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Attainable Sets of Markets: An Overview

by

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Abstract

The attainable set of a market is the set of all utility outcomes which can be achieved by the traders through a redistribution of goods. Attainable sets fall into two broad classes, according to whether the traders' utility functions are concave or are merely quasiconcave. Questions of major interest concern the characterization of these two classes, and the further classification of attainable sets according to their "complexity". The study of games which arise from markets is based on the analysis of attainable sets.

This paper surveys the current state of knowledge concerning these and related issues, and presents a set of open research problems.

1. Definitions and Basic Results

Consider a market consisting of a set $N = \{1,2,\ldots,n\}$ of traders, and an m-dimensional commodity space $I^m = \{(y_1,\ldots,y_m): 0 \leq y_j \leq 1 \text{ for all } j\}$. For any collection $\{u_i\}_{i=1}^n$ of utility functions (real-valued functions on I^m) of the traders, the <u>attainable set</u> of the market is

$$A(u_1,\ldots,u_n) = \{x \in R^n: \; x \leq (u_1(y^1),\ldots,u_n(y^n)) \text{ , where each } y^i \in I^m \text{ and } \Sigma y^i = (1,\ldots,1)\} .$$

We envision the traders arriving at a marketplace with initial commodity holdings which, when pooled, yield the commodity bundle $e^m \equiv (1,\ldots,1)$. Trader i has a preference relation over bundles in R^m_+, which is represented by the utility function u_i. The attainable set of the market is the set of all utility outcomes which can be achieved through some distribution of the available commodities among the traders.

Common assumptions in the study of markets are that the traders' preferences are complete, and that for every y in I^m, the "preference sets" $\{z: z \succsim_i y\}$ are closed and convex. A consequence of these assumptions is that the traders' utility functions are upper-semicontinuous and quasiconcave (see, for example, [16]). A stronger assumption is that the traders' preferences are concavifiable; that is, that they can be represented by concave utility functions [17].

Let U_1 be the collection of all upper-semicontinuous, quasiconcave utility functions, and let U_2 be the subcollection of continuous, concave utility functions. For $k = 1,2$, let $A_k(n)$ be the collection of all n-dimensional attainable sets arising from markets in which the traders' utility functions are in U_k.

Let V be an attainable set in $A_k(n)$. If $u_1,\dots,u_n$ are functions in U_k defined on I^m (for some fixed m) such that $V = A(u_1,\dots,u_n)$, then $\{u_i\}_{i=1}^n$ is a k-representation for V over I^m . The k-complexity of V is the least $m \geq 0$ such that there exists a k-representation for V over I^m .

A set X in R^n is the comprehensive hull of another set Y if $X = \{x \in R^n : x \leq y \text{ for some } y \in Y\}$; in this case, we say that X is generated by Y . Comprehensiveness is implicit in the definition of an attainable set. This embodies the assumption that any trader can unilaterally decrease his own utility. For later reference, we define a corner to be the comprehensive hull of a single point.

The following results are not difficult to prove:

Theorem A1. Every set in $A_1(n)$ is generated by a compact set.

Theorem A2. Every set in $A_2(n)$ is generated by a compact, convex set.

Theorem A1 can actually be extended slightly: if the functions $u_1,\dots,u_n$ are upper-semicontinuous and bounded from below, then $A(u_1,\dots,u_n)$ is compactly generated. The assumption of lower-boundedness cannot be eliminated (see [14]).

There are several useful operations which can be performed on attainable sets. If f and g are real-valued functions on I^m and I^ℓ respectively, define the functions $f \wedge g$ and $f \oplus g$ from $I^{m+\ell}$ to R as follows: if $(x,y) \in I^m \times I^\ell = I^{m+\ell}$, then $(f \wedge g)(x,y) = \min(f(x),g(y))$ and $(f \oplus g)(x,y) = f(x) + g(y)$. Both U_1 and U_2 are closed with respect to the operations $\wedge$ and $\oplus$.

<u>Proposition 1.</u> Suppose V_1 and V_2 are in $A_k(n)$, with $V_1 = A(u_1,\ldots,u_n)$ over I^m and $V_2 = A(w_1,\ldots,w_n)$ over I^ℓ. Take $a > 0$ and $b \in R^n$. Then

$$(1) \quad aV_1 + b = A(au_1 + b_1,\ldots,au_n + b_n) \text{ over } I^m .$$

$$(2) \quad V_1 \cap V_2 = A(u_1 \wedge w_1,\ldots,u_n \wedge w_n) \text{ over } I^{m+\ell} .$$

$$(3) \quad V_1 + V_2 \equiv \{x_1 + x_2 : x_1 \in V_1 \text{ and } x_2 \in V_2\}$$
$$= A(u_1 \oplus w_1,\ldots,u_n \oplus w_n) \text{ over } I^{m+\ell} .$$

Furthermore, these three derived sets are all in $A_k(n)$.

