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LINEAR STOCHASTIC MODELS WITH DECENTRALIZED INFORMATION

by

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1. INTRODUCTION

This paper is an exploratory and incomplete attempt to characterize models where agents are faced with nontrivial problems of signal extraction. Such models have become increasingly important, particularly in modern business cycle theory as exemplified by Lucas (1972, 1975) and Sargent (1979). These models rely upon the notion that agents in a decentralized economy are unable to distinguish real from nominal shocks and temporary changes on their environment from persistent changes. Models have been developed which are capable of explaining a wide variety of phenomena which until recently could not be explained without resorting to a disequilibrium construct. Explaining the observed features of the business cycle (as for example serial correlation of output) presents a major challenge in an equilibrium construct. Three major routes have been taken to attack this problem. The first is to posit the existence of capital (Lucas (1975) , Blinder and Fischer (1979)) which distributes the effects of one period shocks over several periods. The second is to assume the existence of adjustment costs (Sargent (1979)) which tends to produce a capital-like effect by causing shocks in current periods to affect decisions made in future periods. The third is to assume the existence of contracts or fixity of prices and wages and thus distribute the effect of serially uncorrelated shocks over several periods (Taylor (1979)). In general, in adjustment cost models the serial correlation (of say employment) is determined by the technology of adjustment and the nature of the stochastic shocks. Models with capital (such as Lucas (1975)) require that agents do not observe certain aggregate variables (such as the economy-wide capital stock) at the end of each period to generate phenomena such as the gradual correlation output over the cycle.

This paper is an attempt to disentangle the effects of information lags in observing aggregate variables from the other effects discussed above. It deals with situations where agents do not observe certain variables with larger lags as they do others. In particular, it attempts to address the question of how the properties of the system change when information lags change. A related question to the one above is the question of whether agents' information sets can be represented by a small set of variables. Where agents never observe variables of interest to them, the answer is that there does not generally exist a small set of variables that could be said to be sufficient statistics that summarize all past information. The order of the difference equation describing the process is strongly affected by the informational asymmetries that exist.

Throughout this paper, I study only linear models since such models seem to be the ones that are econometrically most useful and are, of course, the only ones that are tractable. Section 2 presents the model and examines a situation with no decentralized information. It is seen that, in general, such structures are straightforward to solve by the method of undetermined coefficients. Section 3 provides an introduction to the case where informational asymmetries are involved. Section 4 attempts an extension to a case where agents never observe certain aggregate variables. This section contains an impossibility theorem that, I believe, has serious implications for this research. The theorem states that, in general, it is not possible to represent the process that agents use in forecasting prices as a rational polynomial. Section 5 concludes the paper.

2. EQUILIBRIUM IN A MODEL WITH COMPLETE INFORMATION

The model used to develop the ideas outlined in the introduction is a simple variant of the one first outlined by Muth (1961). J competitive producers are required to make production decisions $\{q_{it}\}_{i=1}^{J}$ prior to observing a market price P_t. Producers do not observe the market price directly. Instead, they are assumed to see the market price with an error ϵ_{it} which is identically, independently distributed across producers and time. The price which each producer i observes is

$$P_{it} = P_t + \epsilon_{it}$$

The production decision of a firm is linear in the price that is expected to prevail conditional upon the information that the firm possesses Ω_{it}.

$$q_{it} = aE(P_t | \Omega_{it}).$$

This decision rule can, of course, formally be derived by assuming that producers have quadratic cost functions and are profit maximizers.

Average output in the economy is given by

1

$$Q_t = \frac{1}{J} \sum_{i=1}^{J} q_{it}$$

This completes our discussion of the supply side of the model.

The demand for the good is dependent upon the market price P_t and is affected by a covariance stationary stochastic process u_t.

$$Q_t^d = u_t - P_t.$$

All variables here may be interpreted as deviations from the mean. The reader will have noted that we have assigned a coefficient of 1 to the price in the demand equation. This is not quite as arbitrary as it seems, since by an appropriate change in the coefficient a on the production equation and in u_t, the representation above is an equivalent one.

The model presented here may be rationalized in a number of ways. Firms may be thought of as distributed across "islands" with some consumers visiting some firms in any given period. The u_t shock is then interpreted as a shock to tastes and if the number of people who arrive at any firm changes randomly from period to period by ϵ_{it}, then ϵ_{it} is interpreted as as an instant shock. An alternative interpretation is provided by Lucas (1973).

Suppose as in Lucas,

$$q_{it} = b[P_{it} - E[P_t | \Omega_{it}]]$$

and

$$P_t + Q_t = u_t$$

where u_t is a shock to nominal demand and production by each firm is linear in the expected relative price of each firm i.e. relative to the "nominal" price in the economy. Then, by an appropriate relabeling of the variables, this system is seen easily to reduce to the model examined here. Lucas' model, of course was designed to examine how the slope of the aggregate supply schedule changes as the variance of the shocks to nominal demand changes. Our objective will be to ask what kinds of changes in the serial correlation of output result with changes in the information structure of the economy.

