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The Existence of a Strategy Proof Voting Procedure*

by

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Abstract

The study considers committees whose task is to select one alternative from a set of three or more alternatives. Committee members cast ballots which are counted by a voting procedure. The voting procedure is strategy proof if it always induces every committee member to cast a ballot that reveals his preferences. The theorem proved states that no strategy proof voting procedure exists that satisfies the Pareto principle and is not dictatorial.

1. INTRODUCTION

Almost every participant in the formal deliberations of a group or committee of any size realizes that situations may occur where it is advantageous to misrepresent one's preferences when casting one's vote. The most common instances of such situations are general elections when three or more candidates are running for an elective office. Suppose, to be specific, that a Republican, a Democrat, and a minor party candidate are running against each other and a particular voter, Turner, prefers the minor party candidate first, the Republican second, and the Democrat last. Further suppose that Turner subjectively estimates that the minor party candidate has no chance of winning the election but that his second choice, the Republican, has a good chance of defeating his last choice, the Democrat. Turner may then decide to follow the "sophisticated strategy" of voting for the Republican instead of following his "sincere strategy" of voting for the minor party candidate.[1] If he does choose to employ the sophisticated strategy, then he is misrepresenting his preferences; his ballot reveals that he most prefers the Republican to win when in fact he most prefers the minor party candidate to win.

Such decisions to employ sophisticated strategies may inject extraneous influences into the election's outcome. Suppose that Turner is wrong in his estimate that the minor party candidate has no chance of winning: in fact a plurality of the electorate prefers

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[1]Robin Farquharson [4] introduced the terms sophisticated strategy and sincere strategy. Whether Turner, the voter in the example, will in fact decide to employ the sophisticated strategy depends on how likely he thinks it is that his vote will make a difference and how intense his preferences are among the three candidates.

the minor party candidate over the other two candidates. Moreover suppose that a substantial proportion of the minor party candidate's potential plurality make the same error Turner made and also decide to follow the sophisticated strategy of voting for the Republican. The result may be that the minor party candidate loses even though a plurality prefers him. This possibility - which is in the form of a self-fulfilling prophecy - violates the purpose for holding the election because the social decision of who is elected depends not only on the voters' preferences,but also depends on their unreliable subjective estimates of what other voters' preferences are.

Concern on the part of voting theorists with the problem that employment of sophisticated strategies can create has a long history. Duncan Black [2] relates that Jean-Charles de Borda, the eighteenth century originator of the Borda count,[1] when confronted with the possibility that committee members might gain advantages by employing sophisticated strategies retorted, "My scheme is only intended for honest men." More recently Richard Musgrave [6] emphasized that committee members' use of sophisticated strategies can frustrate a democratic government's search for an optimal level of taxation and expenditure. William Vickery [10] evaluated several different voting rules using their vulnerability to sophisticated strategies as one of his criterion. Robin Farquharson [4] developed a theory of voting behavior in which every commiteee member has perfect information concerning every other members' preferences. His theory

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[1]The Bordo count is a well known voting rule. Each member of the committee casts a ballot which ranks the m alternatives in order from one to m. Each alternative is then awarded one point for each first place vote it receives, two points for each second place vote, etc. The points foreach alternative are summed and that alternative with the least total of points is declared the committee's choice.

describes how each committee member in the process of selecting an optimal sincere sophisticated strategy for himself can take into account the effect his own choice of strategy will have on the other committee members' choices of strategies.

Kenneth Arrow [1] recognized that employment of sophisticated strategies creates problems and introduced, but did not pursue, the idea of constructing a strategy proof voting rule:

> Once a machinery for making social choices from individual tastes is established, individuals will find it profitable, from a rational point of view, to misrepresent their tastes by their actions...because such misrepresentation is somehow directly profitable.... Even in the case where it is possible to construct a procedure showing how to aggregate individual tastes into a consistent social preference pattern, there still remains the problem of devising rules of the game so that individuals will actually express their true tastes when they are acting rationally.