Part (1) of the proposition, in combination with Theorems A1 and A2, enables us, without loss of generality, to occasionally restrict our discussions to attainable sets generated by sets lying in the interior of the unit n-cube.

2. Characterization Theorems

Billera and Bixby [2] developed a full converse to Theorem A2.

<u>Theorem B2</u>. Every convex, compactly-generated set in R^n is in $A_2(n)$.

To see the basic idea of their proof, fix a set V which is generated by a convex, compact subset of the unit n-cube. Let $D^i(h)$ be the corner in R^n generated by the point $(1,\dots,-h,\dots,1)$, and consider the set $V^i(h)$, defined as the convex hull of $V \cup D^i(h)$. This set is represented over I^{n-1} by the utility functions $u_j(y) = y_j$ for $j \neq i$, and $u_i(y) = \sup \{x_i : (e^{n-1} - y; x_i) \in V^i(h)\}$, where $e^{n-1} \equiv (1,\dots,1) \in R^{n-1}$. If $h \geq n-1$, then every boundary point of V is a boundary point of $V^i(h)$ for at least one value of i . Therefore $\cap V^i(h) = V$, and by part (2) of Proposition 1, it follows that $V \in A_2(n)$. Incidentally, this construction demonstrates that every set in $A_2(n)$ has complexity no greater than $n(n-1)$.

A precise characterization of $A_1(n)$ is not known. However, two large subsets of $A_1(n)$ have been determined. A compactly-generated set V in R^n is <u>convexifiable</u> if there are continuous, strictly increasing, real-valued functions $g_1,\dots,g_n$ on R , such that

$$V(g_1,\dots,g_n) \equiv \{x \in R^n : x \leq (g_1(z_1),\dots,g_n(z_n)) \text{ for some } z \in V\}$$

is convex. For an attainable set to be convexifiable, there must be <u>some</u> utility representation of the traders' preferences which yields a convex attainable set.

<u>Theorem B1a.</u> Every convexifiable set in R^n is in $A_1(n)$.

The proof of this result is straightforward, and appears in [15]. In essence, one convexifies V , adjoins the corner $D^i(0)$, obtains a Billera-Bixby representation of the new set, and then applies the inverses of the convexifying functions to the constructed utility functions. This procedure is carried out n times, with each trader i distinguished in turn; the resulting markets are then "intersected". In this manner, one obtains a representation of V involving at most $n(n-1)$ commodities.

The following proposition, due to Mantel [11] and Weber [15], provides an inductive description of a family of convexifiable sets. A consequence of the proposition is that the convexifiable sets are dense (in the topology induced by the Hausdorff metric) in the collection of all compactly-generated sets. A set is <u>exponentially convexifiable</u> if for sufficiently large k the functions $g_1,\ldots,g_n$ defined by $g_i(x_i) = 1 - \exp(-kx_i)$ serve to convexify the set. Note that the only compactly-generated sets in R^1 are corners, and are exponentially convexifiable.

<u>Proposition 2.</u> Let C be a compact set in R^n , such that the comprehensive hull of C is exponentially convexifiable. Let $f:R^n \to R$ be twice continuously differentiable, and assume that the first-order partial derivatives of f are negative throughout C . Then the comprehensive hull of the compact set $\{(z,f(z)) \in R^{n+1}: z \in C\}$ is exponentially convexifiable.

If the conditions of the proposition are not satisfied, then the set may in fact fail to be convexifiable. Two examples are illustrated on the following pages.

The set in Figure 1 (which is shown to be non-convexifiable in [10] and [15]) is a member of $A_1(3)$. This can be seen through the following construction, which is based upon a construction given in [10]. Consider the comprehensive hull of the union of the three sets $\{x \in R^3_+ : 2x_i + 2x_j + x_k \leqslant 1\}$ defined by permuting the indices $\{i,j,k\} = \{1,2,3\}$. This set has a representation over I^2, wherein all three traders' utility functions are defined by $u(x_1,x_2) = x_1/(2-x_2)$. Furthermore, the three sets generated by the corner on $(1,1,1/5)$, by $\{x \in R^3_+ : x_1 + x_3 \leqslant 2/5,\ x_2 \leqslant 1\}$, and by $\{x \in R^3_+ : x_2 + x_3 \leqslant 2/5,\ x_1 \leqslant 1\}$, are in $A_1(3)$ by virtue of their convexity. The intersection of these four attainable sets can be mapped into the set in Figure 1 through a positive affine transformation.

