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LOCAL SIMPLE GAMES */

by

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ABSTRACT

A local game is an instantaneous game in which an outcome is a direction in which the social state can feasibly be moved. The equilibrium function which assigns to each possible social state a directional outcome will result in a dynamic process governing the path of the social state. The best known dynamic process generated from local games is the MDP allocation procedure.

In this paper I examine processes that result from local games that are simple games. Thus directional preferences are (myopically) induced, a dominance relation on directions of motion is constructed, the directional core is characterized, conditions for its existence are established, and the convergence properties of the resulting dynamic processes are investigated. The focus is on the two most important simple games, the Pareto (unanimity) game and the majority rule game.

Some of the more interesting results are as follows: (1) the cone of Pareto-undominated directions intersects the cone of Pareto-improving directions, but neither cone necessarily contains the other; (2) the directional core is empty exactly when Schofield's Null Dual condition holds, so that "local cycling" can be said to occur whenever "local social indecision" occurs; (3) the nonemptiness of the majority rule directional core requires the same pairwise symmetry of utility gradients that the constrained voting equilibrium studied by Plott requires, indicating that majority rule directional cores are generically empty; (4) a social planner can always specify a Pareto-undominated direction at each time that causes the social state to converge to the Pareto set; finally, (5) when preferences are euclidean, a social state that moves in any undominated direction will converge to the point core if the point core exits, and it will converge to the local cycling set if the point core does not exist.

1. Introduction

This paper initiates an investigation of processes in which cooperative games determine at each moment the direction of motion of the social state. Attention is focused upon simple games, i.e., upon games in which a coalition can prevent either every direction or no direction from being chosen.

As motivation, consider the problem of constructing a planning procedure to allocate goods. Existing procedures, as surveyed in Tulkens [27], specify a local (instantaneous) game that determines the direction of change of the allocation vector. For example, the MDP procedure of Malinvaud [13] and Dreze and de la Valee Poussin [8] specifies the direction of change at each moment as a function of reported marginal rates of substitution. Assuming that individuals are concerned only with maximizing their rates of utility increase, local games can be constructed so that their directional equilibria will lead the allocation vector to the Pareto set. Various equilibrium concepts have been used. In MDP local games, for example, maxmin equilibria [8], Nash Equilibria ([21],[26]), and certain cooperative equilibria [28] have been shown to cause convergence to the Pareto set.

Whenever a procedure is actually to be implemented, account should be taken of the fact that it will be constrained by rules that are effective in the society. For example, in a dictatorship the social state must move in a direction that is most preferred by the dictator; the dictator will simply not abide by the equilibrium direction of a local game unless it is one of his most preferred directions. Less trivially, whenever unanimous disapproval of a motion can prevent its occurence, the society will only agree to a procedure whose equilibrium direction of motion cannot be feasibly changed to a direction that increases everybody's utility at a greater rate.

These two examples of constraints on motion are not of the type that is usually imposed in the literature on allocation procedures and, furthermore, in areas like tax reform (e.g., Guesnerie [10]) and majority rule dynamics (e.g., McKelvey [19], Schofield [24]). The constraints on motion in these studies can be viewed as resulting from a requirement that any motion must increase the utility of every member of some coalition in a specified set of winning coalitions. The resulting motion must therefore be Pareto-improving if the only winning coalition is the coalition of the whole; this is the constraint imposed in much of the allocation procedure and tax reform literature. If majority coalitions are winning, then motion is constrained to be improving for all members of some majority coalition; this majority-improving property is required of the paths studied in the literature on majority rule dynamics.

In this paper a different type of constraint is studied. We shall say that one direction <u>dominates</u> another at a point if every member of some winning coalition experiences a greater rate of utility increase, or a slower rate of utility loss, if the point moves according to the first direction rather than according to the second direction. With this terminology, the Pareto-improving criterion for motion can be viewed as requiring that any chosen direction of motion dominate the null direction in the Pareto (unanimity) game. Similarly, the majority-improving criterion results from requiring the direction of motion to dominate the null direction in a majority rule game. But now a natural question arises: why are nonnull directions not compared to each other, or rather, why should a Pareto-improving (or a majority-improving) direction be chosen if another direction dominates it?

In this paper it is not required that a direction of motion dominate the null direction. Rather, it is required that a direction of motion not be dominated by any alternative feasible direction, i.e., that the direction of motion be in the core of the local simple game. This constraint is the more natural one to impose in situations where comparisons between nonnull directions can be made. [1/]

Considering again the Pareto game, the difference between the Pareto-improving and the Pareto-undominated criterion is striking: it will be shown that neither property implies the other. Consequently it may be important in many situations to study Pareto-undominated curves instead of Pareto-improving curves. However, the two properties are not completely inconsistent: it will be shown that Pareto-undominated directions always exist that are also Pareto-improving. Attention should be focused upon these directions whenever (1) comparisons between nonnull directions are allowed, and (2) a bias towards stationarity exists in the sense that no motion will occur unless the direction of motion dominates the null direction.

The agenda for the paper is as follows. Directions of motion are defined and two myopic ways of inducing directional preferences from utility functions are discussed in Section 2. In Section 3 the local game and its directional core are defined. Pareto games are studied in Section 4: their cores are characterized, the two ways of inducing directional preferences are shown to usually result in the same core, and the existence of Pareto-improving core directions is demonstrated.

In Section 5 these results are extended to arbitrary simple games. In this section it is also shown, via some results of Schofield [24], that an agenda can be constructed to yield any point near the social state exactly when the directional core at the state is empty. In Section 6 it is shown that directional cores can be expected to be empty in games like majority rule, since severe symmetry conditions are required for the core to be nonempty in these games. Finally, in Section 7 various dynamical results, such as convergence to the Pareto set of Pareto-undominated curves, are presented. Concluding remarks are in Section 8. Some material on cones and all proofs are contained in an appendix.

2. Directions

The feasible set of social states is $X \subseteq R^m$. X is assumed to be closed, convex, and (sometimes) to satisfy

(A) there exists $Q = \{q_1,\ldots,q_J\} \subseteq R^m$ and $\{b_1,\ldots,b_J\} \subseteq R$ such that $X = \{x \in R^m\colon x \cdot q_j \geq b_j$ for all $j = 1,\ldots,J\}$.

Assumption (A) is general enough for many economic applications. For example, the feasible set of an exchange economy is given by a set of linear equality and inequality constraints and hence can be expressed in the form of (A).

For any $x \in X$, the set of <u>feasible directions</u> in which x can be (infinitesimally) shifted is the cone

$$T(x) = \text{closure } \{v \in R^m\colon x + \lambda v \in X \text{ for some } \lambda > 0\}.$$

It will be useful to list here several features of $T(x)$. First, because X is convex, $T(x)$ is a closed convex cone that is always contained in the smallest subspace that contains X. Denoting this subspace by T, it is also true that $T(x) = T$ for all x in the relative interior of X, riX. Secondly, if $c\colon[0,\infty) \to X$ is a differentiable curve, then the tangent vector $\dot{c}(t)$ is contained in $T(c(t))$. Finally, if (A) holds, then

$$T(x) = C(q(x))^*,$$

where $q(x) = \{q_j \in Q\colon x \cdot q_j = b_j\}$ and $C(q(x))^*$ is the nonnegative dual of the cone $C(q(x))$ generated by $q(x)$. [2]

Preferences over the directions in $T(x)$ can be induced from a differentiable utility function $U: R^m \to R$ in the following way. Represent the derivative of U at x by the gradient $u(x) \in R^m$, so that the directional derivative in direction $v \in T(x)$ can be denoted $v \cdot u(x)$. Then a preference ordering $P(x)$ on $T(x)$ is given by

$$v_1 P(x) v_2 \qquad \text{iff} \qquad v_1 \cdot u(x) > v_2 \cdot u(x).$$

This ordering of directions corresponds to an ordering of the differentiable curves through x according to the instantaneous rate at which they increase utility.

An alternative ordering of $T(x)$ can be obtained in a limiting fashion. Suppose that the status quo x can be shifted a distance no greater than $d > 0$ in a single period of a discrete-time process. Then direction v_1 is certainly preferred to direction v_2 if $U(x+\lambda v_1) > U(x+\lambda v_2)$ for all $\lambda > 0$ satisfying $\|\lambda v_1\| < d$ and $\|\lambda v_2\| < d$. In the limit as $d \to 0$, the following preference relation on $T(x)$ is obtained:

$$v_1 \hat{P}(x) v_2 \qquad \text{iff} \qquad \text{there exists } \bar{\lambda} > 0 \text{ for which}$$

$$U(x + \lambda v_1) > U(x + \lambda v_2) \quad \text{for all} \quad 0 < \lambda < \bar{\lambda}.$$

Generally $P(x)$ and $\hat{P}(x)$ order $T(x)$ differently, unless U is affine. Viewed as subsets of $T(x) \otimes T(x)$, the following lemma shows that $P(x)$ is a subset of $\hat{P}(x)$.