Subsequently, Alexander Dummett and Robin Farquharson [3] speculated that this problem is insoluable: "... it seems unlikely that there is any voting procedure in which it can never be advantageous for any voter to vote 'strategically,' i.e. non-sincerely." The purpose of this paper is to prove for the first time that Dummett and Farquharson were correct in their speculation that when a committee is considering three or more alternatives no useful strategy proof voting rule exists.

2. FORMULATION

Let a committee be a set I_n of n, $n \geq 1$, committee members whose task is to select a single alternative from an alternative set S_m of m elements, $m \geq 3$. The preferences P_i of committee member $i \in I_n$ is a complete, transitive, and strict preference relation on S_m. Since P_i is strict, committee members are not permitted to be indifferent between two alternative $x, y \in S_m$. Thus if $x, y \in S_m$ and $i \in I_n$, then $x \; P_i \; y$ means that committee member i prefers that the committee choose alternative x instead of alternative y. Let Π_m represent the collection of all possible preferences and let Π_m^n represent the n-fold cartesian product of Π_m.

The committee makes its selection of a single alternative by voting. Each committee member $i \in I_n$ casts a ballot B_i which is a complete, transitive, strict preference relation, i.e. $B_i \in \Pi_m$. The ballot B_i is a sincere strategy if and only if committee member i has preferences $P_i \equiv B_i$. The ballot B_i is sophisticated strategy if and only if $P_i \neq B_i$. Every committee member may choose to employ either his sincere strategy or any one of his sophisticated strategies because any requirement limiting him to sincere strategies would be unenforceable since ballots are observable while preferences are not.

The ballots are counted by a voting procedure v^{nm}. Formally a voting procedure is a singlevalued mapping whose arguement is the ballot set $B = (B_1, \ldots, B_n) \in \Pi_m^n$ and whose image is the committee's choice, a single alternative $x \in S_m$. Every voting procedure v^{nm} has a domain of Π_m^n and a range of either S_m or some non-empty subset of S_m. Given these definitions, let the three-tuple

$\langle I_n, S_m, v^{nm} \rangle$ be called the committee's description.

A voting rule v is a specified collection of voting procedures v^{nm} where n = 1,2,3,... and m = 3,4,5,.... Thus, given a committee I_n considering an alternative set S_m, each voting rule v uniquely defines a voting procedure $v^{nm} \in v$ which the committee can use to make its choice among the alternatives. In other words, a voting rule is a general rule applicable to any committee while a voting procudure is a specific rule applicable only to committees of a specific size considering a specific number of alternatives.

This formulation of the committee decision problem incorporates two assumptions which particularly merit further comment. First, the committee's task is specified to be selection of a single alternative from a given alternative set. The assumption that the committee is making only one choice excludes from consideration such committee behaviors as logrolling which may occur whenever a committee is making a sequence of choices. Second, the assumption that the committee through the mechanism of its voting rule must select a single alternative contrasts with Kenneth Arrow's [1] and Amartya Sen's [8] [9] specification of set valued choice functions. They made that specification because their focus was social welfare where partitioning the alternative set into classes of equal welfare is a useful result. Nevertheless specification of set valued choice functions (voting rules) is inappropriate here because committees often must choose among mutually exclusive courses of action. For example, a committee can only adopt one budget for a particular activity and fiscal period.

With the basic structure of the committee defined, it is possible to define the concept of a strategy proof voting procedure.