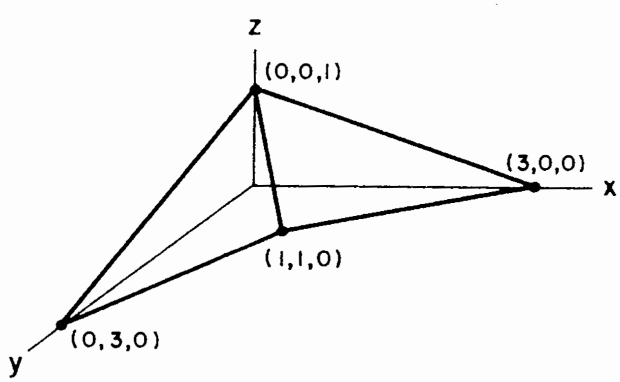

FIGURE 1. A nonconvexifiable set.

It is not known whether the set in Figure 2 is in $A_1(3)$.

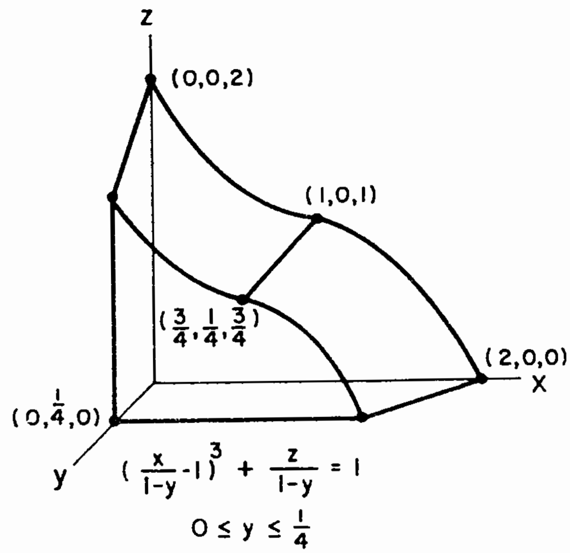

FIGURE 2. $\frac{\partial z}{\partial x} = 0$ for all points $(t, 1-t, t)$.

A set in R^n is <u>finitely generated</u> if it is the union of finitely-many corners. Although such sets contrast sharply with the convexifiable sets of Theorem B1a, we have the following result.

<u>Theorem B1b.</u> Every finitely generated set in R^n is in $A_1(n)$.

This theorem is proved by Weber in [14] via a rather delicate construction which, in analogy with the construction for convexifiable sets, rescales the coordinate axes so as to bring the generating points to the boundary of a convex surface. A result necessary to this construction, which is of some interest in its own right, is the following.

<u>Proposition 3</u>. Take $0 < b < 1/n$, and let $p^0,\ldots,p^n$ be points in R^n of the form $p^k = (b^{\ell_{k,1}},\ldots,b^{\ell_{k,n}})$, where all $\ell_{k,j}$ are non-negative integers. Assume that for some $(t_1,\ldots,t_n) \geqslant 0$ with $\Sigma t_k = 1$, it is the case that $p^0 \geqslant \Sigma t_k p^k$; that is, p^0 lies on or above the convex hull of $\{p^1,\ldots,p^n\}$. Then for some k', $p^0 \geqslant p^{k'}$.

The construction used in the proof of Theorem B1b actually yields a representation in utility functions which are not only quasiconcave, but also monotone increasing and continuous. At one stage of the construction, a continuous function is quasiconcavified (that is, a new function is defined as the infimum of all quasiconcave functions greater than or equal to the original function). Because the domain of the function under consideration is polyhedral, this operation yields another continuous function. Interestingly, this operation of <u>quasiconcavification</u> does not necessarily preserve continuity when applied to a function on an arbitrary compact, convex domain. For example, consider the convex hull in R^3 of the circle $\{(x_1,x_2,0)\colon (x_1-1)^2 + (x_2-1)^2 = 2\}$, and the points $(0,0,1)$ and $(0,0,-1)$. The function $f(x_1,x_2,x_3) = |x_3|$ is continuous on this set, but its quasiconcavification takes the value 1 at the origin, and is 0 at all other points on the circle. The <u>facial dimension</u> of a point x in a set K is the maximal dimension of any convex subset of K of which x is a relative interior point. The k-<u>skeleton</u> of K is the set of all points of K with facial dimension no greater than k; thus, for example, the 0-skeleton of a convex set is its set of extreme points. The crucial implication in the next proposition - that (a) implies (b) - is proved in [18].