<u>Lemma 1</u>: $P(x) \subseteq \hat{P}(x)$. 3/

3. Cooperative Local Games

A cooperative local game is defined by a correspondence

$$F: X \otimes T \otimes 2^N \rightrightarrows T,$$

where $N = \{1,2,\ldots,n\}$ denotes the set of individuals. The set $F(x,v,M)$ is required to be contained in $T(x)$. If $\tilde{v} \in F(x,v,M)$, then M is (said to be) able to block v via $\tilde{v}$ at x. Thus F specifies for each coalition M and for each direction v the set of directions that M can use to guarantee that v is not chosen. [4/]

The interpretation of F is clearer when it is seen how preferences combine with F to form a dominance relation. Given that each $i \in N$ has a continuously differentiable utility function $U_i: R^m \to R$, define $P_i(x)$ as in section 2. For any $M \subseteq N$, let $v_1 P_M(x) v_2$ mean that $v_1 P_i(x) v_2$ for each $i \in M$. Then for any directions v and $\tilde{v}$, $\tilde{v}$ dominates v via M provided

$$\tilde{v} \in F(x,v,M) \quad \text{and} \quad \tilde{v}\, P_M(x) v.$$

The M-core is the set of feasible M-undominated directions,

$$K_M(x) = \{v \in T(x): \tilde{v} P_M(x) v \ \text{ for no } \ \tilde{v} \in F(x,v,M)\},$$

and the core is the set of undominated directions,

$$K(x) = \bigcap_{M \subseteq N} K_M(x).$$

In exactly the same fashion, the cores $\hat{K}_M(x)$ and $\hat{K}(x)$ are defined via the relations $\hat{P}_i(x)$.

A <u>local simple game</u> F is characterized by a collection of <u>winning coalitions</u> $\mathcal{W}$, which is assumed to be nonempty and monotone in the sense that supersets of winning coalitions are also winning, and by the following three properties:

(S1) $F(x,v,M) = \emptyset$ for all $x \in X$, $v \in T$, $M \in 2^N \setminus \mathcal{W}$

(S2) for all $M \in \mathcal{W}$, $v \in T(x)$, $\lambda v \in F(x,0,M)$ for some $\lambda > 0$

(S3) $F(x,v,M) = \{\tilde{v} \in T(x): \|\tilde{v}\| \leq \|v\|\}$ for all $x \in X$, $v \in T \setminus \{0\}, M \in \mathcal{W}$.

Property (S1) specifies that coalitions that are not winning can never block. Property (S2) specifies that a winning coalition, when confronted with the possible choice of the null direction, believes that it can guarantee instead some shift in any direction that keeps the social state in X. These two assumptions seem innocuous.

Assumption (S3), which concerns the set $F(x,v,M)$ for a non-null direction v and a winning coalition M, is more restrictive. Since a winning coalition is traditionally supposed to be able to guarantee the adoption of any feasible direction, $F(x,v,M)$ should be the set of feasible directions at x. However, setting $F(x,v,M) = T(x)$ causes interpretational and mathematical problems. It is not reasonable that directions of arbitrarily large magnitude, which correspond to arbitrarily large rates of motion for the state, be perceived as feasible. Consequently, if $F(x,v,M)$ is interpreted as the set of directions which M perceives that it can guarantee rather than have v adopted, $F(x,v,M)$ should be bounded. A fairly natural bound results from assuming that M, when faced with the possible choice of v, perceives itself as able to guarantee directions in $T(x)$ that have magnitudes no greater than the magnitude of v. For example, in a discrete world, if a social planner announces that he will move the state a certain amount in a particular direction, then any winning coalition should perceive it as feasible to force the planner to instead shift the state the same or smaller amount in another direction. This is the rationale behind (S3). The result of assumption (S3) is that a curve will have an M-undominated tangent at x if and only if another curve can, at x, increase the utility of every member of M at a greater rate only by actually moving the state at a greater rate through x. (Alternatives to (S3) are discussed in section 8.)

Henceforth, except in lemma 2 in the following section, $\mathcal{F}$ will be assumed to be a local simple game satisfying (S1) - (S3).

4. M-Cores

The directional M-cores $K_M(x)$ and $\hat{K}_M(x)$ are characterized and compared in this section. Besides being of use in studying the core, M-cores are of direct interest because they are the actual cores of M-games, i.e., of games defined by a specified coalition M and a set of winning coalitions

$$\mathcal{W}_M = \{M' \subseteq N: M \subseteq M'\}.$$

Two important examples of M-games are the Pareto (unanimity) game ($M = N$) and the dictatorship game ($M = \{i_o\}$).

For any $M \subseteq N$, let $u_M(x) = \{u_i(x): i \in M\}$. Two cones depending on the set $u_M(x)$ will be important:

$$C(u_M(x)) = \{ \sum_{i \in M} \lambda_i u_i(x) : \sum_{i \in M} \lambda_i > 0, \lambda_i \geq 0 \}$$

$$C(u_M(x))^+ = \{ v \in R^m : v \cdot u_i(x) > 0 \text{ for all } i \in M \}$$

These cones are, respectively, the finite cone generated by the set $u_M(x)$ and its positive dual. The directions in $C(u_M(x))^+$ are called <u>M-improving</u> or, if $M = N$, <u>Pareto-improving</u>. The following lemma shows the relationship between M-improving and M-undominated directions.

<u>Lemma 2:</u> $K_M(s) = \{\tilde{v} \in T(x) : t \in C(u_M(x))^+ \Rightarrow \tilde{v} + t \notin F(x, \tilde{v}, M)\}$ for any cooperative local game F.

This lemma implies the intuitive proposition that an M-undominated direction is any feasible direction $\tilde{v}$ that cannot be incremented by an M-improving direction to obtain a direction that M can use to block $\tilde{v}$. More loosely put, $\tilde{v}$ is M-undominated if and only if M regards as infeasible any improvement of $\tilde{v}$, where an improvement of $\tilde{v}$ is precisely the sum of $\tilde{v}$ and an M-improving direction. If F is a simple game, then lemma 2 together with (S3) imply that $K_M(x)$ is a cone.

The exact nature of $K_M(x)$ will depend upon whether x is M-optimal. The set of <u>(infinitesimal) M-optima</u> is the subset of points at which no M-improving direction is feasible:

$$\mathcal{P}(M) = \{x \in X : C(u_M(x))^+ \cap T(x) = \emptyset\} .$$

It is well known that $\mathcal{P}(M)$ is the set of Pareto optimal (for M) points if each U_i is pseudoconcave. Lemma 2 and (S2) immediately imply that the null direction, which corresponds to no movement from x, is M-undominated exactly when x is M-optimal.

<u>Proposition 1</u>: For all $x \in X$, $0 \in K_M(x)$ iff $x \in \mathcal{P}(M)$.

Any M-optimal x can be characterized in terms of the projection of the cone $C(u_M(x))$ onto the tangent cone $T(x)$. For any $u \in R^m$, the projection of u onto $T(x)$ is the $\bar{u} \in T(x)$ that is closest to u:

$$\| u - \bar{u} \| = \min_{v \in T(x)} \| u - v \| .$$

Hence $\bar{u} = u$ if and only if $u \in T(x)$. Let $\overline{C(u_M(x))}$ denote the projection of $C(u_M(x))$ onto $T(x)$. The following lemma characterizes points $x \in \mathcal{P}(M)$ in terms of the projected cone $\overline{C(u_M(x))}$.

<u>Lemma 3</u>: $x \in \mathcal{P}(M)$ iff $0 \in \overline{C(u_M(x))}$.

The following proposition characterizes $K_M(x)$ for most x.

<u>Proposition 2</u>: For all $x \in X$,

(i) $\overline{C(u_M(x))} \subseteq K_M(x)$;

(ii) $K_M(x) = \overline{C(u_M(x))}$ if $x \notin \mathcal{P}(M)$ and if (A) holds;

(iii) $K_M(x) = T(x)$ if $C(u_M(x))^+ \cap T = \emptyset$; and

(iv) $K_M(x) = T$ if $x \in (\text{ri}X) \cap \mathcal{P}(M)$. [5/]

Part (i) of proposition 2 states that the projection onto $T(x)$ of any semipositive linear combination of the utility gradients in $u_M(x)$ is M-undominated. Part (ii) states further that if (A) holds and x is not M-optimal, then only these projections are M-undominated. Part (iii) says that every feasible direction is M-undominated if x is M-optimal and would remain M-optimal even if every direction in T was made feasible. Part (iv) says that every feasible direction is M-undominated if x is

M-optimal and in the relative interior of X. In other words, the coalition M is indifferent among all directions (in the sense of $P_M(x) = \emptyset$) when x is an interior M-optimum, just as an individual with differentiable utility is indifferent among all small movements away from an interior utility optimum. When (A) holds, the only points at which $K_M(x)$ is not characterized by proposition 2 are M-optimal points that are contained in the relative boundary of X and do not satisfy the condition of part (iii).