To summarize the model,

Demand	$Q_t^d = u_t - P_t$	1.1
Production	$q_{it} = aE[P_t \| \Omega_{it}] \quad i=1,\ldots J$	1.2
Supply	$Q_t = \frac{1}{J}\sum_{i=1}^{J} q_{it}$	1.3
Price	$P_{it} = P_t + \epsilon_{it} \quad i=1,\ldots J$	1.4

We now impose some structure on the model. It is well known that if u_t is a covariance stationary linearly indeterministic process then it has a representation (see appendix for definitions of these and related technical terms) of the form

$$u_t = \sum_{s=0}^{\infty} \alpha_s \eta_{t-s} \qquad 1.5$$

where

$$\sum_{s=0}^{\infty} \alpha_s^2 < \infty$$

$$E[\eta_t \eta_{t-s}] = 0 \quad \text{for } s \neq 0$$

and

$$\eta_t = u_t - \pi[u_t | u_{t-1}, u_{t-2}, \ldots]$$

where $\pi[\cdot|\cdot]$ is a projection operator (see appendix for a definition and properties of the projection operator).

and

$$E(\eta_t^2) = \sigma_\eta^2 .$$

We will also assume that u_t has an autoregressive representation.

Furthermore, since the ϵ_{it} are identically, independently distributed across time and firms, we have (if there are a large number of firms)

$$E(\epsilon_{it}) = 0 \qquad 1.6$$

$$E(\epsilon_{it}^2) = \sigma_\epsilon^2$$

$$E(\epsilon_{it} \eta_s) = 0 \quad \text{all } t \text{ and } s, \; i = 1, \ldots J.$$

In what follows, the expectation operator will be replaced by the linear projection operator $\pi[\cdot|\cdot]$. The two operators are identical if η and ϵ

are normally distributed. In any case, since our main interest is in linear models, we are interested in the linear projection operator rather than the (possibly) non-linear expectation operator.

Our first endeavor is to examine the rational expectations equilibrium in the case where the information sets of agents contain the market price and quantities.

Suppose that

$$\Omega_{it} = \begin{bmatrix} P_{t-1}, P_{t-2}, \ldots \\ P_{it-1}, P_{it-2}, \ldots \\ Q_{t-1}, Q_{t-2}, \ldots \end{bmatrix} \qquad 1.7$$

Clearly, in such an environment, the P_{it}'s are irrelevant. In forecasting future prices all agents care about is the market price and quantity and there is no additional information conveyed by the island prices. We will describe such an environment as one with complete information. In such a world, market clearing (equations 1.1, 1.2, 1.3) implies

$$P_t = u_t - \frac{a}{J}\sum_{j=1}^{J} \pi[P_t | P_{t-1}, \ldots, Q_{t-1}, \ldots]$$

$$= u_t - a\pi[P_t | P_{t-1}, \ldots, Q_{t-1}, \ldots]$$

Definition: A Rational expectations equilibrium for the model with complete information is

(a) a pair of covariance stationary, linearly indeterministic process P_t, Q_t

(b) two sets of parameters $\{\gamma_j\}_{j=0}^{\infty}$ $\{\beta_j\}_{j=0}^{\infty}$ with $\sum_{j=0}^{\infty} \gamma_j^2 < \infty$, $\sum_{j=0}^{\infty} \beta_j^2 < \infty$

which satisfy the following

(i) $P_t = u_t - Q_t$

(ii) $Q_t = a\pi[P_t | P_{t-1}, \ldots, Q_{t-1}, \ldots]$

(iii) $P_t = \sum_{s=0}^{\infty} \gamma_s \eta_{t-s}$

(iv) $Q_t = \sum_{s=0}^{\infty} \beta_s \eta_{t-s}$

Condition (i) is market clearing. Condition (ii) requires producers to forecast optimally. Conditions (iii) and (iv) state that an outside observer armed with a knowledge of current and past η should be able to predict the price perfectly. Recall that $u_t = \sum_{s=0}^{\infty} \alpha_s \eta_{t-s}$ and that u_t has an autoregressive representation. It seems intiutively obvious that the price should be in the space spanned by current and past u_t. Futia (1979) calls this the axiom of no divine revelation. The idea is simply that the market price can contain no more information than is available from the cause of uncertainty the stochastic process that drives demand. The following proposition illustrates how one can use the method of undermined coefficients to characterize the equilibrium for this model.

Proposition 1: A unique equilibrium exists for the model described above and is given by

$$\gamma_0 = \alpha_0$$

$$\gamma_s = \frac{\alpha_s}{1+a} \qquad s \geq 1$$

$$\beta_o = 0$$

$$\beta_s = \frac{a\alpha_s}{1+a} \qquad s \geq 1$$

Proof: Since

$$P_t = u_t - Q_t$$

it follows that

$$\pi[P_t | P_{t-1}, \ldots, Q_{t-1}, \ldots] = \sum_{s=1}^{\infty} \gamma_s \eta_{t-s}$$

From market clearing then

$$\sum_{s=0}^{\infty} \gamma_s \eta_{t-s} = \sum_{s=0}^{\infty} \alpha_s \eta_{t-s} - a \sum_{s=1}^{\infty} \gamma_s \eta_{t-s}$$

$$\gamma_0 = \alpha_0$$

$$\gamma_s = \frac{\alpha_s}{1+a}$$

and the values of β_s follow immediately.

