Discussion Paper No. 435

A GENERALIZED HAZARD RATE

by

Morton I. Kamien and Nancy L. Schwartz

August 1980

A GENERALIZED HAZARD RATE

Morton I. Kamien and Nancy L. Schwartz[*]

The "hazard rate," defined as the conditional probability density of the occurrence of an event, given its nonoccurrence to date, has been widely employed in the formulation of economic choice problems under uncertainty. For example, the hazard rate may reflect the intensity of rivalry to introduce a new product or process and collect the innovational rewards; see Kamien and Schwartz (1972, 1974, 1976). Or a decision variable may affect the hazard rate. For instance, the research and development (R&D) budget affects the chance that the R&D effort will succeed in the next moment, given no success to date; see Loury, Dasgupta and Stiglitz, and Lee and Wilde. Comparative statics analysis of the optimal solution to such problems often involves determining how it is affected by a shift in the hazard rate.

In the cited papers, the hazard rate is constant over time, albeit exogenous in the former papers and dependent on an endogenous variable in the latter ones. This corresponds to an analytically tractable exponential distribution. But it seems less than plausible that in the purposeful activity of research and development, the conditional probability of imminent success, given no success to date, would be constant. Rather, this hazard rate should be a nondecreasing function of time, perhaps equal to zero for some initial period and then increasing. If eventual success is not assured, the hazard rate may peak and decline thereafter.

In this paper, we show how the exponential distribution can be replaced by a very general class of probability distributions. The hazard

rate can depend on time as well as on an endogenous parameter (decision variable). Many of the results of the cited papers hold under the more general formulation; see Kamien and Schwartz (1980). One such model and result is shown here.

The Formulation

The hazard rate H(t) associated with a general probability distribution F(t) over the time of occurrence of an event is

$$(1) \qquad H(t) = F'(t)/(1-F(t)),$$

where prime indicates derivative. It is the conditional probability density of occurrence of an event, given nonoccurrence to date. Since $-d \ln(1-F(t))/dt = H(t)$ and $F(0) = 0$, it follows on integrating that

$$(2) \qquad 1-F(t) = e^{-\int_0^t H(s)ds}.$$

Because we are especially interested in the effects of a change in the hazard rate, we write the hazard rate as a nonnegative hazard function u(t) multiplied by a nonnegative hazard parameter h(x), where x is an endogenous variable in the optimization problem being studied:

$$(3) \qquad H(t;x) = h(x)u(t).$$

H is a function of t; x enters parametrically. An increase in the hazard parameter h(x) results in a constant proportionate increase in the hazard rate. The exponential distribution (used exclusively in the previous literature) corresponds to $u(t)=1$; the Weibull distribution is $u(t) = wt^{w-1}$; and the extreme value distribution is $u(t) = we^{wt}$. The exponential distribution is a special case of the Weibull with $w=1$. The hazard rate

increases through time for the extreme value distribution and for the Weibull distribution with w > 1. See Sivazlian and Stanfel for a more extensive discussion.

Using (3), (2) becomes

$$(4) \qquad F(t;x) = 1-e^{-h(x)v(t)} ,$$

where, by definition,

$$(5) \qquad v(t) \equiv \int_0^t u(s)ds .$$

For example, $v(t)=t$ for the exponential distribution, $v(t)=t^w$ for the Weibull distribution, and $v(t)=e^{wt}-1$ for the extreme value distribution. F is viewed as a function of the single variable t, and x is a parameter of that probability distribution. (Thus prime in (6) below indicates derivative with respect to t.) The probability density function related to (4) is

$$(6) \qquad F'(t;x) = h(x)u(t)e^{-h(x)v(t)} .$$

Equations (4)-(6) express a general class of probability functions in which the hazard rate is explicit. The temporal behavior of the hazard rate is governed by the function $u(t)$. Thus one may posit an appropriate form for $u(t)$. And $h(x)$ can be viewed as a shift parameter, available for choice or for parametric analysis. In a special case, h may be a given parameter, independent of all decision variables.

Example

We illustrate by generalizing the model of Loury and Dasgupta and Stiglitz, and showing that one of Loury's results still holds. The time t

of successful completion of research and development by a firm is governed

by probability distribution (4), where x is the firm's lump-sum expendi-

ture on R&D.   (Loury, Dasgupta and Stiglitz assumed u=1; we do not restrict

u.)   The function h is twice continuously differentiable, strictly increas-

ing, and satisfies $h(0) = 0 = \lim_{x \to \infty} h'(x)$ and $h''(x) \gtrless 0$ as $x \lessgtr \bar{x}$, where $\bar{x} \geq 0$,

as Loury posited.   A larger expenditure x raises the conditional probability

density of completion at any date (the hazard rate) and reduces the ex-

pected time

$$E(t;x) = \int_0^\infty e^{-h(x)v(s)} ds$$

to complete the R&D.

