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A SUFFICIENT CONDITION ON f for $f \circ \mu$ TO BE IN pNAD

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The Aumann-Shapley (A-S) pricing rule was axiomatically characterized for a class of cost-functions in [BH] and [M T]. This class consists of all continuously differentiable cost functions with zero fixed cost. However, cost functions are not necessarily continuously differentiable. It is natural to expect the continuous differentiability of cost functions for the case in which the configuration of the means of production changes smoothly. But, if there are components which do not change smoothly one can expect the cost function to have kinks at those points where these components change. When enlarging the production along the segment $[0,\alpha]$, for any given vector α of quantitites produced, it is natural to expect that only a finite number of changes occur.

Let us consider the class G of all cost functions which have only a finite number of kinks in $[0,\alpha]$ for every α in their domain. It is worth mentioning that the equilibrium result obtained in [MT] showing the compatibility of A-S prices with demands, is proven for cost functions which are basically in G (in addition the cost functions are assumed to be non-decreasing and continuous). It is easy to verify that the formula defining the A-S prices in the continuously differentiable case is well defined on the class G as well, and obeys all the given axioms. However, it is an open question whether the result of [BH] and [MT], i.e. whether the given set of axioms uniquely determines the A-S pricing rule, can be extended to this class, (For a class of functions, including certain piecewise continuously differentiable functions, the above characterization was extended in [STZ].)

Since the axiomatic approach cannot be used to justify A-S prices for the class G of cost functions, the approach of [BHR], which is based upon the theory of values of non-atomic games, will be used as a justification for using these prices for the class G. Cost sharing prices using the non-atomic games approach is derived as follows. Let $f: E_+^m \to E$, be a cost function with $f(0) = 0$. Let $\alpha = (\alpha_1,\ldots,\alpha_m)$ be a given vector in the domain of f. The vector α represents the quantities demanded by consumers. Let I be the interval $[0,m]$ and $\wp$ the family of Borel subsets of I. Define the vector of m measures $\mu^\alpha = (\mu_1^\alpha,\ldots,\mu_m^\alpha)$ on $(I,\wp)$ by

$$(1)\quad \mu_i^\alpha(S) = \alpha_i\,\lambda\,(S \cap [i-1,i]), \quad i = 1,\ldots,m,$$

where λ is the Lebesque measure on I.

A coalition S represents a bundle consisting of fractions of the m commodities. For each i, $i = 1,\ldots,m$, $S \cap [i-1,i]$ is that part of the $i^{\underline{th}}$ commodity which is contained in the bundle S, and $\mu_i^\alpha(S)$ is the quantity of the $i^{\underline{th}}$ commodity in the bundle S. For $S = I$ the vector $\mu^\alpha(S)$ is α.

Let us define now a non-atomic game v on $(I,\wp)$ by

$$(2)\quad v(S) = f(\mu_i^\alpha(S),\ldots,\mu_m^\alpha(S))$$

or, for short, $v = f \circ \mu^\alpha$. $v(S)$ is the cost of producing the vector $\mu^\alpha(S)$.

Assume now that v is a game in a space Q of non atomic games on which an Aumann-Shapley value φ exists. Generally, the value distributes the payoff of the grand coalition in such a way that each coalition gets a "proper" share of the payoff according to the contribution it makes. In our case for each S, $(\varphi v)(S)$ is the "proper" share of S in the total cost $v(I) = f(\alpha)$ of producing α. In particular the effect of the $i^{\underline{th}}$ commodity on the cost is $(\varphi v)([i-1,i])$ and the price per unit of the $i^{\underline{th}}$ commodity is therefore chosen to be

$$p_i = \frac{(\varphi v)([i-1,i])}{\alpha_i} .$$

If Q is the well known space of non-atomic games pNAD , then it is known that a unique continuous value φ exists on Q. Hence on such spaces these prices are well defined. This important result, due to Aumann-Shapley and Neyman, is stated as follows.

