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INTERTEMPORAL OPTIMIZATION AND DEMAND ANALYSIS

by

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ABSTRACT

The two basic postulates in the theory of consumption function are that (1) consumption expenditure is proportional to lifetime wealth and (2) the relative prices of consumption goods do not affect consumption expenditure. A complete characterization of the two postulates is given in a continuous-time model with additively separable objective function. It is shown that the two postulates are consistent with budget shares that are nonlinear in consumption expenditure.

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1. Introduction

The demand analysis and the consumption function are two of the most extensively studied areas in the theory of consumer behavior. There are voluminous literature on the estimation of consumer demand functions (see, e.g., Barten (1977) for a survey). Usually the demand functions to be estimated are derived from the consumer's <u>static</u> maximization. One of the main issues in the estimation of the demand functions is how to impose and test the set of restrictions implied by the theory of static optimization. There are even larger literature on the formulation and estimation of consumption functions which has been dominated by the life cycle-permanent income theory of consumption expenditure (see, e.g., Modigliani (1975) for a survey). Although the life cycle-permanent income consumption functions are derived from the consumer's <u>intertemporal</u> utility maximization considerations, the theory of intertemporal optimization itself places virtually no restrictions on the form of consumption function as a function of lifetime wealth or permanent income. Instead, the life cycle-permanent income theory has employed two basic postulates that are not necessarily implied by intertemporal optimization. The first postulate is that consumption expenditure is proportional to lifetime wealth. This has been referred to in the literature as the <u>proportionality postulate</u>. It has been well supported by empirical data for the U.S.

The second postulate, not necessarily explicitly stated in the literature, is that consumption expenditure is independent of the relative prices of its components. We will call this the <u>separability postulate</u>. That the separability postulate has been assumed in the life cycle-permanent income theory and, for that matter, in almost all macroeconomic models is evident from the fact that (1) the life cycle-permanent income consumption function is derived from the intertemporal utility maximization with only one kind of consumption good (see Friedman (1957) and Yaari (1964)), and (2) the usual formulation of consumption function to be estimated does not allow the effect of relative prices. Apart from its obvious intuitive appeal, the merit of the proportionality postulate is that it saves the number of parameters for the consumption function.

The purpose of this paper is to explore how much theoretical restriction one can impose on the demand functions for individual components of consumption expenditure, by maintaining the two basic postulates in the theory of consumption function. This is an important task because there is a vague but widespread belief that the two postulates imply homothetic preference over components of consumption expenditure. Our theoretical apparatus is a continuous-time, deterministic model of intertemporal utility maximization. The objective function is the integral over the lifetime of the *instantaneous utility function*. In this sense the postulated preference over the time path of consumption is *additively separable*. The reason we assume the additive separability is not only that it is consistent with data (see Hall (1978)) but also that it is such a universal assumption, particularly in the theory of economic growth. The only feature that distinguishes our model from the conventional intertemporal utility maximization problem of Yaari (1964) is that the instantaneous utility function in our model depends on many kinds of consumption goods including leisure. This provides a link between the life cycle-permanent income theory of consumption expenditure and the static demand analysis. Section 2 sets up our intertemporal utility maximization problem and derives a condition for optimality.

In section 3, we first define the separability postulate in a formal fashion, and then prove Proposition 1 which gives a necessary and sufficient condition for the separability postulate. The characterization of the class of preferences that admit the separability postulate is carried out in terms of the _indirect utility function_ associated with the instantaneous utility function through the static demand analysis. One might conjecture that the instantaneous utility function must be homothetic for the separability postulate to hold, but our Proposition 1 implies that an indirect utility function that is the sum of the indirect utility function associated with homothetic preference and an arbitrary 0-th homogeneous function of prices of consumption goods is consistent with the separability postulate.