The equilibrium is unique because P_t has a unique representation of the form (iii) by Wold's decomposition theorem (see appendix).

One may wonder whether the values of $Q_{t-1}, \ldots,$ should be included in the information set of agents. This is indeed necessary to avoid existence questions of the kind raised by Futia (1979) and is in any case harmless. A necessary condition for prices to reveal quantities is that the price process P_t as defined (implicitly) from proposition 1 and the definition of an equilibrium have an autoregressive representation. Thus, for example, if

$$\alpha(L) = \sum_{s=0}^{\infty} \alpha_s L^s = \frac{C(L)}{D(L)}$$

where L is the lag operator

and $$C(L) = \sum_{s=0}^{m} C_s L^s$$

$$D(L) = \sum_{s=0}^{n} D_s L^s \quad m, n < \infty$$

the requirement that prices reveal quantities and that an equilibrium exist is that

$$\gamma(L) = \alpha_0 + \sum_{s=1}^{\infty} \frac{\alpha_s}{1+a} L^s$$

be invertible.

Our main intention in the rest of this paper is to vary the information set Ω_{it} and examine the consequences for Q_t and P_t.

3. Models with Decentralized Information

Models such as the one described in section 2 display an output response that exactly mirrors the stochastic process driving demand. Thus, if we supposed that

$$u_t = \frac{\eta_t}{1-\rho L} \qquad |\rho| < 1 \qquad 3.1$$

It can easily be shown that total output Q_t is given by

$$Q_t = \frac{a}{1+a} \left[\frac{1}{1-\rho L} \right] \eta_{t-1} \qquad 3.2$$

Quite different results are obtained if we alter the information set. Suppose now that

$$\Omega_{it} = \begin{bmatrix} P_{it-1}, P_{it-2}, \ldots\ldots \\ P_{t-2}, P_{t-3}, \ldots\ldots \\ Q_{t-2}, Q_{t-3}, \ldots\ldots \end{bmatrix} \qquad 3.3$$

Thus, here we assume that the market price is known to all agents with a lag of one period. In addition, and quite harmlessly for the purposes of this analysis, we assume that aggregate quantity information is also available with a lag of one period. This last assumption has been made to avoid the kinds of existence questions raised earlier. Then, we note that $u_{t-2}, u_{t-3}, \ldots$ (and consequently the respective η's by our assumption that u_t has an autoregressive representation) are included in the information sets of agents.

Now, market clearing is given by

$$P_t = u_t - \frac{a}{J} \sum_{j=1}^{J} \pi[P_t / \Omega_{jt}] \qquad 3.4$$

where

$$\Omega_{jt} = \begin{bmatrix} P_{jt-1}, \ldots\ldots \\ u_{t-2}, \ldots\ldots \end{bmatrix} .$$

We impose a condition similar to the one imposed earlier

$$P_t = \sum_{s=0}^{\infty} \gamma_s \eta_{t-s} \qquad 3.5$$

The idea behind this is that if there are a large number of producers, the island specific shocks tend to "cancel" out across firms and the market price is affected only by the economy wide shocks to demand. For convenience, we repeat the definition of P_{jt} as

$$P_{jt} = P_t + \epsilon_{jt} \qquad j = 1,2 \ldots N \qquad 3.6$$

where ϵ_{jt} is identically, independently distributed across firms and time

$$E(\epsilon_{jt}) = 0 \quad E(\epsilon_{jt}^2) = \sigma_\epsilon^2 \quad E(\epsilon_{jt}\, \eta_s) = 0$$

all j, t, s .

Then, a Rational Expectations equilibrium for a model with a decentralized information set given in 3.3 is a covariance stationary, linearly indeterministic stochastic process $\{P_t\}$ and a set of parameters $\{\gamma_j\}_{j=0}^{\infty}$ with $\sum_{j=0}^{\infty} \gamma_j^2 < \infty$ such that 3.4 and 3.5 are satisfied for the definition provided in 3.6.

We proceed now to characterize the nature of the equilibrium. From 3.5 we have

$$\pi\left[P_t / P_{jt-1}, \ldots\ldots, u_{t-2}, \ldots\ldots\right] = \gamma_1 \pi\left[\eta_{t-1} / P_{jt-1}, \ldots\ldots, u_{t-2}, \ldots\ldots\right] + \sum_{s=2}^{\infty} \gamma_s \eta_{t-s} \qquad 3.7$$

Note that 7 follows since $P_{jt-2}, \ldots\ldots$ is composed of two components: $\epsilon_{jt-2}, \ldots\ldots$ which is orthogonal to $\eta_{t-2}, \ldots\ldots$ and a linear combination of $\eta_{t-2}, \ldots\ldots$.