Consider a committee with description $\langle I_n, S_m, v^{nm} \rangle$. A committee member $i \in I_n$ with preferences $P_i \in \Pi_m$ has an incentive to consider employment of a sophisticated strategy if and only if there exists a set of n-1 ballots $B^i = (B_1, \ldots, B_{i-1}, B_{i+1}, \ldots, B_n) \in \Pi_m^{n-1}$ and a sophisticated strategy $B_i \in \Pi_m$ such that

$$(1) \qquad v^{nm}(B_i, B^i) \; P_i \quad v^{nm}(P_i, B^i).$$

If n = 1, then B^i is the null set. In words, committee member i has an incentive if and only if the other committee members may cast their ballots B^i in such a manner that he can secure for himself a more favorable outcome by playing the sophisticated strategy B_i instead of playing his sincere strategy P_i. The voting procedure v^{nm} is strategy proof if and only if there exists no $i \in I_n$ and no $P_i \in \Pi_m$ such that committee member i has an incentive to consider employment of a sophisticated strategy. Similarly, a voting rule v is strategy proof if and only if every voting procedure $v^{nm} \in v$ is itself strategy proof.

If a voting procedure v^{nm} is strategy proof, then no situation can arise where a committee member $i \in I_n$ can improve the vote's outcome relative to his preferences P_i by employing a sophisticated strategy. Consequently, if a voting procedure v^{nm} is strategy proof, then every set of sincere strategies $P = (P_1, \ldots, P_n) \in \Pi_m^n$ is a Nash equilibrium. If the voting procedure is not strategy proof, then there must exist a set of sincere strategies $P = (P_1, \ldots, P_n) \in \Pi_m^n$ which is not a Nash equilibrium.

3. STATEMENT OF THE THEOREM

For committees which are considering three or more alternatives, we find four classes of strategy proof voting procedures: dictatorial, semidictatorial, imposed, and twin alternative voting procedures. Consider dictatorial voting procedures first. Let the top ranked alternative on the ballot B_i of any committee member $i \in I_m$ be represented by $f(B_i)$. Formally, $f(B_i) = x \in S_m$ is the top ranked alternative on the ballot B_i if and only if for every $y \in S_m$, $y \neq x$, $x\ B_i\ y$. Given this, a voting procedure v^{nm} is dictatorial if and only if there exists a committee member $i \in I_n$ such that $v^{nm}(B) = f(B_i)$ for all ballot sets $B = (B_1,\ldots,B_i,\ldots,B_n) \in \Pi_m^n$, i.e. v^{nm} is dictatorial if the top ranked alternative of committee member i -- the dictator -- is always picked as the committee's choice.

A dictatorial voting procedure is strategy proof because the dictator clearly has no reason to misrepresent his preferences since the committee's choice is always that alternative which he ranks first on his ballot. The other committee members also have no reason to misrepresent their preferences because their ballots have no influence whatsoever on the vote's outcome. This last statement, in agreement with the definition of strategy proofness, assumes that the dictator can not punish those who disagree with him. Otherwise committee members might have reason to curry the dictator's favor through the sophisticated strategy of "agreeing" with him.

The three remaining classes of strategy proof voting procedures all fall into the broader classification of weak alternative excluding voting procedures.[1] A voting procedure v^{nm} is <u>weak alternative excluding</u> if and only if there exists at least one alternative $x \in S_m$ such that $v^{nm}(B) \neq x$ for every ballot set $B = (B_1,\ldots,B_m) \in \Pi^n_m$, i.e. v^{nm} is weak alternative excluding if its range is some proper subset of S_m. Weak alternative excluding voting procedures are presesented by the notation g^{nm}_T where $T \subset \subset S_m$ is its range.

A semidictatorial voting procedure is formally alternative excluding but substantively dictatorial. One committee member is the semidictator. He dominates the committee's decisions except that his range of choice is arbitrarily limited to some set T of included alternatives. The committee's choice is always that element of T which he ranks highest on his ballot, but he is unable to choose as the committee's choice any alternative $x \in (S_m - T)$. Semidictatorial voting procedures are strategy proof for the same resasons that dictatorial voting procedures are strategy proof.

