m_v^1(\chi_2)$.

Proposition 5 Let $v \in NADIFF$ and assume that for $0 \leqslant \alpha \leqslant 1$ $v^*(t)$ is continuous at $t=\alpha$. Then $m_v^{\chi_{[\alpha,1]}}(1) = v^*(1) - v^*(\alpha)$, where $\chi_{[\alpha,1]}$ is the indicator of $[\alpha,1]$. In particular if $\alpha=0$ then $m_v^1(1) = v(I)$.

(In fact we have defined m_v^f only for continuous f but the definition can be obviously extended to all bounded and measurable functions f).

Proof According to [M, p. 538] the limit

$$m_v^{\chi_{[\alpha,1]}}(1) = \lim_{\substack{\tau>0\\ \tau\to 0}} \int_\alpha^1 \frac{v^*(t+\tau)-v^*(t)}{\tau}\,dt$$

exists (there, only games in DIFF are considered, however the proof does not make any use of the additivity property of m_v^f for games v in DIFF). Hence:

$$m_v^{\chi_{[\alpha,1]}}(1) = \lim_{\substack{\tau>0\\ \tau\to 0}} \frac{1}{\tau}\Big[\int_{\alpha+\tau}^{1+\tau} v^*(t)dt - \int_\alpha^1 v^*(t)dt\Big]$$

$$= \lim_{\substack{\tau>0\\ \tau\to 0}} \Big[\frac{1}{\tau}\int_1^{1+\tau} v^*(t)dt - \frac{1}{\tau}\int_\alpha^{\alpha+\tau} v^*(t)dt\Big].$$

From the continuity of $v^*(t)$ at $t=\alpha$ and $t=1$ ($v \in EXT$)

$$m_v^{\chi_{[\alpha,1]}}(1) = v^*(1) - v^*(\alpha).$$

<u>Proposition 6</u> For each $v \in NADIFF$

$$\|m_v^1\|_{IBV} \leq \|v\|_{BV}$$

<u>Proof</u> Let

$$\Omega : 0 = \chi_0 \leq \chi_1 \leq , \ldots \leq \chi_k = 1$$

be a chain of functions from $B_1(I, C)$.

(1) $$\|m_v^1\|_\Omega = \sum_{i=0}^{k-1} |m_v^1(\chi_{i+1}) - m_v^1(\chi_i)| =$$

$$= \sum_{i=0}^{k-1} |\lim_{\substack{\tau>0 \\ \tau\to 0}} \frac{1}{\tau} \int_0^1 [v^*(t+\tau\chi_{i+1}) - v^*(t+\tau\chi_i)]dt|.$$

using $v = v^+ - v^-$

$$\sum_{i=0}^{k-1} |\frac{1}{\tau} \int_0^1 [v^*(t+\tau\chi_{i+1}) - v^*(t+\tau\chi_i)]dt| \leq$$

$$\sum_{i=0}^{k-1} |\frac{1}{\tau} \int_0^1 [(v^*)^+(t+\tau\chi_{i+1}) - (v^*)^+(t+\tau\chi_i)]dt - \frac{1}{\tau} \int [(v^*)^-(t+\tau\chi_{i+1}) - (v^*)^-(t+\tau\chi_i)]dt|.$$

$$\leq \sum_{i=0}^{k-1} | \frac{1}{\tau} \int_0^1 [(v^*)^+(t+\tau\chi_{i+1}) - (v^*)^+(t+\tau\chi_i)]dt|$$

$$+ \sum_{i=0}^{k-1} |\frac{1}{\tau} \int_0^1 [(v^*)^-(t+\tau\chi_{i+1}) - (v^*)^-(t+\tau\chi_i)]dt|$$

$(v^*)^-$ and $(v^*)^+$ are nondecreasing on $B_1(I,C)$ therefore the above inegrals exist. Moreover, the last sums can be written as

$$\frac{1}{\tau} \int_0^1 [(v^*)^+(t+\tau) - (v^*)^+(t)]dt + \frac{1}{\tau} \int_0^1 [(v^*)^-(t+\tau) - (v^*)^-(t)]dt.$$

From the continuity of $(v^*)^-(t)$ and $(v^*)^+(t)$ at t=0 and t=1 the last two summands converge to $(v^*)^+(1)-(v^*)^+(0)$ and $(v^*)^-(1)-(v^*)^-(0)$ respectively as $\tau\to 0$. Hence by (1)

$$\|m_v^1\|_\Omega \leqslant (v^*)^+(1)-(v^*)^+(0)+(v^*)^-(1)-(v^*)^-(0).$$

Since $(v^*)^-(0) = (v^*)^-(0) = 0$ and since $(v^*)^+(1) = v^+(I)$ and $(v^*)^-(1) - v^-(I)$,

$$\|m_v^1\|_\Omega \leqslant v^+(I) + v^-(I) = \|v\|.$$

The last inequality holds for each Ω therefore $\|m_v^1\|_{IBV} \leqslant \|v\|_{BV}$