The first term on the right hand side of 7 needs to be computed. Whittle ((1962), p. 68) provides the signal extraction formula appropriate for this case

$$\pi\left[\eta_{t-1} / P_{jt-1}, \ldots\ldots, \eta_{t-2}, \ldots\ldots\right] = \frac{\sigma_\eta^2}{\sigma_\eta^2 + \sigma_\epsilon^2}\left[\gamma_o \eta_{t-1} + \epsilon_{jt-1}\right] \qquad 3.8$$

Recall that σ_η^2 is the variance of η. The right hand side of 3.8 is observed by agents because

$$P_{jt-1} = \gamma_o \eta_{t-1} + \sum_{s=1}^{\infty} \gamma_s \eta_{t-s} + \epsilon_{jt-1}$$

and since $\eta_{t-2}, \ldots$ is known and P_{jt-1} is observed, it follows that $\gamma_o \eta_{t-1} + \epsilon_{jt-1}$ is known. Equation 3.7 can then be substituted into the production decision rule

$$q_{jt} = a \pi [P_t / \Omega_{jt}]$$

$$\therefore \qquad q_{jt} = a \gamma_1 \frac{\sigma_\eta^2}{\sigma_\eta^2 + \sigma_\epsilon^2}\left[\gamma_o \eta_{t-1} + \epsilon_{jt-1}\right] + a \sum_{s=2}^{\infty} \gamma_s \eta_{t-s}$$

By averaging output across all firms and noting that $\frac{1}{J}\sum_{j=1}^{J} \epsilon_{jt} = 0$, we have

$$Q_t = a\,\gamma_1\,\gamma_o\,\frac{\sigma_\eta^2}{\sigma_\eta^2+\sigma_\epsilon^2}\,\eta_{t-1} + a\sum_{s=2}^{\infty}\gamma_s\,\eta_{t-s} \qquad 3.9$$

Since

$$P_t = u_t - Q_t$$

We can solve for $\{\gamma_j\}_{j=0}^{\infty}$ to get

$$\gamma_o = \alpha_o \qquad 3.10$$

$$\gamma_i = \frac{\alpha_1}{\alpha_o\left[1+\dfrac{a\,\sigma_\eta^2}{\sigma_\eta^2+\sigma_\epsilon^2}\right]}$$

$$\gamma_s = \frac{\alpha_s}{1+a}$$

With these results established, it is interesting to compare them to the results that emerge when there is complete information.

When the market price is known without a lag, the process on output is given by

$$Q_t = \frac{a}{1+a}\sum_{s=1}^{\infty}\alpha_s\,\eta_{t-s} \qquad 3.11$$

When the market price is known with a lag of one period, the process on output is given by

$$Q_t = \frac{a\,\alpha_1}{\alpha_o\left[\dfrac{\sigma_\eta^2+\sigma_\epsilon^2}{\sigma_\epsilon^2}+a\right]}\,\eta_{t-1} + \frac{a}{1+a}\sum_{s=2}^{\infty}\alpha_s\,\eta_{t-s} \qquad 3.12$$

Not surprisingly, the output response is dampened with respect to the innovation in the stochastic process governing demand. It is possible that the nature of the process governing output may change as the information structure in the economy changes. The following example illustrates precisely this possibility.

Suppose u_t follows a first order autoregressive process

$$u_t = \frac{\eta_t}{1-\rho L} \qquad |\rho| < 1$$

With no lags in information, the process on output is

$$Q_t = \frac{a}{1+a}\left[\frac{1}{1-\rho L}\right]\eta_{t-1} \tag{3.13}$$

With a one period lag in knowledge of the price level

$$Q_t = \frac{A[1-\varphi L]}{1-\rho L}\,\eta_{t-1} \tag{3.14}$$

where $$A = \frac{a\,c}{\dfrac{\sigma_\eta^2}{\sigma_\eta^2+\sigma_\epsilon^2} + a}$$

$$\varphi = \frac{\rho(1-\lambda)}{1+a}$$

The nature of the process on output thus changes from a first order autoregressive process to a (1,1) mixed autoregressive-moving average process. It is noted that changes in the information structure can alter the manner in which agents forecast prices and thereby alter the nature of the process governing output.

When the market price is known more than one period later, computation of the equilibrium becomes more burdensome but it is still possible to follow the technique outlined above and characterize the equilibrium.

Suppose

$$\Omega_{it} = \begin{bmatrix} P_{it-1}, \ldots\ldots \\ P_{t-n}, \ldots\ldots \\ Q_{t-n}, \ldots\ldots \end{bmatrix}$$

Define $\Gamma \equiv \begin{bmatrix} \gamma_o & 0 \ldots\ldots 0 \\ \gamma_1 & \gamma_o \\ \cdot \\ \cdot \\ \cdot \\ \gamma_{n-1} & \gamma_{n-2} \cdots \gamma_o \end{bmatrix}$ a lower triangular n x n matrix.

$$\gamma \equiv \begin{bmatrix} \gamma_1 \\ \cdot \\ \cdot \\ \cdot \\ \gamma_n \end{bmatrix} \qquad \pi \equiv \begin{bmatrix} \pi_{t-1} \\ \cdot \\ \cdot \\ \cdot \\ \pi_{t-n} \end{bmatrix}$$

$$\varepsilon_i \equiv \begin{bmatrix} \varepsilon_{it-1} \\ \cdot \\ \cdot \\ \cdot \\ \varepsilon_{it-n} \end{bmatrix}$$