The most useful part of proposition 2 is (ii). It, however, can be simplified for relatively interior x. Let

$$\bar{u}_M(x) = \{\bar{u}_i(x): i \in M\}$$

denote the set of projections of the gradients in $u_M(x)$ onto $T(x)$.

<u>Proposition 3</u>: If $x \in (riX) \setminus \mathcal{O}(M)$, and if (A) holds, then

$$K_M(x) = C(\bar{u}_M(x)).$$

Proposition 3 reveals the relationship between M-undominated and M-improving directions. Suppose for simplicity that x is contained in the interior of X, that x is not M-optimal, and that (A) holds. Then $K_M(x)$ is the cone $C(u_M(x))$ that is generated by the gradients in $u_M(x)$. Hence the cone of M-improving directions is the positive dual of the cone of M-undominated directions. Only if $C(u_M(x))$ is very "narrow" will every M-undominated direction be M-improving, and only if $C(u_M(x))$ is very "wide" will every M-improving direction be M-undominated.

However, M-improving directions in $K_M(x)$ do exist, as the following lemma demonstrates.

<u>Lemma 4</u>: Let $x \in \text{ri}X$, and let $H(\bar{u}_M(x))$ be the convex hull of $\bar{u}_M(x)$. Define a function $\theta_M: X \to R^m$ by $\theta_M(x) = \underset{\theta \in H(u(x))}{\text{Argmin}} \|\theta\|$.

Then $\theta_M(x) \in K_M(x) \cap C(u_M(x))^+$ if $x \notin \mathcal{P}(M)$, and $\theta_M(x) = 0$ if $x \in \mathcal{P}(M)$.

Now the M-core $\hat{K}_M(x)$ can be compared to its "linear approximation" $K_M(x)$. The next proposition indicates that $K_M(x)$ contains $\hat{K}_M(x)$ and, given (A), that the two cores are equal if x is not M-optimal or if x satisfies a particular condition. The condition requires that for every $v \in T$ (not just every $v \in T(x)$), some member of M has $v \cdot u_i(x) < 0$. Thus the two cores can be different only at points that are M-optimal but that are not optimal in this stronger sense. Overall, proposition 4 serves as a justification for dealing solely with $K_M(x)$ in subsequent sections.

<u>Proposition 4</u>: For all $x \in X$,

(i) $\hat{K}_M(x) \subseteq K_M(x)$;

(ii) $\hat{K}_M(x) = K_M(x)$ if $x \notin \mathcal{P}(M)$ and if (A) holds; and

(iii) $\hat{K}_M(x) = K_M(x) = T(x)$ if $C(u_M(x))^* \cap T = \{0\}$.

5. <u>General Simple Games</u>

The results of the previous section can be easily extended to any simple game. The first task is to characterize the directional core

$$K(x) = \bigcap_{M \in \mathcal{W}} K_M(x).$$

A necessary definition is that of the set $\mathscr{P}$ of (infinitesimal) $\mathscr{W}$-optima, which is the subset of X from which no feasible direction is M-improving for any winning coalition M:

$$\mathscr{P} = \bigcap_{M \in \mathscr{W}} \mathscr{P}(M).$$

Also, for any $x \in X$ define

$$\mathscr{W}(x) = \{M \in \mathscr{W}: x \notin \mathscr{P}(M)\}.$$

The next theorem is an immediate consequence of propositions 1 and 2.

Theorem 1: For any $x \in X$,

(i) $0 \in K(x)$ iff $x \in \mathscr{P}$;

(ii) $\bigcap_{M \in \mathscr{W}} \overline{C(u_M(x))} \subseteq K(x)$;

(iii) $K(x) \subseteq \bigcap_{M \in \mathscr{W}(x)} \overline{C(u_M(x))}$ if $x \notin \mathscr{P}$ and (A) holds; and

(iv) $K(x) = T(x)$ if $C(u_M(x))^{+} \cap T = \emptyset$ for all $M \in \mathscr{W}$.

Notice that (iii) is not an equality, as is its counterpart in proposition 2 (ii). This is because (ii) and (iii) do not necessarily "sandwich" $K(x)$, i.e., the containment

$$\bigcap_{M \in \mathscr{W}} \overline{C(u_M(x))} \subseteq \bigcap_{M \in \mathscr{W}(x)} \overline{C(u_M(x))}$$

may be a proper containment. However, (iii) is an equality if

$C(U_M(x))^{+} \cap T = \emptyset$ for all $M \in \mathcal{W}/\mathcal{W}(x)$ (see proposition 2 (iii)). Thus an equality in (iii) is obtained for a relatively interior x, as is indicated in the following consequence of propositions 2 and 3.

<u>Theorem 2:</u> For any $x \in riX$,

(i) $K(x) = \bigcap_{M \in \mathcal{W}(x)} C(\overline{u}_M(x))$ if $x \notin \mathcal{P}$, and

(ii) $K(x) = T$ if $x \in \mathcal{P}$.

The characterizations of $K(x)$ in theorems 1 and 2 provide a hint of a severe problem for games with many winning coalitions, such as majority rule. The problem is that there is no guarantee that the intersections in theorem 1 (iii) and theorem 2 (i) will be nonempty. Hence $K(x) = \emptyset$ is a real possibility, in which case the theory has no predictive or normative value. This problem is treated in more detail in the next section. For now we note simply that $K(x) = K_M(x)$ is nonempty in an M-game, and that in fact $K(x)$ is nonempty whenever the <u>collegium</u>,

$$C = \bigcap_{M \in \mathcal{W}} M,$$

is nonempty. [6]

The consequences of an empty core are significant. If $K(x)$ is empty, then there is extreme latitude for agenda manipulation at x, which will now be discussed.

Call a curve $c: [0,\tau] \to X$ <u>improving from x to y</u> if $x = c(0)$, $y = c(\tau)$, $\dot{c}(t)$ exists on all but a finite subset of $[0,\tau]$, and

$$\dot{c}(t) \in C(u_M(c(t)))^{+}$$

for some $M \in \mathcal{W}$ whenever $\dot{c}(t)$ exists. Thus, at almost every point on the curve, the utility of every member of some winning coalition is strictly increasing.

If $K(x) \neq \emptyset$, then an improving curve through x should not in general be expected to occur, since an undominated direction need not be improving. An improving curve would occur, however, in two special circumstances. First, if at each point in time there is an election between an incumbent and a challenger, and the incumbent can only adopt the status quo (null direction) as his campaign promise, then the challenger will win by adopting a direction that increases the utility of every member of some winning coalition. This process results in an improving curve. Second, an improving curve also results if the local game is restricted by an agenda in which the null direction is pitted against exactly one other direction. If an improving curve from x to y exists, then an agenda-maker can insure that the state will reach y by always choosing the curve's tangent vector as the direction to put up against the null direction.

Schofield [24] provided a condition, the "Null Dual" condition, for there to exist an improving curve from x to every nearby y. Consequently, if the Null Dual condition holds then an agenda-maker can manipulate the agenda to achieve any nearby state that he wants. At interior x, theorem 2 (i) implies that $K(x) = \emptyset$ is equivalent to the Null Dual condition holding at x. Hence the following theorem.

<u>Theorem 3</u>: Suppose $x \in \text{ri}X$. If $K(x) = \emptyset$, then there exists a neighborhood V of x such that for every $y \in V \cap X$, there exists an

improving curve from x to y that stays in V. Conversely, if such a neighbrohood of x exists, then $x \in cl\{y \in X: K(y) = \emptyset\}$.

6. λ-Majority Games

A λ-majority game, where λ is a positive fraction, is a simple game in which the collection of winning coalitions is given by

$$\mathcal{W}_\lambda = \{M \subseteq N: \lambda n \leq |M|\}.$$

Such games are anonymous, since a coalition's power depends only upon its size and not upon the identities of its members. Two common examples are the Pareto game $(\lambda = 1)$ and the (absolute) majority rule $(\lambda = \frac{1}{2})$ game. In this section it is shown that when λ is near one-half, then the utility gradients must satisfy a severe symmetry condition at a point x if any direction is undominated at x. The severity of the condition will imply that the closure of

$$L = \{x \in X: K(x) = \emptyset\}$$

is generically equal to X. Hence an agenda-maker in such games can generally achieve, by theorem 3, practically any state he desires.