<u>Proposition 7</u> Let v be in NADIFF. If $|v^*|(t)$ is continuous for each $0 \leqslant t \leqslant 1$ then for each $f_\alpha o\mu\varepsilon J$ the game $w = (f_\alpha o\mu)\cdot v$ is in NADIFF and

$$(2) \qquad m_w^f(\chi) = f(\alpha)v^*(\alpha)\,\mu^*(\chi) + \lim_{\substack{\tau>0\\ t\to 0}} \int_\alpha^1 f(t)\cdot\frac{v^*(t+\tau\chi)-v^*(t)}{\tau}\,dt$$

<u>Proof</u> According to [M,p.538] the limit

$$\lim_{\substack{\tau>0\\ \tau\to 0}} \int_0^1 \chi_{[0,\alpha]}\,\frac{v^*(t+\tau\chi)-v^*(t)}{\tau}\,dt,$$

exists for each $\chi\varepsilon B(I,C)$ and for each $v\varepsilon$ NADIFF such that $|v^*|(t)$ is continuous on [0,1]. The proof of proposition 2 of [M] will remain true if we replace there the interval [0,t] by the function $f\cdot\chi_{[t,1]}$, where f is bounded function which is continuous at each point in [0,1] but for a set of measure 0 with respect to the measure $d|v^*|(t)$. Moreover, in that case the limit

$$\lim_{\substack{\tau>0\\ \tau\to 0}} \int_0^1 f_\alpha(t)\cdot f(t)\cdot\frac{v^*(t+\tau\chi)-v^*(t)}{\tau}\,dt$$

exists for each continuous function f on [0,1] and for each $\chi\varepsilon B(I,C)$. (Again, the proof there is for games v in DIFF, however, it does not make any use of the additivity property of m_v^f. Thus it is valid for games v in NADIFF).

Therefore, the right hand side of (2) is well defined and it remains to prove that the equality (2) holds. Denote

$$\beta_f(\tau,\chi) = \int_0^1 \left[f(t)\cdot\frac{w^*(t+\tau\chi)-w^*(t)}{\tau} - v^*(\alpha)\cdot f(\alpha)\cdot\mu^*(\chi) - f_\alpha(t)\cdot f(t)\cdot\frac{v^*(t+\tau\chi)-v^*(t)}{\tau}\right]dt$$

It is sufficient to prove that

$$\lim_{\substack{\tau>0\\ \tau\to 0}} \beta_f(\tau,\chi) = 0$$

for any continuous function f on [0,1]. Indeed for each $\tau > 0$ if f_α is continuous from the right then

$$w^*(t+\tau\chi) = \begin{cases} v^*(t+\tau\chi) & t \geq \alpha-\tau\mu^*(\chi) \\ 0 & t < \alpha-\tau\mu^*(\chi) \end{cases}$$

$$w^*(t) = \begin{cases} v^*(t) & t \geq \alpha \\ 0 & t < \alpha \end{cases}$$

Hence

$$\beta_f(\tau,\chi) = \int_{\alpha-\tau\mu^*(\chi)}^{\alpha} f(t)\cdot\frac{v^*(t+\tau\chi)}{\tau}\,dt - \int_0^1 f(\alpha)v^*(\alpha)\mu^*(\chi)dt.$$

This implies

$$(3)\qquad |\beta_f(\tau,\chi)| \leq \int_{\alpha-\tau\mu^*(\chi)}^{\alpha} \left|\frac{v^*(t+\tau\chi)-v^*(\alpha)}{\tau}\right|\cdot|f(t)|dt + \int_{\alpha-\tau\mu^*(\chi)}^{\alpha} \frac{v^*(\alpha)}{\tau}|f(t)-f(\alpha)|dt.$$

$|v^*|(t)$ is continuous at $t=\alpha$, therefore for any $\varepsilon > 0$ there is $\delta_1 > 0$ such that $|v^*|(\alpha+\delta_1)-|v^*|(\alpha-\delta_1) < \frac{\varepsilon}{2M}$, where $M = \sup_{0\leq x\leq 1} f(x)$. Thus, for each

$0 < \tau < \delta_1 \quad [\alpha-\tau\mu^*(\chi) \leq t \leq \alpha \Longrightarrow \alpha-\delta_1 \leq t+\tau\chi \leq \alpha+\delta_1].$

Therefore,

$$\begin{aligned} v^*(t+\tau\chi)-v^*(\alpha) &= (v^*)^+(t+\tau\chi)-(v^*)^-(t+\tau\chi)-(v^*)^+(\alpha)+(v^*)^-(\alpha) \\ &\leq (v^*)^+(\alpha+\delta_1)-(v^*)^-(\alpha-\delta_1)-(v^*)^+(\alpha-\delta_1)+(v^*)^-(\alpha+\delta_1) \end{aligned}$$

$$= |v^*|(\alpha+\delta_1)-|v^*|(\alpha-\delta_1) < \frac{\varepsilon}{2M} .$$

In the same way one can also derive

$$v^*(\alpha)-v^*(t+\tau\chi) \leq |v^*|(\alpha+\delta_1)-|v^*|(\alpha-\delta_1) < \frac{\varepsilon}{2M} .$$