<u>Theorem ([A-S] and [N2])</u>. There exists a unique continuous value φ on pNAD. Moreover, if $v = f\circ\mu, \mu = (\mu_1,\ldots,\mu_n)$ then the derivatives, $\frac{\partial f}{\partial x_i}(t\alpha)$, exist for almost every $t \in [0,1]$ and φv is given by

$$(3) \quad \varphi v = \sum_{i=1}^{m} \left(\int_0^1 \frac{\partial f}{\partial x_i}(t\mu(I))dt\right)\mu_i .$$

I.e., the value is a linear combination of the measures $(\mu_i)_{i=1}^m$ with coefficients $\int_0^1 \frac{\partial f}{\partial x_i}(t\mu(I))dt$. Thus, if $v = f\circ\mu^\alpha$ is in pNAD it follows that

$$(\varphi v)([i-1,i]) = \alpha_i \int_0^1 \frac{\partial f}{\partial x_i}(t\alpha)dt,$$

and the price per unit of the $i^{\underline{th}}$ commodity is chosen to be

$$p_i = \int_0^1 \frac{\partial f}{\partial x_i}(t\alpha)dt.$$

These prices are called A-S prices.

It is our purpose to prove that if a cost function f basically belongs to the class G, then, for each vector μ of m non-atomic probability measures, $f \circ \mu$ is a game in pNAD. This fact which is an interesting result in non-atomic games in its own right provides a justification for using A-S prices for cost functions which are basically in the class G. This is made precise in the following theorem.

Theorem. Let $f: E_+^m \to E^1$, with $f(0) = 0$, and let $\alpha \in E_{++}^m$ [1/].
Assume that

(i) $f \mid_{[0,\alpha]}$ is continuous [2/]

(ii) f is non decreasing

(iii) f is continuously differentiable in a neighbrohood of all but a finite number of points of $[0,\alpha]$.

Then for any vector μ consisting of m non-atomic probability measures with $\mu(I) = \alpha$, $f \circ \mu$ is in pNAD.

1/ $E_{++}^m = \{x \in E^m \mid x_i > 0,\ i = 1,\dots,m\}$

2/ We mean that the function $g:[0,\alpha] \to E^1$ defined for $x \in [0,\alpha]$ by $g(x) = f(x)$ is a continuous function.

Proof of the theorem. The proof uses Neyman's ideas appearing in [N1]. Let $t\alpha_1,\ldots,t\alpha_K$ be the kinks of f on $[0,\alpha]$. We may assume w.l.o.g. that $k = 1$, since the arguments in the general case are completely the same. Let $y = t_1 \cdot \alpha$ and assume first that $0 < t_1 < 1$. For any $\delta > 0$ and for any i, $1 \leq i \leq m$, choose $F^i_\delta : E^1 \to [0,1]$ such that

1) $F^i_\delta(x) = 0$ for $|x - t_1 \cdot \alpha_i| \geq 2\delta$,

2) $F^i_\delta(x) = 1$ for $|x - t_1 \cdot \alpha_i| \leq \delta$,

3) F^i_δ is increasing on $[t_1 \cdot \alpha_i - 2\delta, t_1 \cdot \alpha_i - \delta]$ and decreasing on $[t_i \cdot \alpha_i + \delta, t_1 \cdot \alpha_i + 2\delta]$,

4) F^i_δ is a continuously differentiable function, and

5) $0 \leq F^i_\delta \leq 1$.

Now define the game v_δ by

$$v_\delta(S) = \prod_{j=1}^{m} (F^j_\delta \circ \mu_j)(S),$$

where $\mu = (\mu_1,\ldots,\mu_m)$ is a vector of m NA^+ measures. Since BV is an algebra and $\|f^j \circ \mu_j\| = 2$ we get

(A) $\|v_\delta\| \leq 2^m$.