    There are n identical firms, with independent research efforts, each

seeking the prize P, the expected capital value to the firm upon winning

the race to innovate.   The jth firm spends x_j.   The expected discounted

profit of the nth firm is

$$(7) \qquad W(x_n) = \int_0^\infty Pe^{-rt - \sum_1^n h(x_j)v(t)} h(x_n)u(t)dt - x_n$$

where r is the discount rate.   This is the expected discounted reward, mul-

tiplied by the probability density the nth firm succeeds at t,

$h(x_n)u(t)e^{-h(x_n)v(t)}$, and no others have succeeded by t, $e^{-\sum_1^{n-1} h(x_j)v(t)}$,

(i.e. the probability density that the nth firm wins at t), integrated over

all possible completion times, less the expenditure x_n.   The expenditures

$x_1, \ldots, x_{n-1}$ of rivals are known parameters for the nth firm, independent

of its own choice x_n.

If the optimal choice of x_n is positive and finite (as we assume), it satisfies

(8) $\quad W'(x_n) = Ph'(x_n)\int_0^\infty e^{-rt - \sum_1^n h(x_j)v(t)} u(t)[1-h(x_n)v(t)]dt$

$$- 1 = 0$$

$\quad W''(x_n) \leq 0.$

Since all participants in the race are identical, in equilibrium each satisfies (8) at the same choice of $x_j = x$. In equilibrium, therefore, the optimal expenditure x satisfies

(9) $\quad Ph'(x)\int_0^\infty e^{-rt - nh(x)v(t)} u(t)[1-h(x)v(t)]dt = 1.$

How is the firm's R&D effort x related to the number n of firms in the race, assuming that number is exogenous? To determine this, we apply comparative statics analysis to (9), assuming the second order condition satisfied. This indicates that $\partial x/\partial n$ has the sign of the partial derivative of the left side of (9) with respect to n, namely

(10) $\quad -Ph'h \int_0^\infty e^{-rt-nhv} uv(1-hv)dt.$

To sign this, note the identity

(11) $\quad \int_0^\infty e^{-rt-nhv} uvdt = (1/2)\int_0^\infty e^{-rt-nhv}(r+nhu)v^2 dt$

that may be verified by integration by parts on the left, setting $f = e^{-rt-nhv}$ and $dg = uvdt$ in the formula $\int fdg = fg - \int gdf$. Substituting (11) into (10) and collecting terms gives

$$-Ph'h \int_0^\infty e^{-rt-nhv}[(r + (n-2)hu)v^2/2]dt < 0, \qquad n \geq 2.$$

This shows that the left side of (9) is decreasing in n as well as in x. It follows that an increase in the number n of rivals reduces each firm's equilibrium R&D expenditure level. This extends Loury's result (p. 401) from the exponential distribution over R&D completion time to a general class of probability distributions. (But see Lee and Wilde for a reversal of this result under the alternative hypothesis that R&D expenditure is not lump-sum but rather is a flow that ceases upon innovation by the firm or any rival.)

Summary

We have developed and illustrated the use of a general family of probability distributions in terms of a generalized hazard rate, in hopes that it will be useful to others in achieving less restrictive formulations than previously available.

References

Dasgupta, P. and J. Stiglitz, "Uncertainty, Industrial Structure and the Speed of R&D," The Bell Journal of Economics, Vol. 11, Spring 1980, pp. 1-28.

Kamien, M.I. and N.L. Schwartz, "Timing of Innovations Under Rivalry," Econometrica, Vol. 40, January 1972, pp. 43-60.

_____, "Patent Life and R&D Rivalry," American Economic Review, Vol. 64, March 1974, pp. 183-187.

_____, "On the Degree of Rivalry for Maximum Innovative Activity," Quarterly Journal of Economics, Vol. XC, May 1976, pp. 245-260.

_____, Market Structure and Innovation, forthcoming, Cambridge University Press.

Lee, Tom and Louis Wilde, "Market Structure and Innovation: A Comment," forthcoming, Quarterly Journal of Economics.

Loury, G.C., "Market Structure and Innovation," Quarterly Journal of Economics, Vol. XCIII, August 1979, pp. 395-410.

Sivazlian, B.D. and L.E. Stanfel, Analysis of Systems in Operations Research, Englewood Cliffs: Prentice-Hall, 1975, Ch. 3.