Thus,

$$(4) \qquad |v^*(\alpha) - v^*(t+\tau\chi)| < \frac{\varepsilon}{2M} .$$

Hence if $v^*(\alpha) = 0$ our proof is complete. In case $v^*(\alpha) \neq 0$, from the continuity of f at $t=\alpha$ there exists $\delta_2 > 0$ such that for each $0 < \tau < \delta_2$ and for each t with $\alpha-\tau\mu^*(\chi) \leq t \leq \alpha$ $\quad |f(t)-f(\alpha)| < \frac{\varepsilon}{2|v^*(\alpha)|}$.

Together with (3) and (4) we then get for each $0 < \tau < \min(\delta_1, \delta_2)$

$$|\beta_f(\tau, \chi)| \leq \frac{1}{\tau} \int_{\alpha-\tau\mu^*(\chi)}^{\alpha} \varepsilon \, dt \leq \varepsilon.$$

The proof for the case where f_2 is continuous from the left is similar.

Definition the set DIAG* is the set of all games v in EXT such that the following limit and equality

$$m_v^1(\chi) = \lim_{\substack{\tau>0 \\ \tau\to 0}} \int_0^1 \frac{v^*(t+\tau\chi)-v^*(t)}{\tau} \, dt = 0,$$

exists for each $\chi \in B_1(I, C)$. roughly speaking v is in DIAG* if for each $\chi \in B_1(I, \ )$ the average of the marginal contributions of the ideal coalition χ to the diagnonal $\{f(x) \equiv t \mid 0 \leq x \leq 1\}$ is zero.

Definition A value ϕ on a symmetric subspace Q of EXT is called "strongly diagnonal" if for each $v \in Q \cap \text{DIAG*}$ $\quad \phi v = 0$.

The following proposition shows the connection between DIAG and DIAG*.

Proposition 8 If $v \in$ DIAG* has an extension which is DNA continuous then $v \in$ DIAG*.

Proof $v \in$ DIAG implies the existence of a vector $\mu = (\mu_1, \ldots, \mu_n)$, of NA^1

measures and $\varepsilon > 0$ such that if $U_\varepsilon = \{x\varepsilon E^n_+ | d(x,[\mu(\phi),\mu(I)]) < \varepsilon\}$ then $\mu(S)\varepsilon U_\varepsilon \Longrightarrow v(S) = 0$. We shall show that for each $f\varepsilon B_1(I,C)$ $\mu^*(f)\varepsilon U_\varepsilon \Longrightarrow v^*(f) = 0$. Let us assume that $f\varepsilon U_\varepsilon$ but $v^*(f) \neq 0$. W.ℓ.o.g. let us assume that $v^*(f) > 0$. Denote $B = \{\chi\varepsilon B_1(I, \ )|v^*(\chi) > 0\}$. v^* is DNA continuous therefore B is open in the DNA topology and it contains f. Thus there is a neighborhood B_f of f of the form $B_f = \{\chi\varepsilon B_1(I, \ )|\nu^*(\chi) = \nu^*(f)\}$ for some vector measure ν of measures in NA^1, which is contained in B. Using Lyapunov's theorem for (μ,ν) there is $S\varepsilon C$ such that $(\mu^*, \nu^*)(f) = (\mu,\nu)(S)$. Hence, $\chi_S\varepsilon B_f$ and therefore $\chi_S\varepsilon B$ which implies that$v(S) > 0$. On the other hand $\mu^*(f)\varepsilon U_\varepsilon$ therefore $\mu(S)\varepsilon U_\varepsilon$ and hence $v(S) = 0$. This contradiction establishes the proof of the proposition.

Remark There are games which are not in DIAG although it is natural to include them there. For example consider the game $v = \max(\mu_1, 2\mu_2)$ where μ_1 and μ_2 are two measures in NA^1 which are mutually singular. For any automorphism Θ which preserves μ_2 but not μ_1 (i.e. $\theta^*\mu_2 = \mu_2$ but $\theta^*\mu_1 \neq \mu_1$) the game $w = v - \Theta^*v$ vanishes in a neighborhood of the diagnonal, determined by the vector measure $\mu = (\mu_1, \Theta^*\mu_1, \mu_2)$, except for the origin. i.e. there is a neighborhood U of the half open interval $\left((0,0,0), (1,1,1)\right]$ such that for each $S\varepsilon C$ if $\mu(S)\varepsilon U$ then $v(S) = 0$. Formally $w \notin DIAG$, however it is natural to expect that a diagnonal value ϕ on the linear and symmetric supspace Q(v) that generated by v will vanish on w. It turns out that this is false. With the same technique as in [T] one can prove the existence of a diagonal value γ on Q(v) which satisfies $\gamma v = \frac{2}{3}(\mu_1 + 2\mu_2)$. This implies $\gamma w = \frac{2}{3}(\mu_1 - \Theta^*\mu_1) \neq 0$. On the other hand $w\varepsilon DIAG^*$ (since $v\varepsilon NADIFF$ and $m^1_w = 0$) and therefore each strong diagonal value ϕ on Q(v) will satisfy

$\phi w = 0$.