From well-known formulae for signal extraction (Anderson (1971))

$$E\left[\eta_{t-1}/\Omega_{it}\right] = \sigma_\eta^2 \Gamma \left[\sigma_\eta^2 \Gamma' \Gamma + \sigma_\epsilon^2 I\right]^{-1} \Gamma' \left[\eta + \epsilon_i\right] \qquad 3.15$$

$$\therefore E\left[P_t/\Omega_{it}\right] = \gamma' \sigma_\eta^2 \left[\sigma_\eta^2 \Gamma' \Gamma + \sigma_\epsilon^2 I\right]^{-1} \Gamma' \left[\eta + \epsilon_i\right] \qquad 3.16$$

$$+ \sum_{s=n+1}^{\infty} \gamma_s \eta_{t-s}$$

and by aggregating output and noting that

$$\sum_{j=1}^{J} \epsilon_j = \underline{0} \quad \text{and} \quad P_t = u_t - Q_t$$

we have the following solutions

$$\alpha' - \gamma' = a \sigma_\eta^2 \gamma' \Gamma \left[\sigma_\eta^2 \Gamma' \Gamma + \sigma_\epsilon^2 I\right]^{-1} \Gamma' \qquad 3.17$$

$$\gamma_o = \alpha_o$$

$$\gamma_s = \frac{\alpha_s}{1+a} \qquad a \geq n + 1$$

Equation 11 can be simplified by noting that Γ is nonsingular ($\gamma_o = \alpha_o \neq 0$) to read

$$\gamma' = \alpha' - \frac{\sigma_\eta^2}{\sigma_\epsilon^2} \left[(1+a)\gamma' - \alpha'\right] \Gamma \Gamma' \qquad 3.18$$

A solution to the n unknowns in the n nonlinear equations in 3.12 is a characterization of the equilibrium.

A number of points must be noted at this stage. First, if the η shocks are serially uncorrelated, then regardless of the information structure, output will be a constant. Thus, this model provides no endogenous scheme to induce persistence. If the stochastic process driving demand is serially correlated then it is possible that altering the information structure would alter the nature of the process governing output. It is also possible to analyze models such as Lucas (1973) in this framework by altering the information set Ω_{it} to include the current price in the island.

Let $Q_t + P_t = x_t$ x_t is an aggregate demand shock.

$q_{it} = b[P_{it} - E(P_t/P_{it}, P_{it-1} \ldots]$ and $Q_t = \frac{1}{J} \sum_{j=1}^{J} q_{it}$.

Then this system which was analyzed by Lucas for a situation where x_t followed a random walk can be analyzed readily in the context of the models we have studied.

The same general conclusions follow. Extending the information lag tends to increase the order of the serial correlation of output.

4. Models with No Common Information

Models where markets are completely decentralized so that agents share no common information represent an interesting extension of the ideas discussed in section 3. There are compelling reasons to investigate models of this kind. For example, throughout this paper we have used the idea of a market price. One way in which to view these models is that the market price is a fictional entity. It is simply the average of all prices in the economy. Yet, a knowledge of the market price provides information about demand shocks that a knowledge of the island prices does not. Another is that certain variables (such as the aggregate capital stock) are inherently unobservable due to the difficulties in measuring depreciation. Then, if individual firms can observe their own capital stocks, their investment decisions will depend upon their estimates of aggregate capital stocks since it is the aggregate capital stock that influences the economy wide price level. Thus, each firm must make estimates of the economy wide capital stock without ever knowing its "true" value. It is precisely this mechanism that is used by Lucas in his Equilibrium Theory of the Business Cycle (1975).

The attempt here is to study the stochastic process on prices when no agent can observe the market price level. An interesting result is obtained that there is not generally a finite number of variables which we may characterize as summarizing all past information. The impact of such a result is that agents in such an environment will be concerned with the entire history of prices they have encountered.

In the framework where no agent ever observes the market price,

$\Omega_{it} = [P_{it-1}, P_{it-2}, \ldots\ldots\ldots]$.

The market clearing conditions from 2.1, 2.2, 2.3 are

$$P_t = u_t - \frac{a}{J}\sum_{i=1}^{J} \pi[P_t/P_{it-1}, \ldots\ldots] \qquad 4.1$$

We impose a condition identical to 3.5

$$P_t = \sum_{s=0}^{\infty} \gamma_s \eta_{t-s} \qquad 4.2$$

As before $P_{it} = P_t + \epsilon_{it}$ 4.3

Then, a Rational Expectations Equilibrium is defined as a co-variance stationary, linearly indeterministic process $\{P_t\}$ and a set of parameters $\{\gamma_j\}^{\infty}$ with $\sum_{j=0}^{\infty} \gamma_j^2 < \infty$ such that 4.1 and 4.2 are satisfied for the definition given in 4.3.