Section 4 examines the consequence of imposing the proportionality postulate. Gorman (1964) has shown in a discrete-time multi-period model with many consumption goods that the proportionality postulate does _not_ necessarily imply each component of consumption expenditure is proportional to lifetime wealth. He has done so by characterizing the proportionality postulate in terms of the indirect utility function, but he did not impose the additive separability. Our characterization of the proportionality postulate (Proposition 2) implies that Gorman's conclusion is valid even if the objective function is additively separable. Section 4 also examines the implication of imposing the two postulates simultaneously. Our result (Proposition 3) shows that a necessary and sufficient condition is that the indirect utility function is the sum of the indirect utility function associated with homogeneous instantaneous utility function of degree less than one and an arbitrary 0-th homogeneous function of prices. Therefore, even under the two postulates, the demand functions for individual components of consumption expenditure need not be proportional to consumption expenditure.

Section 5 introduces the utility of terminal wealth in the intertemporal optimization problem. This is what Yaari (1964) called the bequest motive problem. It is shown that the same conclusions as in the previous sections hold with a minor modification for the bequest motive problem. Section 6 briefly states the main conclusion and indicates possible extensions of the model.

2. The Continuous-Time Intertemporal Utility Maximization Problem

The model developed in this section is a generalization of Yaari's (1964) in that the instantaneous utility u depends on many consumption goods and leisure: $u = u(c,\ell)$, where c is the n-dimensional vector of consumption goods and ℓ is leisure. The consumer's plan is a (n+1) - dimensional real-valued function (c_t, ℓ_t) of time over the lifetime [0, T]. (c_t, ℓ_t) is required to be nonnegative for all t in [0, T]. The consumer's lifetime budget constraint is written as

$$(1)\quad A_T \exp\left(-\int_0^T r_t dt\right) = A_0 + \int_0^T w_t\,(\bar{\ell}_t - \ell_t)\exp\left(-\int_0^t r_s ds\right)dt - \int_0^T p_t c_t \exp\left(-\int_0^T r_s ds\right)dt\ ,$$

where A_0 = consumer's initial assets,

A_T = terminal assets,

$\bar{\ell}_t$ = time endowment at time t,

p_t = price vector of consumption goods,

r_t = nominal interest rate,

w_t = nominal wage rate.

The second term on the right-side of (1) represents the present discounted value of labor income. (1) can be rewritten as

$$(2)\quad A_T \exp(-\int_0^T r_t dt) = W_0 - \int_0^T F_t \exp(-\int_0^t r_s ds)dt,$$

where $W_0 = A_0 + \int_0^T w_t \bar{\ell}_t \exp(-\int_0^t rsds)\, dt$,

$F_t = p_t c_t + w_t \ell_t$.

W_0 is the sum of the consumer's initial assets and the present discounted value of time endowments. W_0 will be referred to as <u>lifetime wealth</u>. F_t is consumption expenditure in the usual sense plus the imputed value of leisure. F_t will be referred to as <u>full consumption</u>.[1]

The consumer's preference over the time path $\{c_t, \ell_t\}, t\varepsilon[0, T]$, is represented by the additive separable utility function:

$$(3)\quad V = \int_0^T \alpha_t u(c_t, \ell_t)dt.$$

The function α_t is to be interpreted as the subjective discounting function. The consumer's problem is to maximize (3) with respect to the time path $\{c_t, \ell_t\}$. The constraint is $(c_t, \ell_t) \geq 0$ and $A_T = 0$. This is what Yaari (1964) called the wealth constraint problem. The alternative problem, referred to as the bequest motive problem by Yaari, is to maximize

$$(4)\quad U = \int_0^T \alpha_t\, u(c_t, \ell_t)\, dt + \psi(A_T)$$

with respect to $\{c_t, \ell_t\}$ and A_T subject only to the nonnegativity constraint $(c_t, \ell_t) \geq 0$. The function ψ is to be interpreted as the utility of bequest. For the rest of this section and in sections 3 and 4, we deal with the wealth constraint problem. The bequest motive problem will be discussed in section 5.