Definition A subset B of EXT is invariant if for each $v \varepsilon B$ $m_v^1 \varepsilon B$. If $B \subseteq$ NADIFF we denote by m_B^1 the set of all m_v^1 for $v \varepsilon B$.

Examples the spaces pNA, bv'NA, DIFF and Q^n are all invariance spaces.

Notice that

$$m^1_{PNA} = m^1_{bv^1 NA} = m^1_{DIFF} = NA$$

and all of them contain NA. By proposition 1 and from the linearity of the mapping $m \to m_v^1$ for any $v \varepsilon Q^n$ $m_v^1 = v$.

Remark It is easy to verify that a value ϕ on a symmetric supspace Q of EXT is a strong diagonal value if and only if $\phi v = \phi m_v^1$. Denote by ϕ_D the value on DIFF. Since $\phi_D \mu = \mu$ for any $\mu \varepsilon NA$ ϕ_D is a strong diagonal value.

Definition For any game v the integral of v is denoted by $\int v$ and is defined to be the set of all games w in EXT for which m_w^1 is well defined and $v = m_w^1$. In the same way the integral of the set of games B is denoted by $\int B$ and is defined by

$$\int B = \bigcup_{v \varepsilon B} \int v$$

Remarks (1) From the main theorem of [M] we have $\int NA \subseteq$ DIFF.

In fact one can show that a strictly inclusion holds.

(2) If Q is a linear and symmetric space of games then $\int Q$ is a linear and symmetric space of games in EXT which contains DIAG* (Notice that $\int 0$ = DIAG*).

(3) It might be the case where $\int v = \emptyset$ for $v \varepsilon$ NADIFF. Indeed proposition 2 implies $m_w^1(t) = t\, m_w^1(1)$ for each $w \varepsilon$ NADIF and each $0 \leq t \leq 1$. Thus $m_w^1(t)$ is continuous at t and hence $\int f_\alpha o \mu = \emptyset$ for each $f_\alpha o\mu \ \varepsilon\ J$.

Theorem 9 Let ϕ be a value of a linear and symmetric subspace Q of NADIFF. If

(1) $NA \subseteq Q$.

(2) For each $v \in Q$ $|v^*|(t)$ is continuous on $[0,1]$

(3) For each $0 \leq t \leq 1$ and for each $v \in Q$ $m_v^{\chi[\alpha,1]} \in Q$,

then there exists a strong diagonal value of γ on $(Q \circledast J) \vee \int Q$ which is an extension of ϕ_D on DIFF and which satisfies for each $v \in Q$ and $f_\alpha o \mu \in J$

$$\gamma\big((f_\alpha o \mu)\cdot v\big) = v^*(t)\mu + \phi(m_v^{\chi[\alpha,1]}).$$

Moreover $\|\gamma\| \leq \|\phi\|$.

Proof Any game w in $(Q \circledast J) \vee \int Q$ is of the form

$$w = \sum_{i=1}^{n} (f_{t_i} o \mu_i)\cdot v_i + v$$

where $v \in \int Q$, $v_i \in Q$, $0 \leq t_i < 1$, $\mu_i \in NA^1$ and $1 \leq i \leq m$.

Define $\gamma : (Q \circledast J) \vee \int Q \to FA$ by

$$\gamma w = \sum_{i=1}^{n} v^*(t)\mu_i + \sum_{i=1}^{n} \phi\big(m_{v_i}^{\chi[t_i,1]}\big) + \phi m_v^1.$$

If γ is well defined then by definition it is linear and symmetric. By proposition 5 $(\phi m_v^1)(I) = m_v^1(I) = v(I)$, and for each $1 \leq i \leq m$

$$\gamma\big((f_{t_i} o \mu_i)\cdot v_i\big)(I) = v_i^*(t_i)\mu_i(I) + \big[\phi m_{v_i}^{\chi[t_i,1]}\big](I) =$$

$$= v_i^*(t_i) + m_{v_i}^{\chi[t_i,1]}(1) =$$

$$= v_i^*(t_i) + v_i^*(1) - v_i^*(t_i) = v_i^*(1) = v_i(I).$$

Thus γ is efficient.