A natural manner in which to attach the problem of establishing the existence of an equilibrium is to note that if P_{it} is a covariance stationary, linearly indeterministic process there is a representation of P_{it}, by Wold's decomposition theorem, of the form

$$P_{it} = \sum_{s=0}^{\infty} \beta_s \psi_{it-s} \qquad 4.4$$

where ψ_{it} is fundamental for P_{it}. ψ_{it} is therefore a serially uncor-related random variable. Note though that we would not expect ψ_{it} to be uncorrelated with ψ_{jt} since the ψ_{it}'s presumably incorporate the effects of η.

We now establish that for a class of models of the kind examined here, it is not possible to represent the price process by a limited number of state variables. A different way of making the same point is to ask whether there exist representations of the price process as a rational polynomial when the shifts in market demand u_t has a representation as a rational polynomial i.e., it can be expressed as a ratio of finite order polynomials in the lag operator.

Suppose that $\beta(L)$ defined from 4.4 can be expressed as

$$\beta(L) = \frac{A(L)}{B(L)}$$

$$\text{where } A(L) = A_o + A_1 L + A_2 L^2 + \ldots\ldots + A^m L^m \qquad m < \infty$$

$$B(L) = B_o + B_1 L + \ldots\ldots\ldots + B^n L^n \qquad n < \infty$$

and A(L), B(L) are polynomials in the l g operator L.

Then,

$$B(L)P_{it} = A(L)\psi_{it}$$

Given that A(L) and B(L) are of finite order, it follows that if ψ_{it} is fundamental for P_{it}, it is fundamental for $B(L)P_{it}$ and A(L) has an inverse that is one-sided in the nonnegative powers of L. (Of course, a necessary and sufficient condition for this is that the roots of $\sum_{j=0}^{m} A_j Z^j$ all lie outside the unit circle. This is met by our assumptions on $\beta(L)$, A(L) and B(L)).

With these conclusions, it follows that P_{it} has an autoregressive representation given by

$$P_{it} - \sum_{s=1}^{\infty} \delta_s P_{it-s} = \psi_{it} \qquad 4.5$$

where $\pi[P_{it}/P_{it-1},\ldots] = \sum_{s=1}^{\infty} \delta_s P_{it-s}$

Note that since $P_{it} = P_t + \epsilon_{it}$

$$\pi[P_t/P_{it-1},\ldots] = \pi[P_{it}/P_{it-1},\ldots]$$

Then, market clearing yields

$$P_t = u_t - \frac{a}{J}\sum_{j=1}^{J}\sum_{s=1}^{\infty} \delta_s P_{jt-s} \qquad 4.6$$

But $P_{jt} = P_t + \epsilon_{jt}$

and $\sum_{j=1}^{J} \epsilon_{jt} = 0$

$$\therefore \quad P_t = u_t - a\sum_{s=1}^{\infty} \delta_s P_{t-s} \qquad 4.7$$

$$\therefore \quad [1 + a\delta(L)]P_t = u_t$$

where $\delta(L) = \sum_{s=1}^{\infty} \delta_s L^s$

We now apply 4.2 and note that since

$$P_t = \gamma(L)\eta_t$$

where $\gamma(L) = \sum_{s=0}^{\infty} \gamma_s L^s$

it must be that

$$\gamma(L) = \frac{\alpha(L)}{1+a\delta(L)} \qquad 4.8$$

Also, from 4.5 and the fact that $\beta(L) = \frac{A(L)}{B(L)}$

$$1 - \delta(L) = \frac{B(L)}{A(L)}$$

$$\delta(L) = \frac{A(L) - B(L)}{A(L)} \qquad 4.9$$

Combining 4.8 and 4.9 yields

$$\delta(L) = \frac{A(L)\alpha(L)}{(1+a)A(L) - aB(L)} \qquad 4.10$$

With these results, we may now state the theorem:

Theorem 4.1

Suppose u_t has a purely autoregressive representation

$$\alpha(L) = \frac{1}{G(L)}$$

where $G(L) = g_o + g_1 L + \ldots + g_c L^c \qquad c < \infty$

Then, there does not exist a rational expectations equilibrium with a representation of P_{it} of the form

$$P_{it} = \frac{A(L)}{B(L)} \psi_{it} \qquad 4.11$$

where A(L) is of order m

B(L) is of order n $\qquad n, n < \infty$

and A(L) and B(L) have no common factors.

Proof: Suppose there does exist such a representation. From 4.2 and 4.3.

$$P_{it} = \gamma(L)\eta_t + \epsilon_{it}$$

From 4.10

$$P_{it} = \frac{A(L)\alpha(L)}{(1+a)A(L)-aB(L)} \quad \eta_t + \epsilon_{it}$$

$$P_{it} = \frac{A(L)\alpha(L)\eta_t + [(1+a)A(L)-aB(L))]\epsilon_{it}}{(1+a)A(L)-aB(L)} \qquad 4.12$$

4.12 can be expressed as

$$P_{it} = \frac{N(L)}{D(L)} \psi_{it} \qquad 4.13$$

where $N(L)\psi_{it} = A(L)\eta_t + G(L)[(1+a)A(L)-aB(L))]\epsilon_{it}$

and $D(L) = G(L)[(1+a)A(L)-aB(L)]$

Let $r = \text{Max}\{m,n\}$

It is readily that

$N(L)$ is of order $c + r$

$D(L)$ is of order $c + r$

But $\dfrac{N(L)}{D(L)} = \dfrac{A(L)}{B(L)}$

Since the orders of A(L) and B(L) are m and n respectively, it follows that the order of A(L) must be the same as that of B(L) or $m = n$.

