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CORRELATED EQUILIBRIA IN SOME CLASSES OF TWO-PERSON GAMES

by

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ABSTRACT

A correlated equilibrium in a two-person game is "good" if for every Nash equilibrium there is a player who prefers the correlated equilibrium to the Nash equilibrium. If a game is "best-response equivalent" to a two-person zero-sum game, then it has no good correlated equilibria. But games which are "almost strictly competitive" or "order equivalent" to a two-person zero-sum game may have good correlated equilibria.

1. Introduction

In an interesting recent paper [2], Aumann has pointed out the occurrence of some heretofore unnoticed phenomena in noncooperative games when the players may either correlate their strategy selections or peg their strategy selections on events over which they have differing subjective probabilities. In this paper, we examine further the question of correlated (or partially correlated) equilibria in two-person games, when no differing subjective probabilities are assumed.

Let (A, B) be a bimatrix game and $C = \{C_1, \ldots, C_k\}$ and $D = \{D_1, \ldots, D_\ell\}$ be two partitions of some probability space $(\Omega, \mathcal{B}, P)$. Let $\overline{x} = (\overline{x}^1, \ldots, \overline{x}^k)$ and $\overline{y} = (\overline{y}^1, \ldots, \overline{y}^\ell)$ be k and ℓ (randomized) strategies for players 1 and 2 respectively. Then

$$(\overline{x}/C;\ \overline{y}/D) = \left(\overline{x}^1/C_1, \ldots, \overline{x}^k/C_k;\ \overline{y}^1/D_1, \ldots, \overline{y}^\ell/D_\ell\right)$$

is a correlated equilibrium if:

$$\overline{x}^{i^T} A \sum_{j=1}^{\ell} P(D_j/C_i)\overline{y}^j \geq x^T A \sum_{j=1}^{\ell} P(D_j/C_i)\overline{y}^j$$

for all randomized strategies x for player 1 and for all $i \in \{1, \ldots, k\}$ such that $P(C_i) > 0$; and

$$\sum_{i=1}^{k} P(C_i/D_j)\overline{x}^{i^T} B\overline{y}^j \geq \sum_{i=1}^{k} P(C_i/D_j)\overline{x}^{i^T} By$$

for all randomized strategies y for player 2 and for all $j \in \{1, \ldots, \ell\}$ such that $P(D_j) > 0$.

It should be obvious that ordinary Nash equilibria are correlated equilibria (with trivial partitions). Furthermore, if $(\overline{x}^1, \overline{y}^1), \ldots, (\overline{x}^k, \overline{y}^k)$ are all Nash equilibria, then $\left(\overline{x}^1/C_1, \ldots, \overline{x}^k/C_k;\ \overline{y}^1/C_1, \ldots, \overline{y}^k/C_k\right)$ is a correlated equilibrium whenever $C = \{C_1, \ldots, C_k\}$ is a partition consisting of nonnull events. Through such correlated equilibria, any payoff in the convex hull of the Nash equilibrium payoffs can be achieved. The following example (from [2]) illustrates that the payoff from a correlated equilibrium may be outside the convex hull of the Nash equilibrium payoffs.

<u>Example 1</u>:

6, 6	2, 7
7, 2	0, 0

In this example, the payoffs to the three Nash equilibria are (7, 2), (2, 7), and (14/3, 14/3), respectively; but the payoff (5, 5) is achieved via the correlated equilibrium $\left((1, 0)/C_1, (0, 1)/C_2; (1, 0)/D_1, (0, 1)/D_2\right)$ where the following table lists all the relevant probabilities.

	D_1	D_2
C_1	1/3	1/3
C_2	1/3	0

Another game from [2] has a correlated equilibrium which is better for both players than all Nash equilibria. Thus, even in noncooperative games, a kind of self-enforcing correlation of strategies may be possible and beneficial to both players. (Observations of this sort were made long before Aumann's paper. See, for example, [3] for an extensive treatment of related subjects.) As Aumann points out, however, the beneficial effects of correlated equilibria are not possible in two-person zero-sum games. This is not surprising, since two-person zero-sum games are games of strictly opposed interests, and intuitively there should be nothing to be gained from any form of cooperation.