By proving that γ is positive we would conclude that γ is well defined. Indeed if w is nondecreasing m_w^1 is nondecreasing (Proposition 4), and by Proposition 7,

$$m_w^1 = \sum_{i=1}^{n} v_i^*(t_i)\cdot\mu_i + \sum_{i=1}^{n} m_{v_i}^{\chi[t_i,1]} + m_v^1$$

Since $m_{v_i}^{\chi[t_i,1]}$ and m_v^1 are in Q and since $NA^1 \subseteq Q$ $m_w^1 \varepsilon Q$. ϕ is a value on Q and m_w^1 is non-decreasing, thus $\phi m_w^1 \geq 0$. Now, since the unique value on NA is the identify functional i.e. $\phi\mu = \mu$ for each $\mu \varepsilon NA$ we have

$$(5) \qquad 0 \leq \phi m_w^1 = \sum_{i=1}^{n} v^*(t_i)\mu_i + \sum_{i=1}^{n} \phi(m_{v_i}^{\chi[t_i,1]}) + \phi m_v^1 = \gamma w.$$

Thus, γ is positive and hence γ is a value on $(Q \circledast J) \vee \int Q$. To show that γ is a strong diagonal value denote $u_i = m_{v_i}^{\chi[t_i,1]}$, $1 \leq i \leq m$.

$$\gamma m_w^1 = \sum_{i=1}^{m} v_i^*(t) \cdot \mu_i + \sum_{i=1}^{m} \phi m_{u_i}^1 + \phi m_{m_v^1}^1.$$

Hence by proposition 3 we derive that $\gamma m_w^1 = \gamma w$, which proves that γ is strongly diagonal. γ is an extension of ϕ_D since for each $v \varepsilon DIFF$ $m_v^1 \varepsilon NA$ and $\phi_D v = m_v^1$. On the other hand $DIFF \subseteq \int NA$ and for each $v \varepsilon \int NA$ $\gamma v = \phi m_v^1 = m_v^1 = \phi_D v$.

The inequality $\|\gamma\| \leq \|\phi\|$ is derived by (5) and by Proposition 6 as follows

$$\|\gamma w\|_{BV} = \|\phi m_w^1\|_{BV} \leq \|\phi\| \cdot \|m_w^1\|_{IBV} \leq \|\phi\| \cdot \|w\|_{BV}.$$

Thus the proof is complete.

Remark Condition (3) of Theorem 9 holds, for example, for the spaces pNA, bv'NA, DIFF and Q^n.

Our purpose now is to apply the above theorem to subspaces Q of NADIFF which consists of games which are homogenous of degree 1. To that end we need first the following proposition.

Proposition 10 If $v \varepsilon NADIFF$ is homogenous of degree 1 then

(1) $|v^*|(t)$ is continuous for each $0 \leq t \leq 1$

(2) For each $0 \leq \alpha < 1$ $m^{\chi[\alpha,1]} = (1-\alpha)m_v^1$.

Proof (1) v is homogenous of degree 1, therefore v^- and v^+ are hom. of degree 1. Thus for each $0 \leqslant t \leqslant 1$

$$|v^*|(t) = (v^*)^+(t)+(v^*)^-(t) = t[(v^*)^+(1)+(v^*)^-(1)] = t\,\|v\|.$$

Hence $|v^*|(t)$ is continuous on $[0,1]$.

(2) For each $0 \leqslant \alpha < 1$

$$\alpha \cdot \lim_{\substack{\tau>0 \\ \tau\to 0}} \frac{1}{\tau}\int_0^1 [v^*(t+\tau\chi)-v^*(t)]\,dt = \lim \frac{1}{\tau}\int_0^1 [v^*(\alpha t+\alpha t\chi)-v^*(\alpha t)]\,dt$$

$$= \lim_{\substack{\tau\to 0 \\ \tau\to 0}} \frac{1}{\alpha\tau}\int_0^1 [v^*(s+\alpha\tau\chi) - v^*(s)]\,ds$$

$$= \lim_{\substack{\tau>0 \\ \tau\to 0}} \frac{1}{\tau}\int_0^1 [v^*(s+\tau\chi) - v^*(s)]\,ds.$$

Hence $\alpha \cdot m_v^1 = m_v^{\chi[0,\alpha]}$ or $(1-\alpha)m_v^1 = m_v^{\chi[\alpha,1]}$.

Theorem 11 Let ϕ be a value on an invariant space Q of games in NADIFF which are homogenous of degree one. If Q contains NA then there exists a strong diagonal value γ on $(Q\circledast J) \vee \int Q$ which is an extension of ϕ_D on DIFF. Moreover

(1) $\gamma((f_\alpha o\mu)\cdot v) = \alpha v(I)\cdot\mu + (1-\alpha)\phi m_v^1$,

(2) $\|\gamma\| \leqslant \|\phi\|$.

Proof Follows immediately from theorem 9 and proposition 10.

Corollary 12 Let $Q = Q^n \vee NA$. Then there exists a strong diagonal value γ on $(Q\circledast J) \vee \int Q$ which coincides with ϕ_D on DIFF and with the unique value ϕ_n on Q^n. Moreover

$$\gamma(v_n \cdot f_\alpha o\mu) = \alpha\cdot v(I)\cdot\mu + (1-\alpha)\cdot\frac{\mu_1+\ldots+\mu_n}{n}$$

where $v_n = \min(\mu_1,\ldots,\mu_n)$ and μ_i and μ_j are mutually singular for $i\neq j$.