Let

$$U_\delta = \{\mu(S) \mid S \in C \text{ and } |\mu_j(S) - t_1 \cdot \alpha_j| < 2\delta \ \forall j,\ 1 \leq j \leq m\}.$$

Note that if $\mu(S) \notin U_\delta$ then $v_\delta(S) = 0$. Define

$$v = f \circ \mu \quad \text{and} \quad \tilde{v}_\delta = v_\delta \cdot v .$$

Let $\Omega : \phi = S_0 \subseteq S_1 \subseteq \ldots \subseteq S_k = I$ be an arbitrary chain. Let i_0 be the first index for which $\mu(S_{i_0}) \in U_\delta$ and let j_0 be the last index for which $\mu(S_{j_0}) \in U_\delta$. (If there is no index i with $\mu(S_i) \in U_\delta$ write $i_0 = 0$ and take the summation $\sum_{i=0}^{i_0-1}$ as zero.) Then,

$$\|\tilde{v}_\delta\|_\Omega = \sum_{i=1}^{i_0-1} |\tilde{v}_\delta(S_{i+1}) - \tilde{v}_\delta(S_i)| + \sum_{i=i_0}^{j_0} |\tilde{v}_\delta(S_{i+1}) - \tilde{v}_\delta(S_i)|$$

$$+ \sum_{i=j_0+1}^{m} |\tilde{v}_\delta(S_{i+1}) - \tilde{v}_\delta(S_i)|.$$

From the definitions of V_δ and $\tilde{v}_\delta$ we have

$$\text{(B)} \quad \|\tilde{v}_\delta\|_\Omega = \sum_{i=i_0}^{j_0} |\tilde{v}_\delta(S_{i+1}) - \tilde{v}_\delta(S_i)|.$$

For every $w \in BV$, define

$$\|w\|_{U_\delta} = \sup_\wedge \|w\|_\wedge$$

where the sup ranges over all subchains $\wedge$ with terms S such that $\mu(S) \in U_\delta$. Thus, by (B) we get

$$\|\tilde{v}_\delta\|_\Omega \leqslant \|\tilde{v}_\delta\|_{U_\delta}$$

and since the last inequality holds for every Ω we have

$$\text{(C)} \quad \|\tilde{v}_\delta\| = \|\tilde{v}_\delta\|_{U_\delta}.$$

But $\tilde{v}_\delta = v_\delta \cdot v$, therefore,

$$\|\tilde{v}_\delta\|_{U_\delta} \leq \|v_\delta\|_{U_\delta} \cdot \|v\|_{U_\delta} \leq \|v_\delta\| \cdot \|v\|_{U_\delta} .$$

Together with (A) and (C) we get,

(D) $$\|\tilde{v}_\delta\| \leq \|v_\delta\| \cdot \|v\|_{U_\delta} \leq 2^m \cdot \|v\|_{U_\delta} .$$

Now since f is a non decreasing function and since $v = f \circ \mu$ we have for δ small enough,

$$\|v\|_{U_\delta} = f(y + 2\delta \cdot \alpha) - f(y - 2\delta \cdot \alpha) = f\big((t_1 + 2\delta)\alpha\big) - f\big((t_1 - 2\delta)\alpha\big),$$

and from the continuity of $f|_{[0,\alpha]}$ we get,

$$\|v\|_{U_\delta} \longrightarrow 0, \text{ as } \delta \longrightarrow 0.$$

Together with (D) we then have,

$$\|\tilde{v}_\delta\| \longrightarrow 0, \quad \text{as } \delta \longrightarrow 0 .$$

Define a game $g_\delta \circ \mu$ by

(E) $$g_\delta \circ \mu = f \circ \mu - \prod_{j=1}^{m} (F_\delta^j \circ \mu_j) \cdot f \circ \mu = v - \tilde{v}_\delta .$$

For each $t \neq t_1$, let D_t be a neighborhood of $t\alpha$ on which f is continuously differentiable. Let $E = \bigcup_{\substack{t \neq t_1 \\ 0 \leq t \leq 1}} D_t$ Obviously, E is a neighborhood of $[0,\alpha] / \{t_1 \cdot \alpha\}$.

Let D_δ be the neighborhood of $[0,\alpha]$ defined by,

$$D_\delta = \{x \mid x = \mu(S) \text{ and } |\mu_j(S) - \mu_i(S)| < \delta, \forall\, 1 \leq i, j \leq m\}.$$

Let $x \in R(\mu)$. We shall examine two cases:

I. $x \in U_{\delta/2}$. In this case, by (E), we have $g(x') = 0$ for each x' contained in a small neighborhood of x.

II. $x \in (D_\delta \cap E) \setminus U_{\delta/2}$. In this case f is continuously differentiable in a small neighborhood of x, and so is g_δ.