An imposed voting procedure means that no committee member's ballot has any influence on the vote's outcome. No matter how the committee members vote, one alternative $x \in S_m$ is always selected as the "committee's choice," i.e. the range of an imposed voting includes only the single alternative $x \in S_m$. Imposed voting procedures are strategy proof since committee members' choices of strategies are irrelevant to the outcome.[2]

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[1]For proof that imposed, twin alternative, and semidictatorial voting procedures exhaustively categorize all strategy proof weak alternative excluding voting procedures, see Satterthwaite [7].

[2]One may argue that committee members have no incentive to play any strategy at all, whether sophisticated or sincere. Yet an imposed voting procedure is strategy proof according to the definitions established above.

Twin alternative voting procedures have a range T that contains exactly two elements. The committee chooses among these two alternatives by majority rule.[1] For example, consider a committee which is considering four alternatives, i.e. $S_4 = (s_1, s_2, s_3, s_4)$. If the committee uses a twin alternative voting procedure, then it selects either alternatve s_2 or alternative s_4 depending on which of these two alternatives is ranked above the other on a majority of the ballots. How the committee members rank the excluded alternatives s_1 and s_3 on their ballots has no influence on the outcome.

Twin alternative voting procedures are strategy proof because every voting procedure which selects between two alternatives by majority rule is strategy proof. With two alternatives the committee member has only two choices: vote for or against his preferred alternative. Obviously, unless the voting procedure perversely counts a vote for one alternative as a vote for the other alternative, he has every reason to vote for his preferred alternative no matter how he subjectively estimates the other committee members will vote. Thus twin alternative voting procedures are strategy proof because they, in effect, reduce the committee's decision to a majority rule choice between the two elements of T.

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[1] Within the context of majority rule, the ballots of individual committee members may be weighted unequally. Also more than a simple majority may be needed to choose one or the other of the included alternatives.

All four types of strategy proof voting procedures identified above can be classified as either dictatorial or weak alternative excluding. The question which the theorem answers is: does there exist for committees considering three or more alternatives any strategy proof voting procedures which are not either dictatorial or weak alternative excluding?[1]

<u>Theorem</u>. Consider a committee with description $\langle I_n, S_m, v^{nm} \rangle$ where $n \geq 1$ and $m \geq 3$. A necessary condition for v^{nm} to be strategy proof is that it be either a dictatorial or weak alternative excluding voting procedure.

This is formally a possibility theorem, but its substance is that of an impossibility theorem because no committee with democratic ideals will use a voting procedure that is dictatorial or weak alternative excluding. Dictatorial voting procedures vest all power in one committee member, a distribution of power that is clearly unacceptable. Weak alternative excluding voting procedures are equally unacceptable to democratic committees because the range of the voting procedure is arbitrarily restricted to a subset T of the alternative set S_m. Even if the committee members unanimously rank an excluded alternative $x \in (S_m - T)$ at the top of their ballots then that alternative is not

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[1]A third class of strategy proof committee decision rules exist, but they do not satisfy our definition of a voting procedure because they involve a lottery. Let a lottery be held among the committee members' ballots with each ballot having an equal opportunity of winning. The top ranked alternative on the winning ballot is then declared the committee's choice. This rule is strategy proof, but its probabilistic nature would undoubtedly offend most committees. The example was pointed out to me by Andrew Schotter. For a full discussion of lotteries as social choice mechanisms see Fishburn [5].

selected as the committee's choice. Instead the committee's choice is always an element of T. This outcome violates the Pareto principle and thus proves that weak alternative excluding voting procedures can not guarantee selection of a Pareto optimal committee's choice.

4. PROOF OF THE THEOREM

This paper presents a proof of the theorem only for the special case of committees with descriptions $\langle I_n, S_3, v^{n,3}\rangle$ where $n \geq 1$. Extension of the proof to the general case where $m \geq 3$ is found in Satterthwaite [7]. It is not included here because of its length.

Before beginning the proof's substance the concept of a strong alternative excluding voting procedure must be defined. Its definition is based on a unanimity requirement, condition L.