In what follows, we seek larger classes of "competitive" two-person games which possess no correlated equilibria which are "good". A correlated equilibrium is "good" if for every Nash equilibrium in the game there is some player who prefers the correlated equilibrium to the Nash equilibrium. We find that any game which is "best-response equivalent" to a zero-sum game has this property, but that games in certain other classes, all of which seem to be in some sense competitive, may possess good correlated equilibria. These include the class of "almost strictly competitive" games and the class of games which are "order equivalent" to zero-sum games. (In order that the payoff at a correlated equilibrium be well-defined, events conditioned on must be nonnull. This will be implicitly assumed whenever payoffs to correlated equilibria are to be considered.)

For brevity, whenever the word "game" is used in this paper, it is understood to stand for "two-person game".

2. Best-Response Equivalence

In any game, for a fixed (randomized) strategy by one player, the set of best responses is the set of (randomized) strategies by the other player which maximize his expected payoff when played against the fixed strategy. Let (A, B) and (A', B') be bimatrix games which have identical dimensions, say $m \times n$, with the pure strategies labeled $1, \ldots, m$ and $1, \ldots, n$, respectively, in each game. We associate identically labeled pure strategies in the games and randomized strategies which play associated pure strategies with identical probabilities. The two games are best-response equivalent (b.r.e.) if under some ordering of the pure strategies in both games the set of strategies associated with the set of best responses to any randomized strategy in either game is precisely the set of best responses to the associated strategy in the other game. Since Nash equilibria may be defined completely in terms of best responses, the set of Nash equilibria in a game may be associated with the set of Nash equilibria in any b.r.e. game. Similarly for correlated equilibria.

Lemma 1: In any zero-sum game (A, -A), the expected payoff to either player at any correlated equilibrium is exactly the value of the game for that player.

Proof: Let $\left((\bar{x}^1/C_1, \ldots, \bar{x}^k/C_k), (\bar{y}^1/D_1, \ldots, \bar{y}^\ell/D_\ell)\right)$ be a correlated equilibrium and let (x^*, y^*) be a Nash equilibrium.

$$\sum_{i=1}^{k} P(C_i)\bar{x}^{i^T} A \sum_{j=1}^{\ell} P(D_j/C_i)\bar{y}^j \geq \sum_{i=1}^{k} P(C_i)x^{*^T} A \sum_{j=1}^{\ell} P(D_j/C_i)\bar{y}^j = x^{*^T} A \sum_{j=1}^{\ell} P(D_j)\bar{y}^j \geq x^{*^T} Ay^*.$$

But

$$\sum_{i=1}^{k} P(C_i)\bar{x}^{i^T} A \sum_{j=1}^{\ell} P(D_j/C_i)\bar{y}^j = \sum_{j=1}^{\ell} P(D_j)\bar{y}^{j^T} A^T \sum_{i=1}^{k} P(C_i/D_j)\bar{x}^i$$

which is by similar reasoning $\leq x^{*^T} Ay^*$. ||

Corollary 1. In the above proof, $\sum_{i=1}^{k} P(C_i)\bar{x}^i$ and $\sum_{j=1}^{\ell} P(D_j)\bar{y}^j$ are also Nash equilibrium strategies.