Thus for every $x \in (D_\delta \cap E) \cup U_{\delta/2}$ there is a small neighborhood on which g is continuously differentiable. Since $(D_\delta \cap E) \cup U_{\delta/2}$ is a neighborhood of $[0,\alpha]$ we thus have proved that for small δ, g_δ is continuously differentiable in a neighborhood of the diagonal $[0,\mu(I)]$. Moreover by (E) we get that $g_\delta \circ \mu \in BV$.

<u>Proposition</u> Let $h: R(\mu) \to \mathbb{R}^1$ be a continuously differentiable function on some convex neighborhood U of the diagonal $[0,\mu(I)]$ such that $h \circ \mu \in BV$, then $h \circ \mu \in pNAD$.

<u>Proof</u> For each $\epsilon > 0$, there exists a polynomial p_ϵ such that

$$\| p_\epsilon - h \|_{C^1(U)} < \epsilon/m,$$

where $\| p_\epsilon - h \|_{C^1(U)}$ is the C^1 norm of $p_\epsilon - h$ on U. Thus

$$\| p_\epsilon \circ \mu - h \circ \mu \|_U < \epsilon .$$

For a proof see [AS, p. 43]. Define for each $S \in \mathcal{C}$, with $\mu(S) \in U$,

$$w(S) = (p_\epsilon \circ \mu)(S) - (h \circ \mu)(S) .$$

By lemma 45.4 of [A-S, p. 270] w can be extended to a game $\tilde{w}$ such that

$$\| \tilde{w} \| = \| w \|_U < \epsilon .$$

Now the game D which is defined by

$$D = \tilde{w} - (h \circ \mu - p_\epsilon \circ \mu),$$

belongs to DIAG. (Notice that $D(S) = 0, \forall\ S \in \mathcal{C}$, such that $\mu(S) \in U$). Hence

$$h \circ \mu - \tilde{w} \in \text{pNAD} .$$

The distance between $h \circ \mu$ and pNAD (in the BV-norm) is thus less than ϵ. Since this is true for each $\epsilon > 0$ and since pNAD is a closed subspace of BV we get that $h \circ \mu$ is in pNAD.

By the above proposition we get that

$$\tilde{v}_\delta = v - g_\delta \circ \mu,$$

which together with (A) and (C) implies that $\| v - g_\delta \circ \mu \| \to 0$, as $\delta \to 0$. Since pNAD is a closed space and since $g_\delta \circ \mu \in$ pNAD (for small δ) $v \in$ pNAD, and the proof is completed.

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For the case in which $t_1 = 0$, the argument of the proof is the same. However the functions F^i_δ are chosen to coincide with some F_δ which satisfies

$1)^*$ $F_\delta(s) = 0 \quad x \leq \delta$,

$2)^*$ $0 \leq F_\delta(s) \leq 1 \quad \delta \leq x \leq 2\delta$,

$3)^*$ $F_\delta(s) = 1$, $x > 2\delta$,

$4)^*$ F_δ is monotonic increasing on $[\delta, 2\delta]$,

and

$5)^*$ F_δ is a continuously differentiable function.

One can weaken slightly the conditions of Theorem by allowing α to be in the boundary of E^m_+ .

<u>Corollary</u> . Let $f: E^m_+ \to E^1$, with $f(0) = 0$. Let $\alpha \in E^m_+$ and assume the three conditions of Theorem 11 for $f|_{L^\alpha}$ instead of for f. Then $f \circ \mu \in$ pNAD, for every vector μ of m NA^+ measures with $\mu(I) = \alpha$.

<u>Proof</u>. Assume that $M^\alpha = \{i_1, \ldots, i_k\}$. Let $e_1, \ldots, e_m$ be the unit

vectors of E_+^m . Define $g: E_+^k \to E^1$ by

$$g(x) = f\left(\sum_{j=1}^{k} x_j e_{i_j} \right) .$$

g satisfies the three conditions of the Theorem which completes the proof.

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