The wealth constraint problem (and the bequest motive problem) can be simplified by introducing the indirect utility function associated with $u(c_t, \ell_t)$. Let the indirect utility function $g(p_t, w_t, F_t)$ be defined as the ordinary fashion as

$$(5)\quad g(p_t, w_t, F_t) = \max_{c_t, \ell_t} u(c_t, \ell_t) \quad \text{s.t.} \quad p_t c_t + w_t \ell_t = F_t.$$

Now we observe that the constraint (2) (with $A_T = 0$) involves (p_t, c_t) only through full consumption F_t. Therefore the solution $\{c_t, \ell_t\}$, $t \in [0, T]$, for the wealth constraint problem can be obtained in two steps. The first step is to solve the static utility maximization problem (5) for any given full consumption F_t. This gives (c_t, ℓ_t) as a function of (p_t, w_t, F_t). The second step is to allocate full consumption over time:

$$(6)\quad \max_{\{F_t\}} \int_0^T \alpha_t\, g(p_t, w_t, F_t)\, dt$$

$$\text{s.t.} \qquad F_t \geq \text{ for all } t \text{ in } [0,T]$$

$$W_0 = \int_0^T F_t \exp\left(-\int_0^t r_s ds\right) dt.$$

Since the first step is entirely standard, we will henceforth focus on the second step (6).

To avoid unnecessary technical complications, we make the following assumptions throughout the paper.

Assumption 1. α_t is positive and continuous for all t in [0,T].

Assumption 2. g is continuous in (p,w,F).

Assumption 3. $(p_t, w_t)$is continuous for all t in [0,T].

Assumption 4. g is twice differentiable in F and the second derivative $g_{FF} = \partial^2 g/\partial F^2$ is negative for all (p,w,F). Under these conditions, it is possible to show the following:

(i) The solution $\{F_t\}$ to (6) is unique if it exists.

(ii) A necessary and sufficient condition for the full consumption plan $\{F_t\}$ to be optimal is

$$(7) \quad \exp(-\int_t^T r_s ds)\ \alpha_t\ g_F(p_t,\ w_t,\ F_t) = k \quad \text{if } F_t > 0,$$

where k is independent of t.[2]

(iii) The solution $\{F_t\}$ is continuous for all t in [0,T].

It is routine to show (i) above from the concavity of g in F. That (7) is a necessary condition can be proved in much the same way as in Yaari (1964). A proof of sufficiency can be found, e.g., in Hadley and Kemp (1971). We can also follow Yaari's argument to show (iii).

3. The Separability Postulate

In most macroeconomic models, all kinds of consumption goods are aggregated into a single composite good and the consumption function is the demand for that composite good for consumption purposes. The rationale for this is that the relative prices of the components of consumption expenditure are not important in the determination of total consumption expenditure. In models where there are more than one consumption goods, the usual practice is to assume that total consumption expenditure is independent of the relative prices. The situation is most clearly exemplified in the standard international trade model with traded and nontraded goods (see, e.g., Dornbusch (1974)). The demand functions for traded goods and nontraded goods are written as

$$D_T(p_T, p_N, F) \quad \text{and} \quad D_N(p_T, p_N, F)$$

where p_T, p_N, and F are the price of traded goods, the price of nontraded goods, and total consumption expenditure. Total expenditure F in turn is a function of interest rate r, income y, and assets A:

$$F = F(r, y, A).$$

We note here that F is independent of the relative price p_T/p_N.

On the empirical side, no attempt has been made to allow for the relative price effect in the formulation and estimation of consumption functions. Since this sort of separability -- the absence of the relative price effect -- is so widely accepted and taken for granted, it is very important to examine what restriction the separability places on the demand functions for each component of total consumption expenditure. This section defines the separability postulate in a rigorous fashion and then characterize it in terms of the indirect utility function.

The solution to (6) can be written in a general form:

(8) $$F_t = F(W_0, \{p_t, w_t, r_t\}, t, T), \quad 0 \leq t \leq T,$$

where $\{p_t, w_t, r_t\}$ represents the time path of (p_t, w_t, r_t) over the interval $[0,T]$. We say that full <u>consumption is separable from the relative prices</u> if there exists an index $\phi_t = \phi(p_t, w_t)$, linear homogeneous in (p_t, w_t), for which F_t is written as

(9) $$F_t = F(W_0, \{\phi_t, r_t\}, t, T), \ 0 \leq t \leq T.$$

That is, the full consumption plan $\{F_t\}$ as of time 0 depends on current and future prices and wage rate <u>only through</u> the price index ϕ. The following proposition completely characterize the class of preferences that admits the separability postulate.