The symmetry condition will follow from a lemma that is proven via the following remarks. For any $x \in X$ and $\tilde{v} \in T(x)$, define a cone

$$\tilde{T}(x,\tilde{v}) = \{v \in T(x): v \cdot \tilde{v} < 0\}.$$

Suppose t is contained in $\tilde{T}(x,\tilde{v})$. Then it follows easily that

$$\|\tilde{v} + \lambda t\| < \|\tilde{v}\|$$

for some small $\lambda > 0$, so that $\tilde{v} + \lambda t$ is contained in $F(x,\tilde{v},M)$ if M is a winning coalition. Consequently, lemma 2 implies that $\tilde{v}$ is undominated if and only if λt, and hence t, is not M-improving. Thus we can restate lemma 2 as

Lemma 5: $\tilde{v} \in K(x)$ if and only if $C(u_M(x))^+ \cap \tilde{T}(x,\tilde{v}) = \emptyset$ for every $M \in \mathcal{W}_\lambda$.

In words, lemma 5 states that a direction $\tilde{v}$ is undominated at x if and only if x would be $\mathcal{W}_\lambda$-optimal if the set of feasible shift directions were $\tilde{T}(x,\tilde{v})$ rather than $T(x)$. Therefore we can apply the conditions obtained in [17] for x to be $\mathcal{W}_\lambda$-optimal when x can only shift in directions contained in $\tilde{T}(x,\tilde{v})$.

Definitions are required. For a nonzero $\tilde{v} \in T$, a pair of gradients $\{\bar{u}_i(x),\bar{u}_j(x)\}$ is <u>pairwise symmetric about $\tilde{v}$</u> provided that

(PS) neither $\bar{u}_i(x)$ nor $\bar{u}_j(x)$ is a nonnegative multiple of $\tilde{v}$, but there does exist $\alpha_i > 0$ and $\alpha_j > 0$ such that

$$\alpha_i\bar{u}_i(x) + \alpha_j\bar{u}_j(x) \in \{0,\tilde{v}\} .$$

In other words, $\bar{u}_i(x)$ and $\bar{u}_j(x)$ are symmetric about $\tilde{v}$ provided that neither one is contained in the ray generated by $\tilde{v}$, but that that ray is contained in the acute angle between $\bar{u}_i(x)$ and $\bar{u}_j(x)$. Notice that the nonnegative linear dependency of the three gradients, $\bar{u}_i(x)$, $\bar{u}_j(x)$ and $\tilde{v}$, is in some sense "unlikely" if the dimension of T is greater than two.

To apply [17], we define the following coalitions:

$$R_+(x,\tilde{v}) = \{i \in N: \bar{u}_i(x) = \alpha\tilde{v} \text{ for some } \alpha \geq 0\}$$

$$R_-(x,\tilde{v}) = \{i \in N: \bar{u}_i(x) = \alpha\tilde{v} \text{ for some } \alpha < 0\}.$$

Individuals in $R_+(x,\tilde{v})$ will not find any direction in $T(x,\tilde{v})$ to be improving, whereas individuals in $R_-(x,\tilde{v})$ will find every direction in $T(x,\tilde{v})$ to be improving. Lemma 5 and results in [17] now directly imply

Theorem 4: Suppose $x \in riX$. If $\tilde{v} \in K(x)$ in a λ-majority game, then

(1) $\quad |R_+(x,\tilde{v})| - |R_-(x,\tilde{v})| > (1-2\lambda)n$. 7/

Furthermore, if

(2) $\quad |R_+(x,\tilde{v})| - |R_-(x,\tilde{v})| \leq 1 + (1-2\lambda)n$,

then $\tilde{v} \in K(x)$ in a λ-majority game if and only if both (1) holds and the set $\{\bar{u}_i(x) : i \notin R_+(x,\tilde{v}) \cup R_-(x,\tilde{v})\}$ can be partitioned into pairs that are pairwise symmetric about $\tilde{v}$. 8/

To clarify the meaning of theorem 4, consider the case of majority rule $(\lambda = \frac{1}{2})$ with n odd. Inequality (1) then says that $R_+(x,\tilde{v})$ must contain at least one member more than $R_-(x,\tilde{v})$. Since n is odd, it is always possible to find some $\tilde{v}$ for which (1) holds. However, unless T is one-dimensional, inequality (2) can be expected to hold for all $\tilde{v} \in T$, since it is "unlikely" that $R_+(x,\tilde{v})$ will contain at least two members more than $R_-(x,\tilde{v})$ for any $\tilde{v} \in T$. If (2) does hold and $\tilde{v}$ is undominated, then theorem 5 implies that an extremely strong symmetry condition must be satisfied by the gradients of all individuals not in $R_+(x,\tilde{v})$ or $R_-(x,\tilde{v})$.

The conditions of theorem 4 have recently been used by Schofield [25] to show that $K(x) \neq \emptyset$ is "uncommon" in a formal sense. Specifically, he investigates games with $\lambda = \frac{1}{2}$ and $X = R^m$, where $m > 2$ if n is odd and $m > 3$ if n is even. In these cases he shows that if the n-tuple of utility functions is contained in a particular dense subset of $\prod_{i=1}^{n} C^2$, then L is dense in R^m. Thus theorem 4 implies that L is *generically* dense in a majority game if the dimension of the space is greater than two (n odd) or three (n even).

7. Dynamics

This section is concerned with the curves that are generated when the social state moves continuously in undominated directions. More formally, define an absolutely continuous function $c:[0,\infty) \to X$ to be *undominated* if the derivative $\dot{c}(t)$ is contained in $K(c(t))$ almost everywhere possible, i.e., at almost every t for which $c(t) \notin \text{cl } L$. [9/] Of course, such curves are of little interest in games like majority rule, since then $\text{cl}L$ is generically equal to X. However, there is no problem in local games with nonempty collegiums, since then L is empty. Call an undominated curve *M-undominated* (*Pareto undominated*) if the local game is an M-game (Pareto game).

Three questions are of interest. First, to guard against vacuity, do undominated curves exist? Second, does an undominated curve exist that converges to some "nice" set? This question is asked by a planner who desires to arrive at a social optimum, but who knows that at each moment his actions will be constrained by the political realities embodied in the function F. If it is known only that motion of the state will, whenever possible, be so constrained, then predictions regarding future states require a positive answer to the third question: do <u>all</u> undominated curves converge to a well-defined set?

The existence question can be answered affirmatively. Assume for simplicity that X is a subspace, i.e., that $X = T$. For some interval $S = (s_1, s_2)$, where $0 < s_1 < s_2$, define

$$K_M^S(x) = \{v \in K_M(x) : s_1 \leq \| v \| \leq s_2\}.$$

Let $HK_M^S(x)$ denote the convex hull of $K_M^S(x)$, and define a correspondence

$$G(x) = \begin{cases} \bigcap_{M \in \mathcal{M}} HK_M^S(x) & \text{if } x \notin \text{cl}\, L \\ \{v \in T(x) : \| v \| \leq s_2\} & \text{if } x \in \text{cl}\, L. \end{cases}$$

Now consider solutions of the dynamic process

(PI) $\qquad \dot{x} \in G(x)$ and $x(0) = x_0$

where a solution is a function $c:[0,\infty) \to X$ such that $c(0) = x_o$, $\dot{c}(t)$ exists and satisfies (P1) almost everywhere, and c is absolutely continuous on finite intervals. The following argument shows that any solution c to (P1) is undominated. If $c(t) \in \mathcal{P}$ at time t, then $\dot{c}(t)$ is undominated because, by theorem 2 (ii), $K(c(t)) = T$. If $c(t) \notin \mathcal{P} \cup \mathrm{cl}L$ then $\dot{c}(t)$ is again undominated because

$$\dot{c}(t) \in G(c(t)) = \bigcap_{M \in \mathcal{S}(c(t))} HK_M^S(c(t))$$

$$\subseteq \bigcap_{M \in \mathcal{S}(c(t))} K_M(c(t)) = K(c(t)).$$

Consequently solutions of (P1) are undominated curves.

Still assuming that $X = T$, a theorem of Castaing and Valadier [4] (theorem A1 in [5]) implies the existence of solutions to (P1) if G is bounded and upper hemicontinuous, and if for each x, $G(x)$ is convex, compact and nonempty. By the definition of G, every condition except upper hemicontinuity is obviously satisfied. The upper hemicontinuity of G at any $x \in \mathrm{cl}\,L$ is obvious, since then $G(x)$ contains $G(y)$ for every $y \in X$. If $x \notin \mathrm{cl}\,L$ then the upper hemicontinuity of G at x follows from the following lemma.