Proof The space $Q = Q^n \vee NA$ is invariant space that contains NA. Moreover $v = m_v^1$ for each $v\in Q$. By [T] there exists a (unique) value ϕ_n on Q. Hence by Theorem 11 there exists a strong diagonal value which is an extension of ϕ_D

on DIFF such that for each $v \in Q^n$

$$\gamma\big((f_\alpha o\mu)\cdot v\big) = \alpha v(I)\mu + (1-\alpha)\ \phi_n\ m_v^1.$$
$$= \alpha v(I)\mu + (1-\alpha)\ \phi_n v.$$

This together with the fact

$$\phi_n(\min\ (\mu_1,\ldots,\mu_n)) = \frac{\mu_1+\ldots+\mu_n}{n}$$

completes the proof of the theorem.

Definition A market game is a game in EXT which is supper-additive and homogenous of degree 1. Denote by MA the set of all market games.

Proposition 13 Any market game is in NADIFF. Moreover for each bounded measurable (Borel) function g on [0,1] and for each $v \in MA$

$$\lim_{\substack{\tau>0\\ \tau\to 0}} \int_0^1 g(t)\cdot \frac{v^*(t+\tau\chi)-v^*(t)}{\tau}\,dt = \int_0^1 g(t)dt \lim_{\substack{\tau>0\\ \tau\to 0}} \frac{v^*(t+\tau\chi)-v^*(t)}{\tau}$$

Proof Follows from [M,p.540].

Definition Let NF be the closure in the BV-norm of the set of all games in NADIFF which are function of finite number of NA measure. Let F be defined in the same way except that the BV-norm is replaced by the sup-norm.

Proposition 14 NF is invariance subspace of NADIFF and $v - m_v^1$ is in $DIAG^* \cap$ NADIFF.

Proof Let $v \in NF$. Let $(v_n)_{n=1}^\infty$ be a sequence of games in NADIFF of the form

$v_n = f_n o\mu_n$ where μ_n is a vector of finite number of NA measures such that

$$\| v_n - v\|_{BV} \to 0 \text{ as } n\to\infty.$$

$m_{v_n}^1$ is a function of μ_n since if $\chi_1, \chi_2 \varepsilon B_1(I,\quad)$ and if $\mu_n^*(\chi_1) = \mu_n^*(\chi_2)$

$v_n^*(t+\tau\chi_1) = f_n\big(t\mu_n(I) + \tau\mu_n^*(\chi_1)\big) = f_n\big(t\mu_n(I) + \tau\mu_n^*\ (\chi_2)\big) = v_n^*(t+\tau\chi_2).$

Therefore $m_{v_n}^1(\chi_1) = m_{v_n}^1(\chi_2)$ and $m_{v_n}^1\ \varepsilon F.$ Now, by Proposition 6

$$\| m_{v_n}^1 - m_v^1\|_{IBV} = \| m_{v_n}^1 - v\|_{IBV} \leqslant \| v_n - v\|_{BV} \to 0 \text{ as } n\to\infty.$$

Since for each $w \varepsilon BV$ $\|w\|_{BV} \geq \|w\|_{sup}$

$$\|m^1_{v_n} - m^1_v\|_{sup} \to 0 \text{ as } n \to \infty,$$

hence $m^1_v \varepsilon$ F and m^1_v is DNA continuous. Let us prove now that $m^1_v \varepsilon$ EXT. Notice first that $m^1_v \varepsilon$ BV since $v \varepsilon BV$ and $\|m^1_v\|_{IBV} \leq \|v\|_{BV}$. Now, m^1_v, is homogenous of degree 1 (Proposition 2) therefore $(m^1_v)^+$ and $(m^1_v)^-$ are homogenous of degree 1. Hence, for each $0 \leq t \leq 1$

$$|m^1_v| \ (t) = (m^1_v)^+(t) + (m^1_v)^-(t) = t \ \|m^1_v\|_{IBV}.$$

Thus $|m^1_v|(t)$ is continuous in t and $m^1_v \varepsilon$ EXT. Proposition 2 implies that m^1_v is in NADIFF and

$$m^1_{m^1_v - v} = m^1_{m^1_v} - m^1_v = 0.$$

Thus $v - m^1_v \ \varepsilon$ DIAG* $\cap$ NADIFF.

Theorem 15

(1) The space MA $\cap$ NF is invariant

(2) Each $v \varepsilon$ MA $\cap$ NF is of the form $w + m^1_v$ where $w \varepsilon$ DIFF $\cap$ DIAG*.