<u>Proposition 1</u>. The separability postulate (9) holds for any time path $\{p_t, w_t, r_t\}$, $t \,\varepsilon [0, T]$, if and only if the indirect utility function takes the form

(10) $$g(p, w, F) = b[F/\phi(p,w)] + \mu(p, w)$$

(or some affine transformation of it), where μ is nonincreasing, quasiconvex, and homogeneous of degree 0 in (p, w), ϕ is nondecreasing, quasiconcave, and linear homogeneous in (p,w), and b is an arbitrary monotone-increasing concave function.

<u>Proof</u>. Sufficiency is obvious if we notice that, under (10), the optimization problem is equivalent to:

(6') $$\max_{\{F_t\}} \int_0^T \alpha_t b(F_t/\phi_t)dt \quad \text{s.t. the same constraints as in (6).}$$

To prove necessity, we look at the necessary condition (7). Now change the time path $\{p_t, w_t\}$ in such a way that both the time path $\{\phi_t\}$ and prices at time t, (p_t, w_t), remain unchanged. If the separability postulate (9) is to hold, F_t and hence the left-side of (7) should not change. Since this must be true for any t in [0, T], k in (7) depends on the time path $\{p_t, w_t\}$ only through the time path $\{\phi_t\}$. Now change the time path $\{p_t, w_t\}$ in such a way that prices at time t, (p_t, w_t), change with $\{\phi_t\}$ unchanged. Since the right-side of (7) does not change and since F_t should not change under the separability hypothesis, it must be that $g_F(p, w, F)$ depends on (p, w) only through ϕ. Therefore g_F must take the form: $g_F = a(F, \phi)$. Solving this partial differential equation and using the usual properties of the indirect utility function (homogeneity, monotonicity and quasiconvexity) we obtain (10). Q.E.D.

Using the price index ϕ, the constraint in (6) or (6') can be written as

$$W_0/\phi_0 = \int_0^T (F_t/\phi_t) \exp[- \int_0^t (r_s - \pi_s) ds] dt , \tag{11}$$

where $\pi_t = \dot{\phi}_t/\phi_t$. So the definition of the separability postulate can also be given by

$$F_t = \phi_t \, f(W_0/\phi_0, \{\rho_t\}, t, T) \tag{9'}$$

where $\rho_t = r_t - \pi_t$ is the "real" rate of interest. Note that μ is irrelevant in the definition of the real rate.

The first term on the right-side of (10) is the indirect utility function associated with a homothetic instantenous utility function. The second term μ therefore represents a deviation from homothetic preference. If μ is not independent of (p,w), then the demand functions for each component of F can be nonlinear in F.

4. The Proportionately Postulate

One of the fundamental assumptions in the life cycle-permanent income theory of consumption is that the proportionality postulate consumption expenditure is proportional to lifetime wealth. In our model, it can formally be defined as

(12) $F_t = \gamma_t W_0$, γ_t is independent of W_0, $0 \leq t \leq T$

The question we ask is: how restrictive is the proportionality postulate? In the single consumption good case where the instantaneous utility depends only on one kind of consumption good, Yaari (1964) has shown that the proportionality postulate holds if and only if the instantaneous utility function takes the form ℓn C or C^{β} . Friedman's (1957) original derivation of the proportionality postulate was based on a two-period, single consumption good model where the preference over the first period consumption and the second period consumption is homothetic. Thus some sort of homotheticity assumption may seem necessary for the proportionality postulate to hold. This has led many people to believe that each component of consumption expenditure must be proportional to lifetime wealth. This belief, if correct, would imply, for example, that the non-linear Engel curve and the proportionality postulate are mutually inconsistent.