<u>Lemma 6</u>: The correspondence K_M^S is upper hemicontinuous on $\mathrm{ri}X$.

Consequently, undominated curves exist, at least if $X = T$.

The second question, which concerns the ability of a planner to find an undominated curve that converges to a "nice" set, can also be answered affirmatively if attention is restricted to M-games. For a given coalition M, consider a welfare function $W: X \to R$ defined by

$$W(x) = \phi\,(U_1(x), U_2(x), \ldots, U_n(x)),$$

where ϕ is twice continuously differentiable and has partial derivatives satisfying $\phi_i > 0$ if $i \in M$ and $\phi_i = 0$ if $i \notin M$. Each U_i and hence W is also assumed to be twice continuously differentiable. The gradient $w(x)$ of $W(x)$ is given by

$$w(x) = \sum_{i \in M} \lambda_i(x) u_i(x),$$

where $\lambda_i(x) = \phi_i(U_1(x), \ldots, U_n(x))$. Consequently $w(x) \in C(u_M(x))$, so that proposition 2(i) implies that the projection $\bar{w}(x)$ of $w(x)$ onto $T(x)$ is M-undominated. Therefore a solution to the dynamic process defined on X by

(P2) $\quad \dot{x} = \bar{w}(x)$ and $x(0) = x_o$

will be an M-undominated curve remaining in X. The existence of a solution to (P2) that has right-hand derivatives everywhere is guaranteed by corollary A3 in [5], and its uniqueness is provided by a result in [11].

The equilibrium set of process (P2), $E = \{x \in X: \tilde{W}(x) = 0\}$, is contained in $\mathcal{P}(M)$ by lemma 3. Now, if utility functions are pseudoconcave then $\mathcal{P}(M)$ is the set of Pareto optimal points for M. Consequently, the planner's problem is solved if solutions to this process are quasistable, i.e., if the limit points of solutions are contained in E.

The quasistability of solutions to various gradient processes like (P2) is exactly what is shown in d'Aspremont and Tulkens [7]. Three technical assumptions are all that are needed for their proofs to be applicable to (P2). One is that W be bounded above on X, and a second is that X be compact. To insure the continuity of solutions in x_o, which is required for Lyapunov convergence arguments, an assumption is made that keeps solutions off the relative boundary of X ($\tilde{W}$ may be discontinuous there). The third assumption is then something of the sort that $\lim_{k\to\infty} W(x_k) = -\infty$ whenever $\{x_k\} \subseteq riX$ is a sequence converging to a point in rbX. In a commodity space, for example, this assumption takes the form of assigning infinitely low social welfare to any allocation in which any individual consumes a zero amount of any good. Given these assumptions, arguments using W as an increasing Lyapunov function can be constructed along the lines of [7] to show the quasistability of (P2) whenever $x_o \in riX$.

Solutions to (P2) may not be individually rational, that is, they may result in decreasing utility for some individuals. This cannot happen if the curve is M-improving, i.e., if $\dot{x}$ is almost always an M-improving direction.

Solutions to an alternative process,

(P3) $\quad \dot{x} = \theta_M(x)$ and $x(0) = x_o$

will be both M-improving and M-undominated, by lemma 4. If θ_M can be shown to be Lipschitzian, and if the compactness, boundedness, and boundary assumptions mentioned above are satisfied, then solutions to (P3) will exist and their limit points will be in $\mathcal{P}(M)$ and be unanimously preferred to x_o .

The third question concerns the quasistability of all undominated curves. Now, clearly, since undominated directions can have arbitrarily small magnitudes, undominated curves may "move" too slowly to ever get anywhere. To avoid this problem, the magnitude of an undominated $\dot{x}$ must be bounded away from zero whenever zero is not itself undominated. This has already been accomplished in the definition of G in the process (P1): $G(x)$ is a compact set that contains zero if and only if either x is ω-optimal or x is contained in the closure of L. Also, by letting s_1 be very small and s_2 very large, the class of undominated curves satisfying (P1) will be fairly broad. Consequently, the third question is revised to asking whether process (P1) is quasistable, i.e., whether the limit points of all solutions of (P1) are contained in the set $\mathcal{P} \cup \mathrm{cl}\, L = \{x \in X: 0 \in G(x)\}$.

An affirmative answer to this revised question can be given in a special case. It shall be assumed that all individuals have utility functions with circular indifference curves, i.e., that each individual i has preferences that are representable by

(E) $\qquad U_i(x) = (x - p_i) \cdot (p_i - x)$ for some $p_i \in R^m$.

Such preferences have long served as prototypical in voting models (e.g., in Kramer [12] and McKelvey [19]). The strength of assumption (E) is revealed in the following lemma.

<u>Lemma 7</u>: Suppose that $X = T$, that each U_i satisfies (E), and that $x \in X$. Then $v \in K_{N}(x)$ if and only if $x + \lambda v \in \mathcal{P}(M)$ for some $\lambda \geq 0$. Furthermore, $z \in \mathcal{P}$ implies that $z - x \in K(x)$.

One implication of lemma 7 is that L is empty if $\mathcal{P}$ is nonempty and (E) holds. Also, for every $M \in \mathcal{M}$ an undominated direction must "point" from x to $\mathcal{P}(M)$. This fact can be used to show that the Euclidean distance from $x(t)$ to $\mathcal{P}(M)$ is decreasing along an undominated path. Consequently, this distance can serve as a Lyapunov function in a long and tedius convergence argument, based on the Lyapunov theorem of [5] (theorem 6.1), to establish

<u>Theorem 4</u>: If $X = T$ and each U_i satisfies (E), then the limit points of any solution of (P1) are contained in $\mathcal{P}$ if $\mathcal{P} \neq \emptyset$, or in cl L if $\mathcal{P} = \emptyset$.[10/]

Hence, for example, given the special class of utility functions, all Pareto-undominated curves converge to the Pareto set. This is not an obvious convergence, since even with these special utility functions the utility of some individuals may decrease along portions of (Pareto) undominated curves. Because the Pareto-improving property is lacking, no immediate function of utilities will serve as a Lyapunov function with which to prove the quasistability of (P1) in general. I conjecture, however, that (P1) is quasistable for a fairly broad class of utility functions.

8. Summary and Concluding Remarks

The basic premise of the paper is that the motion of the social state is constrained by the blocking ability of coalitions in the society. This blocking power, as represented by the correspondence $F(x,v,M)$, may have either a normative or positive origin. One fundamental consequence is that predicted (or suggested) directions of motion need not be Pareto-improving. This conclusion deos not depend upon the simple game assumptions (S1) - (S3) that are used throughout most of the paper. Only if each $F(x,v,\{i\})$ contains the origin will an undominated direction necessarily be Pareto - improving. Similarly, in general simple games, undominated directions of motion need not increase the utility of all members of any winning coalition.

The model requires individuals to evaluate directions in terms of the instantaneous rate at which they increase utility. The objective functions of more rational individuals should take the form

$$\int_0^\infty U_i(x(t),t)dt \quad \text{or} \quad U_i(\lim_{t\to\infty} x(t)),$$

depending upon whether the process occurs in "real time" to generate a utility flow, or whether it is only the final state that is of concern. The myopic maximization of dU_i/dt does not correspond to the complicated problems of maximizing these objective functions in a supergame setting, but it may be regarded as a tractable approximation, a result of "bounded rationality."

The bulk of the paper concerns simple local games, which are defined by assumptions (S1) - (S3). Assumption (S3), which requires that a proposed direction be dominated only by directions of smaller magnitude than itself, is somewhat *ad hoc*. However, any better specification of $F(x,v,M)$ for winning coalitions can only be made in the context of a particular game; no specification can escape being arbitrary in a general setting. The specification chosen in (S3) has the advantage of treating all directions symmetrically, which is a natural assumption to make in a general model.

No results would be drastically changed if (S2) and (S3) are replaced by

$$(S') \qquad F(x,v,M) = \{\tilde{v} \in T(x) : \| \tilde{v} \| \leq s(x)\},$$

and only directions of magnitude less than $s(x)$ are considered feasible. (This is essentially the approach taken in [16].) The interpretation then would be that $s(x)$ is the upper bound on the speed of the social state at x. If (S') is assumed, then for relatively interior x the core is the solid $s(x)$-ball if $x \in \mathcal{P}$, or, if $x \notin \mathcal{P}$, the portion of the surface of the $s(x)$-ball that intersects with

$$\bigcap_{M \in \mathcal{W}(x)} C(\overline{u}_M(x)).$$

In other words, the directional core obtained via (S') is either a truncation or a thin cross-section of the cone that is the core when (S2) and (S3) hold.