N(L) and D(L) have exactly c common factors.

Case1: Suppose the c common factors are $G(L)$.

$$\text{Then } B(L) = \frac{D(L)}{G(L)} = (1+a)A(L)-aB(L)$$

$$\text{or } A(L) = B(L)$$

which contradicts the assumption that $A(L)$ and $B(L)$ have no common factors.

Case 2: Suppose at least one common factor divides $(1+a)A(L)-aB(L)$ but not $G(L)$. But this factor is a factor of $N(L)$.

$$\text{Since } N(L)\psi_{it} = A(L)\eta_t + G(L)[(1+a)A(L)-aB(L)]\epsilon_{it}$$

it follows that the common factor must be a factor of $A(L)$. Hence, it must be a factor of $B(L)$ which contradicts the assumption that $A(L)$ and $B(L)$ have no common factors.

Q.E.D.

Whenever u_t has a purely autoregressive representation, the process governing the price will not have a representation in equilibrium as a rational polynomial. While the conjecture that such a rational polynomial representation exists if u_t has a mixed moving average-autoregressive representation is surely reasonable, it does not seem to be true for a (1,0) or (1,1) process. In general forecasts can be improved by using all past variables of the island price or the innovation in the island price ψ_{it}.

To bring the focus of this theorem into sharper relief, note that in Friedman's Permanent income hypothesis, agents never observe their permanent incomes. Yet there do exist in that model variables that can be said to summarize all past information. If y_t represents permanent income and x_t is measured income, the Permanent Income hypothesis states that

$$y_t = y_{t-1} + w_t \qquad w_t \text{ is serially uncorrelated}$$

$$x_t = y_t + \epsilon_t \qquad \epsilon_t \text{ is serially uncorrelated and independent of } w_t$$

Then, we may write

$$x_t = \frac{w_t}{1-L} + \epsilon_t$$

or $\quad (1-L)x_t = w_t + \epsilon_t(1-L)$

and there is a representation of the form

$$(1-L)x_t = (1-\rho L)v_t \qquad v_t \text{ serially uncorrelated}$$

Note that in this model to predict x_t (and by implication y_t since w_t cannot be predicted from past x_t) it is sufficient to know x_{t-1} and v_{t-1}. In this sense x_{t-1} and v_{t-1} may be said to summarize all past information. The key to understanding Theorem 4.1 lies then not in the fact that agents never observe the market price but rather in the fact that the collective decisions about production made by agents affects the nature of the stochastic process facing them. It is this element that causes our model to differ from the one generated by the Permanent Income hypothesis. Thus, in our model agents will not always be able to collapse their information sets into a small number of key variables.

The remainder of this section is devoted to characterizing the equilibrium and attempting to prove that an equilibrium exists. We will assume that P_{it} has an autoregressive representation.

$$P_{it} = \sum_{s=1}^{\infty} \delta_s P_{it-s} + \psi_{it} \qquad 4.14$$

A version of 4.8 continues to hold

$$[1+a\delta(L)]\gamma(L) = \alpha(L) \qquad 4.15$$

and we require that

$$\pi[P_t/P_{it-1},\ldots] = \delta(L)P_{it} \qquad 4.16$$

i.e., that $\{\delta_s\}_{s=1}^{\infty}$ minimizes

$$E[P_t - \sum_{s=1}^{\infty} \delta_s P_{it-s}]^2 \text{ over all } \{\delta_s\}_{s=1}^{\infty} \text{ such that } \sum_{s=1}^{\infty} \delta_s^2 < \infty.$$

Thus, if we define an initial set of parameters $\{\delta_j\}_{j=0}^{\infty}$, then use this set to determine the optimal expectation rule γ, a new set δ can be obtained from 4.14 and the process repeated. If this mapping has a fixed point an equilibrium exists.

From 4.14 and 4.16

$$\pi[P_t/P_{it-1},\ldots] = \sum_{s=1}^{\infty} \delta_s \sum_{j=0}^{\infty} \gamma_j \eta_{t-s-j} + \sum_{s=1}^{\infty} \delta_s \epsilon_{it-s}$$

The mean square error of the forecast which is to be minimized is

$$MSE = E[P_t - E(P_t/P_{it-1},\ldots)]^2 = \sum_{s=0}^{\infty} (\gamma_s - \sum_{i=1}^{s} \delta_i \gamma_{s-i})^2 \alpha_\eta^2 + \sum_{s=1}^{\infty} \delta_s^2 \alpha_\epsilon^2$$

Then

$$\frac{\partial MSE}{\partial \delta_j} = \sum_{s=j}^{\infty} (\gamma_s - \sum_{i=1}^{s} \delta_i \gamma_{s-i}) \delta_{s-j} \alpha_\eta^2 + 2\gamma_j \alpha_\epsilon^2 = o \qquad 4.17$$

Using 4.15 which yields

$$a\sum_{i=1}^{s} \delta_i \gamma_{s-i} = \alpha_s - \gamma_s$$

together with 4.17, we get

$$\gamma_k = \alpha_k - \frac{\sigma_\eta^2}{\sigma_\epsilon^2} \sum_{i=1}^{k} \gamma_{k-i} \sum_{s=0}^{\infty} \gamma_s [(1+a)\gamma_{s+i} - \dot{\gamma}_{s+i}] \qquad 4.18$$

An equilibrium for the model with no common information exists if there is a set $\{\gamma_k\}_{k=0}^{\infty}$ which satisfies 4.18.