This belief was shown false by Gorman (1964) who characterized the class of non-homothetic preferences that admit the proportionality postulate. The source of the false belief, essentially, was the confusion between two consumption goods in a given period and consumption in two different periods. However, Gorman's result was established in a discrete-time model where the additive separability of preference is not imposed. Now, it is fair to say that the additive separability is a widely accepted postulate in modern economics. In particular, the application of optimal control theory -- deterministic or stochastic -- is impossible if the objective function is not additively separable. Furthermore, there is empirical evidence in favor of additive separable preferences (see Hall (1978)). Therefore, it is important to see if there exists a class of additively separable, non-homothetic preferences that admit the proportinality postulate. The following proposition is a complete characterization of the proportionality postulate in terms of the indirect utility function.

Proposition 2. The proportionality postulate holds for any time path $\{p_t, w_t, r_t\}$ if and only if the indirect utility function takes the form

$$g(p, w, F) = \frac{\delta(p, w)}{\beta(p, w)}\left[\frac{F}{\phi(p, w)}\right]^{\beta(p,w)} + \mu(p, w) \tag{13}$$

(or some affine transformation of it), where δ, β and m are homogeneous of degree 0 in (p, w), and ϕ is a linear homogeneous function of (p, w), and β is less than one for all (p, w).

Proof. Since the proof of sufficiency is straightforward, we give only the proof of necessity which is a slight modification of Yaari (1964). Suppose (12) holds. By differentiating the necessary condition (7) with respect to W_0, we obtain

$$\frac{g_{FF}\gamma_t W_0}{g_F} = \frac{W_0\ \partial k/\partial w_0}{k} \tag{14}$$

Consider a special case where (p_t, w_t) is constant over time at (p, w) and where $\exp(-\int_t^T r_s ds)\ \alpha_t$ is not constant over time. Since k is constant over time, γ_t cannot be constant over time in view of (7). But since the rightside of (14) is independent of t, the left-side of (14) must be independent of t for any W_0. It follows that $g_{FF}\ F/g_F$ must be independent of F:

$$(15) \qquad \frac{g_{FF}F}{g_F} = \bar{\beta}(p,w).$$

(13) is the solution to the partial differential equation (15) with $\beta = \bar{\beta} + 1$ under the 0-th homogeneity of q in (p, w, F). Q.E.D.

Remark 1. Of course δ, β, ϕ and μ must be such that g satisfies monotonicity and quasiconcavity in (p, w).

Remark 2. The log case

$$(16) \qquad g(p, w, F) = \delta(p, w)\ \ell n\ (F/\phi) + \mu$$

is included in (13) as a special case where β tends to zero.

The class of preferences represented by (13) is fairly general. If the instantaneous utility function takes the form

$$(17) \qquad u = \frac{1}{\beta}[h(c, \ell)]^{\beta},$$

where h is linear homogeneous in (c, ℓ), then its indirect utility function is

$$(18) \qquad g = \frac{1}{\beta}(F/\phi)^{\beta},$$

which is a special case of (13) where β, λ and μ are all independent of prices (p, w). In this case each component (c, ℓ) of full consumption F _is_ proportional to F. If β and μ are independent of (p, w) and δ is not, then (c, ℓ) is linear in F with some intercept term which is a function of (p, w). If either β or μ is not independent of (p, w), then (c, ℓ) is <u>nonlinear</u> in F.

Comparing Propositions 1 and 2, we can easily see that the proportionality is neither necessary nor sufficient for the separability postulate. The consequence of imposing the two postulates simultaneously is indicated by the following proposition.

<u>Proposition 3</u>. A necessary and sufficient condition for the separability postulate (9) and the proportionality postulate (12) to hold simultaneously is that the indirect utility function takes the form

$$g(p, w, F) = \frac{1}{\beta}\left[\frac{F}{\phi(p,w)}\right]^{\beta} + \mu(p, w) , \tag{19}$$

where $\beta < 1$ is a constant independent of (p, w), ϕ is nondecreasing, quasiconcave and linear homogeneous in (p, w), and μ is nonincreasing, quasiconvex, and homogeneous of degree 0 in (p, w).