Both the (S') and the (S2) and (S3) formulations suffer the disadvantage of depending upon the Euclidean norm.[11/] Of course, any norm could have been used. To eliminate norm dependence completely, (S') and the set of feasible directions could be generalized to include all directions v that have magnitude no less than some function $s(x, v/\|v\|)$. This would serve to make the maximum possible speed depend upon the direction of motion. With this specification, the perfectly general lemma 2 still holds. Furthermore, it can also be shown that $K(x)$ consists only of directions v that are of magnitude $s(x, v/\|v\|)$ when x is relatively interior and not $\mathcal{U}$-optimal. Otherwise a characterization of $K(x)$ is not obtainable at this level of generality.

Given the assumptions (S1) - (S3), some results should be emphasized. First, it was shown that "projected gradient" processes result in Pareto-undominated curves. Consequently, procedures can be found that result in Pareto-undominated curves that converge to the Pareto optimal set. In fact, some of the planning procedures that have been proposed to allocate public and private goods, such as one proposed in footnote 4 of [13], do so via Pareto-undominated curves. This follows from the demonstration of d'Aspremont and Tulkens [7] that some of these procedures are actually projected gradient processes. There are two important exceptions, however. The MDP procedure is not a process definable by the projection of a gradient onto the feasible set X; consequently the MDP procedure does not result in undominated curves.[12/] This shortcoming is severe, however, only if (S3) and the myopic inducement of directional preferences are reasonable in the application of an MDP planning procedure. Secondly, the directions of tax reform in Guesnerie [10] and Fogelman, Guesnerie and Quinzii [9] are merely required to be Pareto-improving, which, as Weymark [29] demonstrates, does not imply that they are Pareto-undominated. Being Pareto-dominated is probably a more important shortcoming for a tax reform direction than an MDP direction, since tax reform is probably a real-time process in which myopic behavior can be expected.

It was also shown that every member of a class of Pareto-undominated curves defined by (P1) converges to the Pareto set. Unfortunately this result was obtained under the assumption that utility functions are of the Euclidean form (E). Further work needs to be done to generalize this theorem. A second caveat is that although a similar convergence to $\mathcal{P}$ or L (depending upon which set is nonempty) was shown to occur in general simple games when (E) is assumed, and can even perhaps be shown without (E), such convergence is not important in games like majority rule. In these games the convergence to L is trivial, since the pairwise symmetrical location of gradients around an undominated direction imply that generically, X is equal to the closure of L.

The final result to be stressed is the connection made in section 5 between Schofield's [24] Null Dual condition and the existence of directional cores: the Null Dual condition holds at relatively interior x if and only if an undominated direction does not exist at x. Since Null Dual is the condition sufficient for an agenda-maker to have the ability to achieve any point near the current social state, the implication is that agenda manipulation can occur if the directional core is empty, i.e., if society is "locally undecided." Also by theorem 4, if society is "locally decided" at x in the sense of undominated directions existing at all points in a neighborhood of x, then an agenda-maker is not free to obtain any point near the current state.

FOOTNOTES

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This paper is an extensive revision and improvement of [16], which is a chapter in my dissertation [15]. A similar idea was explored in [18] in the context of a two candidate election. I have received valuable comments on various versions of this paper from Charles Plott, John Ferejohn and Randy Calvert. Especially valuable have been the comments of John Weymark. I also acknowledge the support of John Ledyard and NSF Grant No. SOC78-12884.

1/

Tulkens and Zamir [28] also define a dominance relation on nonnull directions. Essentially, a direction is dominated in their framework if some coalition can achieve, by trading within itself via the rules of an MDP procedure, a direction that it unanimously prefers. Thus, in contrast to the local games here, their local games are not simple and do not reflect social rules that are independent of the MDP rules.

2/

See the Appendix for definitions and results relating to cones.

3/

Although it will not be needed, it can also be shown that $\hat{P}(x)$ is contained in the closure of $P(x)$.

4/

Wilson [30], via the concept of an "effectiveness" relation, first expressed formally the idea that the set of alternatives that a coalition can use to block the choice of a proposed alternative may depend upon which alternative is being proposed.

5/

For any $A \subseteq R^m$, riA, rbA, and clA will denote the relative interior of A, the relative boundary of A, and the closure of A, respectively.

6/

Brown [2], [3] investigates the relationship between acyclic dominance relations and collections of decisive sets that have nonempty collegiums.

7/

More complicated necessary conditions are also derived in [17] and presented in [6] and [16]. These conditions put lower bounds on the number of pairs of gradients that are pairwise symmetric about $\tilde{v}$ when (2) does not hold.

8/ If [illegible] $= 1/2$ and n is odd, then theorem 4 follows from lemma 5 and Plott [20]; Plott [20] first derived the pairwise symmetry conditions that are necessary and sufficient for a constrained equilibrium when (2) holds and [illegible] $= 1/2$ and n is odd.

9/ The derivative $\dot{c}(t)$ is not required to be undominated when $c(t)$ is in the relative boundary of L, even though undominated directions exist there. (L can be shown to be open in X.) This is done for technical reasons, namely, to obtain the continuity conditions sufficient for the existence of undominated curves.

10/ The proof of theorem 4 is not included in the appendix because of its tedious length and "special case" nature. See [16] for a rudimentary sketch.

11/ I am indebted to John Weymark for this observation.

12/ Consider the MDP process for a pure exchange economy, as described in chapter 8 of Malinvaud [14]. Let there be two people and three goods. Then the allocation space can be considered as R^6, where odd-numbered coordinates represent person 1's consumption and even-numbered coordinates represent person 2's consumption. The feasible set X is the compact subset of R^6 that satisfies the nonnegativity constraints and the budget balance equalities. Suppose that at some $x \in \text{ri}X$ the two gradients are $u_1(x) = (1,0,6,0,2,0)$ and $u_2(x) = (0,1,0,2,0,4)$. Then, if both people are weighted equally at $\delta_1 = \delta_2 = 1/2$, the MDP procedure sets $\dot{x} = (-5,5,2,-2,-1,1) = z$. However, it is straightforward to find a direction $v \in T(x)$ such that $\|v\| < \|z\|$ and $u_i(x) \cdot (v - z) > 0$ for $i = 1,2$. In fact, $\bar{v} = (-5,5,32,-32,-41,41)/21$ satisfies $\bar{v} \in T(x)$, $\|\bar{v}\| \simeq 3.4 < 7.8 \simeq \|z\|$, and $u_i(x) \cdot \bar{v} = u_i(x) \cdot z = 5$ for $i = 1,2$. Thus, by letting $\dot{x} = 2.3\bar{v}$, the rate of utility increase for both individuals can be made more than twice that which they achieve in the MDP procedure, and this is achieved by moving the allocation in a different direction at a no greater rate.

Appendix

A1. Mathematical Results

Mathematical terms that are not standardized are defined here, and some useful results are listed.

A cone is a set $K \subseteq R^m$ for which $k \in K$ implies $\lambda k \in K$ for all $\lambda > 0$. This definition allows a cone to either contain or not contain the origin. A convex cone is a cone K for which $k_1, k_2 \in K$ imply $k_1 + k_2 \in K$. The cone $C(A)$ generated by a set $A = \{a_1, \ldots, a_n\} \subseteq R^m$ is defined as the set of semipositive linear combinations of the elements of A:

$$C(A) = \{ \sum_{i=1}^{n} \lambda_i a_i : \lambda_i \geq 0, \quad \sum_{i=1}^{n} \lambda_i > 0 \} .$$

Notice that $C(A)$ may or may not contain the origin, and that the closure of $C(A)$ is $C(A) \cup \{0\}$, a closed convex cone containing A.

If $A \subseteq R^m$ is any set, then the convex hull of A is

$$HA = \{ x : x = \sum_{i=1}^{n} \lambda_i a_i, \ \lambda_i \geq 0, \ \sum_{i=1}^{n} \lambda_i = 1 \ , \ a_i \in A \ , \ n \geq 1 \} .$$

HA is the smallest convex set containing A.

The positive dual of any set $A \subseteq R^m$ is defined as

$$A^+ = \{ k \in R^m : a \cdot k > 0 \quad \text{for all} \quad a \in A \} .$$

A^+ is a convex cone and may be empty. The nonnegative dual of A is

$$A^* = \{ k \in R^m : a \cdot k \geq 0 \text{ for all } \ a \in A \} .$$

A^* is a closed convex cone that contains the origin. It is easy to show that $A^+ \subseteq A^*$ and, if $A^+ \neq \emptyset$, that $clA^+ = A^*$.