Proof: Since y^* is a best response to x^* and $x^{*^T} Ay^* = x^{*^T} A \sum_j P(D_j)\bar{y}^j$, so must $\sum_j P(D_j)\bar{y}^j$ be a best response to x^*. If x^* were not a best response to $\sum_j P(D_j)\bar{y}^j$, then for some strategy x

$$x^{*^T} A \sum_j P(D_j)\bar{y}^j < x^T A \sum_j P(D_j)\bar{y}^j = \sum_i P(C_i)x^T A \sum_j P(D_j/C_i)\bar{y}^j \leq \sum_i P(C_i)\bar{x}^{i^T} A \sum_j P(D_j/C_i)\bar{y}^j.$$

A contradiction. Similarly, $\sum_i P(C_i)\bar{x}^i$ and y^* are best responses to each other. Hence, by interchangeability of Nash equilibria, so are $\sum_i P(C_i)\bar{x}^i$ and $\sum_j P(D_j)\bar{y}^j$. ||

In games which are b.r.e. to zero-sum games, Nash equilibria are interchangeable; but the payoffs to equilibria may not all be the same. It suffices for our purposes, however, that the following holds.

Lemma 2: Let (A, B) be b.r.e. to a zero-sum game. Let p_1 be the payoff to player 1 at the Nash equilibrium (x^*, y^*). Let p_2 be the payoff to player 2 at the Nash equilibrium $(\bar{x}, \bar{y})$. Then (p_1, p_2) is the payoff for the Nash equilibrium $(\bar{x}, y^*)$.

Proof: Immediate from interchangeability of Nash equilibria.

Theorem 1: If a game (A, B) is b.r.e. to a zero-sum game, then it has no good correlated equilibria.

Proof: Let $(\bar{x}^1/C_1, \ldots, \bar{x}^k/C_k)$, $(\bar{y}^1/D_1, \ldots, \bar{y}^\ell/D_\ell)$ be a correlated equilibrium and let (x^*, y^*) be any Nash equilibrium in (A, B). From the proof of lemma 1, it is clear that x^* is a best response to $\sum_j P(D_j/C_i)\bar{y}^j$ whenever $P(C_i) > 0$. Hence

$$\sum_i P(C_i)\bar{x}^{i^T} A \sum_j P(D_j/C_i)\bar{y}^j = \sum_i P(C_i)x^{*^T} A \sum_j P(D_j/C_i)\bar{y}^j = x^{*^T} A \sum_j P(D_j)\bar{y}^j .$$

But $\left(x^*, \sum_j P(D_j)\bar{y}^j\right)$ is a Nash equilibrium by corollary 1. Similarly,

$$\sum_j P(D_j)\bar{y}^{j^T} B^T \sum_i P(C_i/D_j)\bar{x}^i = y^{*^T} B^T \sum_i P(C_i)\bar{x}^i$$

and $\left(\sum_i P(C_i)\bar{x}^i, y^*\right)$ is a Nash equilibrium. The result now follows from lemma 2. ||

It is not necessary that a game be b.r.e. to a zero-sum game in order that it have no good correlated equilibria, however.

Example 2:

0, 0	2, 2
2, 2	0, 0

0, 0	2, 2
1, 1	0, 0

Neither game in example 2 is b.r.e. to a zero-sum game, since Nash equilibria are not interchangeable in either game. Still, the equilibrium payoff (2, 2) in both games is at least as good for both players as any correlated equilibrium payoff. It is difficult to think of either of these games as "competitive" in any sense of the word. Thus we are not suggesting that absence of good correlated equilibria qualifies a game as competitive.

Before closing this section, we should remark that performing a positive linear transformation of the payoffs to either player in a game does not alter the best response set for any strategy. The proof of this remark follows immediately from the definitions. The next example shows, however, that if two games are b.r.e., they may differ by more than just positive linear utility transformations.

Example 3:

G =

1, -1	0, -2
0, 0	-5, 1

G' =

1, -1	2, -2
0, 0	-1, 1

In example 3, G is b.r.e. to the zero-sum game G', but there is no positive linear transformation of the payoff matrix of player 1 in G which yields that in G'.

3. Other Classes of "Competitive" Games

In this section we show that two classes of games which may be thought of as "competitive" may nevertheless have good correlated equilibria.

The class of almost strictly competitive (a.s.c.) games was introduced in [1], where some of the same properties were established for this class that are also true of zero-sum games. Example 4 below, however, is an a.s.c. game with a correlated equilibrium which is better for both players than every Nash equilibrium.