<u>Proposition 4</u>. Let K be a compact, convex set in R^n. The following three assertions are equivalent:

(a) All k-skeletons of K are closed (for all $0 \leq k \leq n$).

(b) The convex hull of any relatively open subset of K is relatively open.

(c) The quasiconcavification of any continuous function on K is continuous.

3. Complexity

The constructions used to prove Theorems B1a and B1b treat the n traders symmetrically (and in a sense, simultaneously), and require the use of $n(n-1)$ commodities. With only slight modification, both constructions can be carried out inductively, by treating the traders in sequence and progressively using one commodity, then two more,..., then $(n-1)$ more. This approach, which is detailed by Billera and Weber in [6], yields the following theorem.

<u>Theorem C1</u>. If V in $A_1(n)$ is convexifiable, or is finitely generated, then the 1-complexity of V is at most $n(n-1)/2$.

A similar approach to the problem of 2-complexity is often, but not always, successful. Consider a set V in $A_2(n)$. Assume there is a set C contained in the boundary of V , which generates V and satisfies the following condition: there exists a closed set $Q \subset \{q \varepsilon R^n: \Sigma q_i = 1, q > 0\}$ such that for each $x \varepsilon C$, there is a $q \varepsilon Q$ for which $\Sigma q_i x_i \geq \Sigma q_i y_i$ for all $y \varepsilon V$. In this case, we say that V is (uniformly) <u>positively supported</u>. The collection of positively supported sets in $A_2(n)$ is dense in $A_2(n)$.

<u>Theorem C2a</u>. If V in $A_2(n)$ is positively supported, then the 2-complexity of V is at most $n(n-1)/2$.

The best currently-known universal upper bound on 2-complexity is due to Kalai [8]. Using a construction essentially different from those previously mentioned in this paper, he establishes the following result.

<u>Theorem C2b</u>. The 2-complexity of any set in $A_2(n)$ is at most $(n-1)^2 - (n-2) = n^2 - 3n + 3$.

In view of Theorem C2a, it seems reasonable to conjecture that no set in $A_2(n)$ has 2-complexity greater than $n(n-1)/2$. Although supporting evidence is much weaker concerning lower bounds, it is commonly believed that there are sets in $A_2(n)$ requiring at least $n(n-1)/2$ commodities in their representations. The argument is that in complicated settings, it may be necessary to have a commodity "linking" each pair of traders. The construction used to prove Theorem C2b is of precisely this nature when specialized to the case $n=3$ (see [9]).

The problem of establishing a lower bound for the complexity of an attainable set seems quite different from the upper-bounding problem. The only general result is due to Kalai and Smorodinsky [9].

<u>Theorem C2c</u>. For every $n \geq 3$, there is a set in $A_2(n)$ of complexity n .

The set referred to in Theorem C2c is generated by the convex hull of the $(n+1)$ points $(1,0,\ldots,0),\ldots,(0,\ldots,0,1),(1/2,\ldots,1/2)$ in R^n . It can be shown that for any given trader, when the utility outcome is $(1/2,\ldots,1/2)$ there must be a commodity received only by him, as well as commodities shared exclusively by him with each of the other traders. Since each trader must receive a positive amount of <u>some</u> commodity at this utility outcome, there must be at least n commodities present in any representation of the set. A specific n-commodity representation is given by the utility functions $u_i(x_1,\ldots,x_n) = \frac{1}{2}[x_i + \min x_j]$.

A more detailed look at one aspect of the complexity issue is provided by Billera and Bixby [5] in a characterization of sets in $A_2(n)$ which can be represented using a single commodity.

<u>Theorem C2d.</u> Let $V \varepsilon A_2(n)$ be generated by a set in the unit n-cube I^n , and assume that $\sup\{x_i : x \varepsilon V\} = 1$ for each $i \varepsilon N$. (Every set in $A_2(n)$ can be affinely mapped into a set satisfying these conditions.) Then the complexity of V is at most 1 if and only if there are continuous, convex, nondecreasing functions $h_i:[0,1] \to [0,1]$, with $h_i(0) = 0$, such that

$$V = \{x \varepsilon I^n : \Sigma h_i(x_i) = 1\} - R^n_+ .$$

A geometric statement which follows from this result is that there is a convex rescaling of the coordinate axes which maps V into a set generated by a compact, convex subset of the unit simplex.