The following five lemmas will be useful; proofs of the lemmas are standard and can be found in Bazarra and Shetty [1] as (A1) lemma 3.1.8 (ii), (A2) lemma 3.1.8 (iii), (A3) theorem 3.1.11, (A4) theorem 2.4.4 , and

(A5) theorem 2.4.14, respectively.

Lemma A1: $A^* = (\mathrm{cl} A)^*$.

Lemma A2: $A_1 \subset A_2$ implies $A_2^* \subseteq A_1^*$.

Lemma A3: If $K \subseteq R^m$ is a nonempty convex cone, then $K^{**} = \mathrm{cl}(K)$.

Lemma A4 (Projection Theorem): Let $T \subseteq R^m$ be a closed convex set, and let $u \in R^m$. Then there is a unique $\bar{u} \in T$, called the projection of u onto T, that satisfies

$$\| u - \bar{u} \| = \min_{v \in T} \| u - v \| .$$

Furthermore, $\bar{u} \in T$ is such a minimizing point iff

$$(u-\bar{u}) \cdot (v-\bar{u}) \leqq 0 \quad \text{for all} \quad v \in T. \; .$$

Lemma A5 (Separation Theorem) : Let $S_1 \subseteq R^m$ and $S_2 \subseteq R^m$ be nonempty and convex sets such that $\mathrm{ri}S_1 \cap \mathrm{ri}S_2 = \emptyset$ Then there exists a nonzero $u \in (-S_1)^* \cap S_2^*$.

A2. Proofs of Results

Proof of lemma 1: Suppose $(v_1, v_2) \in P(x)$. Then $(v_1 - v_2) \cdot u(x) > 0$. Let $f(\lambda) = U(x+\lambda v_1) - U(x+\lambda v_2)$, and observe that $f'(0) = (v_1-v_2) \cdot u(x) > 0$. Since $f(0) = 0$, this indicates that $f(\lambda) > 0$ for small $\lambda > 0$, i.e., that $(v_1, v_2) \in \hat{P}(x)$. Hence $P(x) \subseteq \hat{P}(x)$.

Proof of lemma 2: $\tilde{v} \in K_M(x)$

iff $v P_M(x) \tilde{v}$ for no $v \in F(x, \tilde{v}, M)$

iff no $v \in F(x, \tilde{v}, M)$ satisfies $(v-\tilde{v}) \cdot u_i(x) > 0$ for all $i \in M$

iff $t = (v-\tilde{v}) \notin C(u_M(x))^+$ for any $v \in F(x, \tilde{v}, M)$

iff $v = \tilde{v}+t \notin F(x, \tilde{v}, M)$ for any $t \in C(u_M(x))^+$.

Proof of lemma 3: If $C(u_M(x))^+ = \emptyset$, then obviously $x \in \mathcal{P}(M)$ and, furthermore, it is well-known that $0 \in C(u_M(x))$ (e.g., lemma 1.3 in Schofield [24]). Since $0 \in T(x)$, this shows that $0 \in \overline{C(u_M(x))}$. Hence we must only consider the case $C(u_M(x))^+ \neq \emptyset$. Suppose $x \in \mathcal{P}(M)$. Then $T(x) \cap C(u_M(x))^+ = \emptyset$. Both $T(x)$ and $C(u_M(x))^+$ are nonempty and convex. Hence, by a separation theorem (lemma A5), there exists a nonzero $u \in (-T(x))^* \cap C(u_M(x))^{+*}$. By lemma A4, $u \in (-T(x))^*$ implies $\bar{u} = 0$. Also, since the closure of $C(u_M(x))^+$ is $C(u_M(x))^*$ and the closure of $C(u_M(x))$ is $C(u_M(x)) \cup \{0\}$, lemmas A1 and A3 imply

$$\begin{aligned} C(u_M(x))^{+*} &= (\mathrm{cl}\,C(u_M(x))^+)^* \\ &= C(u_M(x))^{**} \\ &= \mathrm{cl}(C(u_M(x))) = C(u_M(x)) \cup \{0\}. \end{aligned}$$

Hence $u \in (-T(x))^* \cap C(u_M(x))$, so that $0 = \bar{u} \in \overline{C(u_M(x))}$. Conversely, suppose that $0 \in \overline{C(u_M(x))}$. Then there exists, by lemma A4, $u \in C(u_M(x))$ such that $u \cdot v \leq 0$ for all $v \in T(x)$. This implies, since u is a semipositive combination of the gradients in $u_M(x)$, that $T(x) \cap C(u_M(x))^+ = \emptyset$. Hence $x \in \mathcal{P}(M)$.

Proof of proposition 2: (i) Let $\bar{u} \in \overline{C(u_M(x))}$. By proposition 1 and lemma 3, $\bar{u} \in K_M(x)$ if $\bar{u} = 0$. So assume $\bar{u} \neq 0$. Suppose $v = \bar{u} + t \in F(x, \bar{u}, M)$ for some $t \neq 0$. Then $||v|| \leq ||\bar{u}||$, which implies that $t \cdot \bar{u} < 0$. By lemma A4, there exists $u \in C(u_M(x))$ such that $(u - \bar{u}) \cdot (v - \bar{u}) = (u - \bar{u}) \cdot t \leq 0$. Therefore $t \cdot u \leq t \cdot \bar{u} < 0$, so that $t \notin C(u_M(x))^+$. Thus, by lemma 2, $\bar{u} \in K_M(x)$.

(ii) Let $\hat{v} \in T(x) \setminus \overline{C(u_M(x))}$. We must show that $\hat{v} \notin K_M(x)$. Since

$x \notin \mathcal{F}(M)$, $\tilde{v} \notin X_M(x)$ if $\tilde{v} = 0$. Hence we can assume $\tilde{v} \neq 0$. Let $q'(x) = \{q_j \in q(x) : \tilde{v} \cdot q_j = 0\}$ and $C = C(u_M(x) \cup q'(x))$. Suppose that $\tilde{v} \in C$. Then $\tilde{v} = u + q$, where $u = 0$ or $u \in C(u_M(x))$, and $q = 0$ or $q \in C(q'(x))$. Because $q = 0$ or $q \in C(q'(x))$, $\tilde{v} \cdot q = 0$. Hence $0 \neq \|\tilde{v}\|^2 = \tilde{v} \cdot (u+q) = \tilde{v} \cdot u$ implies $u \neq 0$. Hence $u \in C(u_M(x))$. Now, by assumption (A), $T(x) = C(q(x))^*$. Hence $q \cdot v \geq 0$ for any $v \in T(x)$, so that

$$(u - \tilde{v}) \cdot (v - \tilde{v}) = q \cdot (\tilde{v} - v) = - q \cdot v \leq 0$$

for any $v \in T(x)$. This proves that $\tilde{v}$ is the projection of u onto $T(x)$. Hence $\tilde{v} \in \overline{C(u_M(x))}$, a contradiction. Therefore $\tilde{v} \notin C$. If $C^* \cap \{-\tilde{v}\}^+ = \emptyset$, then $C^* \subseteq \{\tilde{v}\}^*$, which implies the contradiction $\tilde{v} \in \{\tilde{v}\}^{**} \subseteq C^{**} = \mathrm{cl}C = C \cup \{0\}$ (see lemmas A1 - A3). Hence there exists

$$t_1 \in C^* \cap (-\tilde{v})^+.$$

Observe that $t_1 \cdot u_i(x) \geq 0$ for all $i \in M$. Now, since the cone $C(u_M(x))^+ \cap T(x) = C(u_M(x))^+ \cap C(q(x))^*$ is nonempty, there exists

$$t_2 \in C(u_M(x))^+ \cap C(q(x))^*$$

small enough so that $(t_1 + t_2) \cdot \tilde{v}$ is still negative. Let $t_3 = t_1 + t_2$. Now let $t = \mu t_3$, where $\mu > 0$ is chosen small enough that

$$\|\tilde{v} + t\|^2 - \|\tilde{v}\|^2 = \mu t_3 \cdot (\mu t_3 + 2\tilde{v}) \leq 0,$$

which can be done because $t_3 \cdot \tilde{v} < 0$. Also, μ can be chosen so that

$$(\tilde{v} + t) \cdot q_j = (\tilde{v} + \mu t_3) \cdot q_j \geq 0$$

for every $q_j \in q(x)$, since the following implications hold:

$$q_j \in q(x) \setminus q'(x) \Rightarrow \tilde{v} \cdot q_j > 0, \quad \text{and}$$

$$q_j \in q'(x) \Rightarrow \tilde{v} \cdot q_j = 0,\ t_1 \cdot q_j \geq 0, \quad \text{and} \quad t_2 \cdot q_j \geq 0.$$

Therefore $\tilde{v} + t \in C(\varphi(x))^* = T(x)$. Hence, since $\|\tilde{v} + t\| \le \|\tilde{v}\|$, (S3) implies $\tilde{v} + t \in F(x,\tilde{v},M)$. Also, as $t_1 \in C^* \subseteq C(u_M(x))^*$ and $t_2 \in C(u_M(x))^+$, $t = \mu(t_1+t_2) \in C(u_M(x))^+$. Therefore, by lemma 2, $\tilde{v} \notin K_M(x)$.