Let (A, B) be a bimatrix game. A twisted equilibrium pair in (A, B) is a Nash equilibrium pair for (-B, -A). A bimatrix game is a.s.c. if the set of payoffs to equilibrium points equals the set of payoffs to twisted equilibrium points and if the set of equilibrium points intersects the set of twisted equilibrium points.

Example 4:

6, 6	2, 7	-1, 6.5	-1, 1
7, 2	0, 0	-1, 1.4	-1, 3
6.5, -1	1.4, -1	0, 0	0, 0
1, -1	3, -1	0, 0	0, 0

The verification that example 4 is a.s.c. and has a good correlated equilibrium is found in the Appendix. Note that the class of a.s.c. games neither contains nor is contained in the class of games which are b.r.e. to zero-sum games. Examples 3 and 4 provide the verification.

Order equivalence of two-player zero-sum games has received much attention. See [4] and [5] for treatments of several topics in this area. Here we go somewhat outside of their scope in extending the definition to non-zero-sum games as well. Two games (A, B) and (A', B') are order equivalent if the ordering of elements in each row or column of A is the same as the ordering of the elements in the corresponding row or column of A'. Similarly for B and B'. Though one can in general identify the Nash equilibria in order equivalent games only when the equilibria are pure, it nevertheless seems that games which are order equivalent to zero-sum games are in a strong sense competitive. On the other hand, example 5 is order equivalent to a zero-sum game but possesses a good correlated equilibrium.

Example 5:

12, 0	0, 12	3, 6	1, 8
0, 12	12, 0	1, 8	3, 6
8, 1	6, 3	10, -10	-10, 10
6, 3	8, 1	-10, 10	10, -10

See the Appendix for details. Note that the class of games which are order-equivalent to a zero-sum game neither contains nor is contained in the class of games which are b.r.e. to zero-sum games. Examples 5 and 6 provide the verification.

Example 6:

2, -2	2, -2	1, -1
1, 1	0, 0	-2, 2
0, 0	1, 1	-2, 2

Example 6 is clearly b.r.e. to the zero-sum game

2, -2	2, -2	1, -1
1, -1	0, 0	-2, 2
0, 0	1, -1	-2, 2

but not order equivalent to any zero-sum game, since Nash equilibria of the subgame

1, 1	0, 0
0, 0	1, 1

are not interchangeable.

3. A Humorous Example

Example 7:

$-\infty$, $-\infty$	3, 1	0, 2
1 , 3	0, 0	1, $-\infty$
2 , 0	$-\infty$, 1	0, 0

If infinite payoffs are permitted and if an infinite payoff received with probability zero has zero utility, then one can easily check that example 7 has no Nash equilibria. But the following correlated equilibrium

$$\left((1,\ 0,\ 0)/C_1,\ (0,\ 1,\ 0)/C_2;\ (1,\ 0,\ 0)/D_1,\ (0,\ 1,\ 0)/D_2\right)$$

yields finite utilities to both players whenever $P(C_1 \cap D_1) = 0$, and

$$P(C_1 \cap D_2) > 0,\ P(C_2 \cap D_1) > 0,\ P(C_2 \cap D_2) > 0.$$

Appendix

Example 4

In order to show that example 4 is a.s.c., we first observe that in the twisted game

-6, -6	-7, -2	-6.5, 1	-1, 1
-2, -7	0, 0	-1.4, 1	-3, 1
1, -6.5	1, -1.4	0, 0	0, 0
1, -1	1, -3	0, 0	0, 0

the first two strategies for each player are dominated by the last two. Thus, any independent randomizations over the third and fourth strategies for both players is a Nash equilibrium in the twisted game, or a twisted equilibrium in example 4; and there are no other twisted equilibria.

Such strategy combinations are also Nash equilibria in example 4. That there are no other Nash equilibria in example 4 in pure strategies is easily checked. With the aid of table 1 below we shall show that there are no other Nash equilibria in randomized strategies either. The first row of the table is a list of all possible strategy combinations by player 2. The second cell in each column lists the set of pure-strategy best responses over all randomizations (with positive probabilities) of the strategy combinations lying above.