Proof Let v be in MA $\cap$ NF. By Proposition 13 MA $\subseteq$ NADIFF and for each $\chi \varepsilon B_1(I,C)$ and $t>0$

$$m^1_v(\chi) = \lim_{\substack{\tau > 0 \\ \tau \to 0}} \frac{1}{\tau} \left[v^*(t+\tau\chi) - v^*(t) \right].$$

Together with the super-additivity of v*, for each χ_1, χ_2 in $B_1(I,C)$ such that $\chi_1 + \chi_2 \ \varepsilon B_1(I,C)$

$$m^1_v(\chi_1 + \chi_2) = \lim_{\substack{\tau > 0 \\ \tau \to 0}} \frac{1}{\tau} \left[v^*(t+\tau(\chi_1 + \chi_2)) - v^*(t) \right] \geq$$

$$\geq \lim_{\substack{\tau>0\\ \tau\to 0}} \frac{1}{\tau}\left[v^*(\tfrac{t}{2}+\tau\chi_1) - v^*(\tfrac{t}{2})\right] +$$

$$+ \lim_{\substack{\tau>0\\ \tau\to 0}} \frac{1}{\tau}\left[v^*(\tfrac{t}{2}+\tau\chi_2) - v^*(\tfrac{t}{2})\right].$$

Hence

$$m_v^1(\chi_1+\chi_2) \geq m_v^1(\chi_1) + m_v^1(\chi_2),$$

and thus m_v^1 is superadditive. $m_v^1 \varepsilon$ NF (Proposition 14) Hence $m_v^1 \varepsilon$ MA $\cap$ NF. Now, by Proposition 13 for each continuous function f on [0,1]

$$m^f_{v-m_v^1} = m_v^f - m^f_{m_v^1} = m_v^f - m_v^1 \cdot {}_0\!\int^1 f(t)dt = m_v^f - m_v^f = 0.$$

Therefore $m^f_{v-m_v^1}$ is additive and $v - m_v^1 \;\varepsilon$ DIFF.

Theorem 11 can be restated for supspace Q of market games that are spanned by games which are function of finite number of measures as follows.

<u>Theorem 16</u> Let ϕ be a value on a supspace Q of MA$\cap$NF that contains NA. Then there exists a strong diagonal value γ on $(Q\circledast J)\vee\int Q$ which is an extension of ϕ_D on DIFF. γ obeys

(1) $\gamma\big((f_\alpha \circ \mu)\cdot v\big) = \alpha\; v(I)\cdot\mu + (1-\alpha)\phi m_v^1$

(2) $\|\gamma\| \leq \|\phi\|$.

The rest of the paper is conceptually connected to the previous discussion however it is completely independent. Denote by H' the set of all games in F which are homogenous of degree one and NA continuous at 1. H'$\cdot$J is the set of all games of the form $(f_\alpha \circ \mu)\cdot v$ where $f_\alpha \circ \mu \varepsilon J$ and $v \varepsilon H'$. Let H'J be the minimal linear and symmetric space that contains H'$\cdot$ J. It turns out that the measure $v(I)\cdot\mu$ that distributes the amount v(I) among the players according to their political power only, defines a value on H'J.

Theorem 17

(1) A value ϕ on H'J does exist. ϕ satisfies $\phi\big((f_\alpha o\mu)\cdot v\big) = v(I)\cdot\mu$.

(2) A semi-value $\bar{\phi}$ on H'J does exist. $\bar{\phi}$ satisfies $\bar{\phi}\big((f_\alpha o\mu)\cdot v\big) = \alpha\cdot v(I)\cdot\mu$.

Proof Each $w\varepsilon H'J$ is of the form

$$w = \sum_{i=1}^{n} (f_{t_i} o\mu_i)v_i$$

where $v_i \varepsilon H'$, $f_{t_i} o\mu_i \varepsilon J$, $1 \le i \le n$. Let us define ϕ and $\bar{\phi}$ on H'J by

$$\phi w = \sum_{i=1}^{n} v_i(I)\,\mu_i, \qquad \bar{\phi} w = \sum_{i=1}^{n} t_i v_i(I)\mu_i$$

By definition if ϕ is well defined then it is linear symmetric and efficient, and if $\bar{\phi}$ is well defined then it is linear and symmetric. Hence in order to complete the proof of theorem 17 it is sufficient to prove that if w is non-decreasing then both $\Sigma\, v_i(I)\mu_i \ge 0$ and $\Sigma\, t_i v_i(I)\cdot\mu_i \ge 0$ (providing that we also prove that ϕ and $\bar{\phi}$ are well defined). Denote $N = \{1, 2, \ldots, n\}$. Let us partition N into sets $N_1, N_2, \ldots, N_L$ according to the jumps location i.e.

$$N = \bigcup_{i=1}^{L} N_i \quad N_i \cap N_j = \emptyset \text{ for } i \ne j \text{ and}$$

$$t_i < t_j \iff \exists\, k, \exists\, \ell \quad 1 \le k < \ell \le L \; [i\varepsilon N_k,\ j\varepsilon N_\ell].$$