The proof of this proposition is obvious from the comparison of (10) and (13). The system of demand functions that come out of (19) is:

$$\begin{aligned} c_i &= \phi_i F/\phi - \mu_i\ \phi^{\beta}\ F^{1-\beta}. \qquad (i = 1, \ldots, n) \\ \ell &= \phi_w F/\phi - \mu_w\ \phi^{\beta}\ F^{1-\beta}, \end{aligned} \tag{20}$$

where $\phi_i = \partial\phi/\partial p_i$ and $\phi_w = \partial\phi/\partial w$. The notable restriction embodied in the above system is that F enters each equation with the same power from $F^{1-\beta}$.

5. <u>The Bequest Motive Problem</u>

We now turn to the bequest motive problem. Recent work by Kotlikoff and Summers (1980) shows that the bequest motive accounts for the vast majority of aggregate U.S. saving. Therefore the bequest motive problem deserves a separate discussion. Using the indirect utility function, we can write the bequest motive problem as:

$$\text{(21)} \qquad \max_{\{F_t\},\, A_T} \int_0^T \alpha_t\, g(p_t,\, w_t,\, F_t)\, dt + \psi(A_T)$$

$$\text{subject to} \qquad F_t \geq 0, \quad 0 \leq t \leq T\ , \quad \text{and}$$

$$A_T \exp(-\int_0^T r_t dt) = W_0 - \int_0^T F_t \exp(-\int_0^t r_s\, ds)dt.$$

It should be noted that the utility of bequest, ψ is independent of the time path of prices and interest rate between time 0 and T. The same argument as in Yaari (1964, p. 309) shows that

$$\text{(22)} \qquad \exp\,(-\int_t^T r_s\, ds)\ \alpha_t g_F(p_t,\, w_t,\, F_t) = \psi'\ (A_T)\ \text{if}\ F_t > 0$$

is a necessary (and sufficient) condition for optimality. It is quite easy to show that Proposition 1 applies to the bequest motive problem as well, using exactly the same argument as in the proof of the proposition with k replaced by $\psi'\ (A_T)$.

To prove a result analogous to Proposition 2, suppose that the proportionality postulate (12) is true. (12) and the lifetime budget constraint in (21) immediately imply that the terminal wealth A_T also is propotional to the initial wealth W_0, i.e., $A_T = \lambda W_0$. Now, differentiate (22) with respect to W_0 and obtain

$$\text{(23)} \qquad \frac{g_{FF}\gamma_t W_0}{g_F} = \frac{\lambda W_0 \psi''(\lambda W_0)}{\lambda'(\lambda W_0)} \quad .$$

Again, using the same argument as in Yaari (1964, p. 314), we can easily prove that $\psi(x)$ must take the form: $\ell n\ x$ or x^{β}. Hence the right-side of (23) is equal to β. We notice that β is independent of (p,w) because the utility of bequest, ψ, is a function of A_T alone. The solution of the partial differential equation with the right-side set equal to β is (13) with β independent of (p,w). Thus we have established:

Proposition 2'. The proportionality postulate holds for any time path $\{p_t, w_t, r_t\}$ if and only if the indirect utility function is written as (13) with β independent of (p,w).

Proposition 3 holds without any modifications for the bequest motive problem.

6. Conclusion and Extensions

The conclusion that comes out of this exercise should be clear. If one believes the two postulates in the life cycle-permanent income theory of consumption, one should impose the implied restriction (20) along with the set of restrictions implied by static utility maximization in the empirical analysis of consumer demand.

The model can be extended in a couple of directions. Money can be introduced by making the utility function depend on money balances. In this case the relevant notion of full consumption must include the opportunity cost of holding money. The model can also be extended to encompass risky assets in a way indicated by Merton (1971). See Hayashi (1980) for more details on these extensions.

Footnotes

1. If leisure is not a choice variable, which will happen if the labor market is in the state of excess supply, then full consumption should be interpreted as the usual notion of consumption expenditure, and the time path $\{\ell_t\}$ should be treated as a given parameter just like the time path $\{p_t, w_t, r_t\}$. Under this modification, the whole discussion in the paper will carry over to the case of exogenous labor supply.

2. k may depend on W_0 and the time path $\{p_t, w_t, r_t\}$.

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