1 2	1 3	1 4	2 3	2 4	3 4	1 2 3	1 2 4	1 3 4	2 3 4	1 2 3 4
1 2 3 4	2 3	2 3	4	4	3 4	1 2 3 4	1 2 3 4	2 3	4	1 2 3 4

Table 1

For example, by varying the probabilities with which player 2 plays his first and second pure strategies, any of player 1's pure strategies can be made a best response. From the symmetry of the game, the same table results when the players are reversed. Thus we can eliminate from consideration {2, 3}, {2, 4} and {2, 3, 4} as either player's active set at any Nash equilibrium; since the only best response to these is {4}, which can never be in equilibrium (except against {3}, {4} or {3, 4}). But this observation allows us to eliminate {1, 3}, {1, 4}, and {1, 3, 4} as well. Next, notice that strategies 1 and 2 are never both best responses to a particular probability combination of {1, 2}, {1, 2, 3}, {1, 2, 4}, or {1, 2, 3, 4}. Thus {1, 2} is eliminated. But {1, 2, 3}, {1, 2, 4} and {1, 2, 3, 4} can all be eliminated for similar reasons. Hence the set of Nash equilibria of example 4 equals its set of twisted equilibria, and example 4 is therefore a.s.c. with unique Nash equilibrium payoff (0, 0).

Consider now the correlated equilibrium of example 1 of this paper. If these correlated randomizations are repeated on the first two strategies for each player in example 4, it is easy to check that a correlated equilibrium results with payoff (5, 5).

Example 5

Example 5 is order equivalent to the zero-sum game

12, -12	-12, 12	3, -3	1, -1
-12, 12	12, -12	1, -1	3, -3
8, -8	6, -6	10, -10	-10, 10
6, -6	8, -8	-10, 10	10, -10

.

Furthermore, the following correlated equilibrium yields expected payoff (5, 5):

$$\left((\tfrac{1}{2}, \tfrac{1}{2}, 0, 0)/C_1, (0, 0, \tfrac{1}{2}, \tfrac{1}{2})/C_2; (\tfrac{1}{2}, \tfrac{1}{2}, 0, 0)/D_1, (0, 0, \tfrac{1}{2}, \tfrac{1}{2})/D_2\right)$$

where the following table of probabilities applies:

	D_1	D_2
C_1	1/3	1/3
C_2	1/3	0

.

We must show that no Nash equilibrium yields both players as much. First, note that there are no pure-strategy equilibria. Table 2 presents the best-responses for player 1 in this game as did Table 1 for example 4.

1 2	1 3	1 4	2 3	2 4	3 4	1 2 3	1 2 4	1 3 4	2 3 4	1 2 3 4
1 2 3 4	1 3	1 4	2 3	2 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4

Table 2

Table 3 presents the best responses for player 2 (the symmetry of example 4 being absent).

1 2	1 3	1 4	2 3	2 4	3 4	1 2 3	1 2 4	1 3 4	2 3 4	1 2 3 4
1 2 3 4	2 4	2 3	1 4	1 3	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4

Table 3

From the tables, it is immediate that $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, and $\{2, 4\}$ cannot be the active sets for either player at a Nash equilibrium. If player 1 plays his first and second pure strategies with probabilities 1/2 each and player 2 does likewise with his third and fourth strategies, the resulting Nash equilibrium yields payoff vector (2, 7). Similarly, if the players reverse the above strategies, a Nash equilibrium yielding (7, 2) results. Combining the above, as in example 1, results in the Nash equilibrium pair

$$\left((1/3,\ 1/3,\ 1/6,\ 1/6),\ (1/3,\ 1/3,\ 1/6,\ 1/6)\right)$$

yielding (14/3, 14/3). These are the only Nash equilibria in the game (to check this requires some tedious arithmetic which we shall spare the reader); and the correlated equilibrium is therefore good.