Now for each $1 \le k \le L$ let us partition N_k according to the majority measures. i.e. $N_k = \bigcup_{r=1}^{\ell_k} N_k^r \quad N_k^r \cap N_k^s = \emptyset$ for $r \ne s$ and

$$\forall i,j\ \mu_i = \mu_j \iff \exists\, m,\ 1 \le m \le \ell_k\ (i,\ j\ \varepsilon\ N_k^m).$$

For each m, $1 \le m \le \ell_k$, let us choose a representative i in N_k^m and let us denote $\eta_k^m = \mu_i$. Let $\eta_k = (\eta_k^1, \ldots, \eta_k^{\ell_k})$ and let k, $1 \le k \le L$ be fixed. η_k consists of ℓ_k different NA^1 measures. Therefore there exists a coalition $T\varepsilon\, C$ such that $\eta_k^i(T) \ne \eta_k^j(T)$ for each $i \ne j$, $1 \le i, j \le \ell_k$ (for a proof see the proof of Proposition 8.11 of [A-S]). W.ℓ.o.g. let us assume that

$$\eta_k^1(T) < \eta_k^2(T) < \cdots < \eta_k^{\ell_k}(T).$$

For any $\varepsilon > 0$ define g_ε in $B_1(I,\mathbb{C})$ by

$$g_\varepsilon = \varepsilon\chi_T + (1-\varepsilon)\chi_I.$$

For each $1 \leq i < j \leq \ell_k$

$$(\eta_k^i)^*(g_\varepsilon) < (\eta_k^j)^*(g_\varepsilon).$$

Therefore, since $g_\varepsilon \longrightarrow 1$ in the NA topology as $\varepsilon \to 0$

$$|(\eta_p^q)^*(1-g_\varepsilon)| < \min_{t_i \neq t_j} |t_i - t_j|,$$

for each $1 \leq p \leq L$ and $1 \leq q \leq \ell_p$.

Let us fix j_0, $1 \leq j_0 \leq \ell_k$ and let us choose $0 < \beta_0 < 1$ such that

$$\eta_k^{j_0}(\beta_0 \cdot g_\varepsilon) = t_k.$$

Assume that f_{t_i} is continuous from the left on $[0,1]$ for each $1 \leq i \leq n$. Since $w \in F$ is nondecreasing w^* is nondecreasing on $B_1(I,\mathbb{C})$ and thus for each $\beta > \beta_0$

(6) $$0 \leq w^*(\beta \cdot g_\varepsilon) - w^*(\beta_0 \cdot g_\varepsilon) = \sum_{i \in \bigcup_{p=1}^{k-1} N_p} [(f_{t_i} \circ \mu_i^*) \cdot v_i^*](\beta \cdot g_\varepsilon) +$$

$$+ \sum_{i \in \bigcup_{j=j_0}^{\ell_k} N_k^j} [(f_{t_i} \circ \mu_i^*) \cdot v_i^*](\beta \cdot g_\varepsilon) - \sum_{i \in \bigcup_{p=1}^{k-1} N_p} [(f_t \circ \mu_i^*)](\beta_0 \cdot g_\varepsilon)$$

$$= \sum_{i \in \bigcup_{j=j_0+1}^{\ell_k} N_k^j} [(f_{t_i} \circ \mu_i^*) \cdot v_i^*](\beta_0 \cdot g_\varepsilon).$$

For each i, $1 \leq i \leq n$, v_i is homogenous of degree 1 thus if $\beta \to \beta_0$, $\beta > \beta_0$ we have

$$\beta_0 \cdot \sum_{i\varepsilon N_k^{j_0}} [(f_{t_i} o \mu_i^*) \cdot v_i^*](g_\varepsilon) \geq 0$$

If $\varepsilon > 0$ is small enough such that $\mu_i(g_\varepsilon) > t_i$ for each $1 \leq i \leq n$

$$\sum_{i\varepsilon N_k^{j_0}} v_i^*(g_\varepsilon) \geq 0.$$

v_i^* is NA continuous in 1 hence if ε tends to zero we have

$$\sum_{i\varepsilon N^{j_0}} v_i(I) \geq 0.$$

By the definition of $N_k^{j_0}$

$$\sum_{i\varepsilon N_k^{j_0}} v_i(I)\mu_i \geq 0.$$

The last inequality holds for each j_0, $1 \leq j_0 \leq \ell_k$. Therefore

$$\sum_{i\varepsilon N_k} v_i(I)\mu_i \geq 0.$$

and

$$\sum_{i\varepsilon N_k} t_{m_k} v_i(I)\mu_i \geq 0,$$

where $t_{m_i} = t_i$ for each $i\varepsilon N_k$. The last two inequalitites hold for each k therefore

$$\phi w = \sum_{i=1}^{n} v_i(I)\mu_i \geq 0$$

and

$$\phi \bar{w} = \sum_{i=1}^{n} t_i v_i(I) \mu_i \geq 0$$

Hence the proof is complete. In case there are i's for which f_{t_i} is continuous from the right on [0,1] we will use (6) twice, once for $\beta > \beta_0$ and once for $\beta < \beta_0$.

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