(iii) Assuming $C(u_M(x))^+ \cap T = \emptyset$, we must show that any $\tilde{v} \in T(x)$ is contained in $K_M(x)$. If $\tilde{v} \notin K_M(x)$, then lemma 2 implies the existence of $t \in C(u_M(x))^+$ such that $v = \tilde{v} + t \in F(x,\tilde{v},M) \subseteq T(x) \subseteq T$. Therefore, $t = v - \tilde{v} \in T$, since both v and $\tilde{v}$ are contained in T. This contradiction shows that $\tilde{v} \in K_M(x)$.

(iv) Part (iv) follows from (iii), since $x \notin \mathcal{P}(M)$ implies that $C(u_M(x))^+ \cap T(x) = \emptyset$ and $x \in \mathrm{ri}X$ implies that $T(x) = T$.

<u>Proof of Proposition 3</u>: Because $T(x) = T$ is a subspace, each $u_i(x)$ can be uniquely expressed as a sum $u_i(x) = \bar{u}_i(x) + u_i^{\perp}(x)$, where $u_i^{\perp}(x)\cdot v = 0$ for any $v \in T$. Hence $T \cap C(\bar{u}_M(x))^+ = T \cap C(u_M(x))^+$. Thus lemma 2 implies that each $u_i(x)$ can be replaced by $\bar{u}_i(x)$ without altering $K_M(x)$. Consequently, by proposition 2 (ii), $K_M(x) = \overline{C(\bar{u}_M(x))} = C(\bar{u}_M(x))$.

<u>Proof of lemma 4</u>: Since $x \in \mathrm{ri}X$, nothing is changed if we assume that each $u_i(x) = \bar{u}_i(x)$ (see the proof of proposition 3). Hence by lemma 3, $x \in \mathcal{P}(M)$ iff $0 \in \overline{C(\bar{u}_M(x))} = C(\bar{u}_M(x))$. But $0 \in C(\bar{u}_M(x))$ iff $\theta_M(x) = 0$. Hence $x \in \mathcal{P}(M)$ iff $\theta_M(x) = 0$. Now assume that $x \notin \mathcal{P}(M)$. Because $\theta_M(x)$ is the vector with minimum norm in the convex hull of $\bar{u}_M(x)$, it follows (lemma A4) that $\theta_M(x)\cdot(\bar{u}_i(x) - \theta_M(x)) \ge 0$ for all $i \in M$. Hence $\theta_M(x)\cdot\bar{u}_i(x) \ge \theta_M(x)\cdot\theta_M(x) > 0$ for all $i \in M$. Therefore $\theta_M(x) \in C(u_M(x))^+$. This completes the proof, since $\theta_M(x) \in C(\bar{u}_M(x)) = K_M(x)$ by proposition 2(i).

<u>Proof of Proposition 4</u>: Lemma 1 directly implies (i). To show (ii), suppose $x \in X \setminus \mathcal{P}(M)$ and $\bar{u} \in K_M(x)$. Since (A) holds, proposition 2 (ii) implies that $\bar{u}$ is the projection onto $T(x)$ of some $u \in C(u_M(x))$. Assume

that $v \in T(x)$ satisfies $v\hat{P}_M(x)\bar{u}$ and $v \in F(x,\bar{u},M)$. Lemma 1 implies then that $\bar{u}P_i(x)v$ for no $i \in M$. Hence $(v-\bar{u}) \cdot u_i(x) \geq 0$ for all $i \in M$. Thus, since $u \in C(u_M(x))$, $(v-\bar{u}) \cdot u \geq 0$. Therefore, by lemma A4, $\bar{u} \cdot (v-\bar{u}) \geq (v-\bar{u}) \cdot u \geq 0$. But since $v \in F(x,\bar{u},M)$ implies $\|v\| \leq \|\bar{u}\|$, the Cauchy-Schwarz inequality implies that $\bar{u} \cdot (v-\bar{u}) \leq 0$. Hence $\bar{u} \cdot v = \bar{u} \cdot \bar{u}$, which implies that $v = \bar{u}$ because $\|\bar{u}\| \geq \|v\|$. This contradiction implies $K_M(x) \subseteq \hat{K}_M(x)$. (ii) now follows from (i). To show (iii), let $\tilde{v} \in T(x)$ and suppose $C(u_M(x))^* \cap T = \{0\}$. Then, since $C(u_M(x))^+ \subseteq C(u_M(x))^*$, proposition 2(iii) implies $\tilde{v} \in K_M(x)$. If $v\hat{P}_M(x)\tilde{v}$, then, by an argument based on lemma 1, $t = v - \tilde{v} \in C(u_M(x))^*$. Hence $t \notin T$, which implies $v = \tilde{v} + t \notin T(x)$. Therefore $\tilde{v} \in \hat{K}_M(x)$. Consequently $T(x) = K_M(x) = \hat{K}_M(x)$.

<u>Proof of theorem 3</u>: Schofield [24] proves the existence of V if $x \in \mathcal{P}$ and his "Null Dual" condition holds, i.e., if

$$\bigcap_{M \in \mathcal{D}(x)} C(\bar{u}_M(x)) = \emptyset .$$

Thus theorem 2 implies the existence of V if $K(x) = \emptyset$. Schofield [23] shows that if V exists, then x is a limit point of the set of points satisfying the Null Dual condition, which means by Theorem 2 that x is a limit point of $\{ y \in X : K(y) = \emptyset \}$.

<u>Proof of lemma 6</u>: Let $\tilde{x} \in \text{ri}X$ and suppose $\{x_k\} \subseteq \text{ri}X$ is a sequence converging to $\tilde{x}$. Suppose too that $\{v_k\}$ is a sequence converging to $\tilde{v}$ such that each $v_k \in K^S_M(x_k)$. Because the range of K^S_M is contained in a compact set, K^S_M is uhc if we can show $\tilde{v} \in K^S_M(\tilde{x})$. Assume that $\tilde{v} \notin K^S_M(\tilde{x})$. Then, since $s_1 \leq \|\tilde{v}\| \leq s_2$, $\tilde{v} \notin K_M(\tilde{x})$. Consequently, lemma 2 implies the existence of $t \in C(u_M(\tilde{x}))^+$ such that $\tilde{v} + t \in F(\tilde{x},\tilde{v},M)$. Therefore $\tilde{v} + t \in T(\tilde{x}) = T$, which implies that $t \in T$. Also,

$\|\tilde{v} + t\| \leq \|\tilde{v}\|$ and $t \neq 0$ implies that $v_k \cdot t < 0$ for large k. But $t \in T$ and $v_k \cdot t < 0$ imply that $v_k + \lambda_k t \in T$ and $\|v_k + \lambda_k t\| \leq \|v_k\|$ for some $\lambda_k > 0$. This implies, by lemma 2 and (S3), that $t \notin C(u_M(x_k))^+$ for all large k. However, because the utility gradients are assumed to be continuous and because $t \in C(u_M(\tilde{x}))^+$, $t \in C(u_M(x))^+$ for large enough k. This contradiction proves the lemma.

Proof of lemma 7: Because $X = T$, $K_M(x) = C(\overline{u}_M(x))$. By (E), $\overline{u}_i(x)$ is just a positive multiple of $\overline{p}_i - x$, where $\overline{p}_i$ is the projection of p_i onto T. Consequently,

$$K_M(x) = C(\{\overline{p}_i - x : i \in M\}) = C(\ H\{\overline{p}_i : i \in M\} - \{x\}\ ).$$

But it is easy to show that $\mathcal{P}(M) = H\{\overline{p}_i : i \in M\}$. Therefore $K_M(x) = C(\mathcal{P}(M) - \{x\}\ )$, which proves that $v \in K_M(x)$ iff $x + \lambda v \in \mathcal{P}(M)$ for some $\lambda \geq 0$. It follows that $z - x \in K(x)$ if $z \in \mathcal{P}$, since $\mathcal{P} = \bigcap_{M \in \mathcal{D}} \mathcal{P}(M)$.

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