5. Conclusion

It is evident that this is a far from finished piece of work. It is clear that modern business cycle theory relies heavily on Lucas' pioneering idea that in a decentralized economy agents cannot wait until all conceivable information is available before committing themselves to economic decisions. From the standpoint of refutability of such economic theories the distinction between the impact of imperfect information and the impact of other variables (such as the irreversibility of capital) upon economy wide aggregates must be carefully drawn. It is clear that the hypothesis of incomplete information can lead to explaining such observed phenomena as the Phillips curve. This paper demonstrates, I believe, that with serially correlated disturbances lags in information can lead to serially correlated output movements.

The most important result of this paper is that when the market price is never made known to agents then agents will use the entire history of observables for forecasting purposes. From a modeling perspective, it creates several problems since the "state space" on which agents' decision rules may be defined is infinite dimensional. From a practical perspective, if we are to examine the costs and benefits of collecting and disseminating more information more frequently (e.g., data on several different price indices or data on the capital stock) it is clear that we need to understand how the nature of the stochestic process governing key variables will change. This paper is a first, albeit modest, step in that direction.

Appendix

This appendix is meant to be an informal guide to some of the results used in the text. More formal and rigorous treatments are found in Sargent (1979), Whittle (1962), Anderson (1971), Hansen (1979).

A random variable x is a Borel measureable function on a probability space (W, Ω, μ)

$$x: W \rightarrow R$$

The space of random variables can be normed by $\| x \| = [E(x^2)]^{\frac{1}{2}}$ where

$$E(x) = \int_{\Omega} x \, d \mu$$

The norm can be defined to be an inner product norm and the space of random variables (generally designated by $L^2(W)$) can be shown to be a Hilbert space.

A linear subspace $H \subset L^2(W)$ is a subspace such that if $x, y \in H$; $a_1 . a_2 \in R$ then

$$a_1 x + a_2 y \in H .$$

A closed linear subspace $\bar{H} \subset L^2(W)$ is the smallest subspace containing H which contains all its limit points.

It is then possible to define an operator

$\pi[\cdot | \bar{H}] : L_2 \rightarrow \bar{H}$ the projection of L_2 onto H. The operator π is a continuous linear operator. The operator is defined from Theorem 1.

Theorem 1: (Kolmogorov and Fomin pp 158) For $x \in L_2$ there is unique representation of the form

$$x = h + h'$$

where $h \in \bar{H}$, a closed linear subspace of L_2, and $h' \in \bar{H}'$ where every element of $\bar{H}'$ is orthogonal to every element of $\bar{H}$, i.e., $E(h' h) = 0$.

Then we define the projection operator $\pi[x \mid \bar{H}] \equiv h$. Theorem 2 follows immediately.

Theorem 2. The projection operator $\pi(x \mid \bar{H}) = y$ minimizes $E[(x - y)^2]$ for all $y \in \bar{H}$.

Throughout the text when we refer to $\pi[x \mid y_{t-1}, y_{t-2}, \ldots]$ we mean the projection $\pi[x \mid \bar{H}]$ where $\bar{H}$ is the smallest closed linear subspace with $y_{t-i} \in \bar{H}$. By the space spanned by y_{t-i} we will mean $\bar{H}$.

The closure property is important since it is not necessarily true that

$$\pi[P_t \mid P_{t-1}, \ldots] = \sum_{s=1}^{\infty} \delta_s P_{t-s}$$

A useful theorem in this context is the Wold decomposition theorem which states that if P_t is a covariance stationary, linearly indeterministic process, then

$$P_t = \sum_{s=0}^{\infty} \beta_s \Psi_{t-s}$$

where $E[\Psi_t \Psi_{t-s}] = 0 \qquad s \neq 0$

and $\Psi_t = P_t - \pi[P_t \mid P_{t-1}, \ldots]$

and $\sum_{s=0}^{\infty} \beta_s^2 < \infty$.

$$\pi[P_t \mid P_{t-1}, P_{t-2} \ldots] = \pi[P_t \mid \Psi_{t-1}, \ldots]$$

Ψ_t is said to be fundamental for P_t, if in addition $E(\Psi_t^2) = 1$.

If $\beta(L) = \sum_{s=0}^{n} \beta_s L^s$ is of finite order, then the requirement that Ψ_t is fundamental for P_t is equivalent to the condition that P_t have an autoregressive representation.

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