4. Market Games

Much of the work on attainable sets is an outgrowth of efforts to characterize those n-person games which can arise from markets. The first such effort was that of Shapley and Shubik [13]. They sought to show that certain games which are pathological with regard to the von Neumann-Morgenstern solution theory can occur in non-pathological economic settings.

A game (with transferable utility) is a real-valued function v on the subsets (coalitions) of a set N of players. Consider a market with trader set N, and assume that each trader i has a continuous, concave utility function u_i on R^m_+, and an initial endowment $\omega^i \in R^m_+$ of goods. The associated (transferable utility) market game is defined for each coalition S in N by

$$v(S) = \max \{\Sigma u_i(y^i): \text{ all } y^i \geq 0 \text{ , and } \sum_{i\in S} y^i = \sum_{i\in S} \omega^i\} .$$

For any coalition T in N, a T-balanced collection $\{\gamma_S\}_{S\subset T}$ is a set of non-negative numbers for which $\sum_{i\in S} \gamma_S = 1$ for every $i \in T$. A game is totally balanced if for all T, and all T-balanced collections $\{\gamma_S\}$,

$$\sum \gamma_S v(S) \leq v(T) .$$

Theorem D3. A game is a (transferable utility) market game if and only if it is totally balanced.

If utility is freely transferable among the traders, the attainable set of utility outcomes for any coalition is generated by a simplex. Such attainable sets are quite simple (cf. Theorem C2d). Indeed, Shapley and Shubik showed that every market game arises from a market involving at most n commodities. Hart [7] has recently refined this result by presenting a construction which employs only $(n-1)$ commodities, and by showing that no lesser number will suffice for the "unanimity" game (defined by $v(N) = 1$, and $v(S) = 0$ for all $S \subsetneq N$).

In many (perhaps in most) markets, utility is not freely transferable. The concepts we have just defined can be extended to cover this more general case.

A <u>game</u> (without transferable utility) with player set N is a correspondence V which assigns to each coalition S a set $V(S) = C_S - R_+^S$, where $R^S \equiv \{x \in R^n : x_i = 0 \text{ for } i \notin S\}$ and $C_S \subset R^S$ is nonempty, compact, and convex. Corresponding to any market with trader set N there is a <u>market game</u>, defined for all $S \subset N$ by

$$V(S) = \{x \in R^S : x_i \leq u_i(y^i) \text{ for all } i \in S \text{, where each } y^i \geq 0 \text{ and } \sum_{i \in S} y^i = \sum_{i \in S} \omega^i\} .$$

(After normalization of the commodity space, $V(S)$ is the attainable set of a market with trader set S.)

A game is <u>totally balanced</u> if for all coalitions T in N, and all T-balanced collections $\{\gamma_S\}$,

$$\sum \gamma_S V(S) \subset V(T) .$$

Billera and Bixby [3] were able to prove the following partial analogue of Theorem D3.

Theorem D2a. Every market game is totally balanced. Let V be totally balanced, and assume that each $V(S)$ is a polyhedron. Then V is a market game.

Subsequently, Mas-Colell [12] provided another characterization result. A game V is totally balanced with slack if for every coalition T, and every T-balanced collection $\{\gamma_S\}$ in which $\gamma_T = 0$,

$$\sum \gamma_S V(S) \subset \text{Int } V(T) ,$$

where Int $V(T)$ denotes the relative interior of $V(T)$. The set of games which are totally balanced with slack is open and dense in the set of all totally balanced games.

Theorem D2b. Every game which is totally balanced with slack is a market game.

The Mas-Colell construction is distinctive, in that it uses the operation of market addition presented in part (3) of Proposition 1. Neither the Billera and Bixby nor the Mas-Colell construction bounds the number of commodities needed to represent a market game.

The definition of a game can be weakened further, by requiring only that each $V(S)$ is compactly generated. One can then ask what games arise from markets in which the traders' utility functions are quasiconcave and upper-semicontinuous. There are no known results in this area.

5. Research Problems

1. Are all n-dimensional compactly-generated sets members of $A_1(n)$? (For example, is the set of Figure 2 attainable?)

2. What is the maximum complexity of sets in $A_1(n)$? in $A_2(n)$? (Although it is tempting to conjecture that both answers are $n(n-1)/2$, no n-dimensional attainable sets of complexity greater than n are currently known.)

3. Are all totally-balanced games market games? What types of games arise from quasiconcave markets? Is the number of commodities needed to represent any n-person market game bounded?

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