> Condition L. Consider a committee with description $\langle I_n, S_m, v^{nm}\rangle$ and let T be the range of v^{nm}. Let $\Phi_T(B_i)$ be that alternative $x \in T$ such that for all $y \in T$, $y \neq x$, $x\, B_i\, y$. The voting procedure v^{nm} satisfies condition L is and only if for every ballot set $B = (B_1, \ldots, B_n) \in \Pi^n_m$ such that $\Phi_T(B_1) = \Phi_T(B_2) = \ldots = \Phi_T(B_n)$, then $v^{nm}(B) = \Phi_T(B_1)$.

In other words, if a voting procedure satisfies condition L and if the committee members' ballots unanimously rank alternative $x \in T$ higher than every other alternative $y \in T$, then it will select alternative x as the committee's choice. Given this definition, a voting procedure v^{nm} is strong alternative excluding if and only if it is weak alternative excluding and also satisfies condition L. Strong alternative excluding voting procedures are represented by the notation h^{nm}_T where T is its range.

Condition L is very helpful in the proofs that follow because every weak alternative excluding voting procedure that is stragegy proof must also be strong alternative excluding. Lemma one establishes this assertion.

Lemma 1. Consider a committee with description $\langle I_n, S_m, g_T^{nm}\rangle$ where $n \geq 2$ and $m \geq 3$. A necessary condition for the weak alternative excluding voting procedure g_T^{nm} to be strategy proof is that it also be strong alternative excluding.

Proof: Suppose g_T^{nm} is strategy proof and does not satisfy condition L. Therefore for some $x \in T$ there exists a ballot set $C = (C_1,\ldots,C_n) \in \Pi_m^n$ such that $\Phi_T(C_1) = \Phi_T(C_2) = \ldots = \Phi_T(C_n)$ and $g_T^{nm}(C) \neq \Phi_T(C_1)$. By definition $\Phi_T(C_1)$ is an element of the range of g_T^{nm}. Therefore a ballot set $D = (D_1,\ldots,D_n) \in \Pi_m^n$ exists such that $g_T^{nm}(D) = \Phi_T(C_1)$. Consider the sequence of ballot sets:

$$(C_1, C_2, \ldots, C_n),$$
$$(D_1, C_2, \ldots, C_n),$$
$$\cdots$$
$$(D_1, \ldots, D_{i-1}, C_i, C_{i+1}, \ldots, C_n),$$
$$(D_1, \ldots, D_{i-1}, D_i, C_{i+1}, \ldots, C_n), \tag{2}$$
$$\cdots$$
$$(D_1, \ldots, D_{n-1}, C_n),$$
$$(D_1, \ldots, D_{n-1}, D_n).$$

An $i \in I_n$ must exist such that

$$g_T^{nm}(D_1, \ldots, D_{i-1}, C_i, C_{i+L}, \ldots, C_n) = x \quad \text{and} \tag{3}$$

$$g_T^{nm}(D_1, \ldots, D_{i-1}, D_i, C_{i+1}, \ldots, C_n) = \Phi_T(C_1) \tag{4}$$

where $x \in T$ and $x \neq \Phi_T(C_1)$. Let committee member i have preference $P_i \equiv C_i$, an assumption which implies that he prefers alternative $\Phi_T(C_1)$ to every other element of T. Therefore his favored strategy is clearly the sophisticated strategy D_i rather than his sincere strategy C_i. Thus, contrary to assumption, g_T^{nm} is not strategy proof. Consequently g_T^{nm} must satisfy condition L in order to be strategy proof.||

This completes the proof's preliminaries. Mathematical induction on n, the number of committee members, is the means by which the proof is accomplished. Lemma two begins the inductive chain by proving that if a committee has a single member (n=1), then every strategy proof voting procedure is either dictatorial or strong alternative excluding. Lemma four completes the inductive chain by proving that if all strategy proof voting procedures for committees with n members are either dictatorial or strong alternative excluding, then all strategy proof voting procedures for committees with n+1 members are either dictatorial or strong alternative excluding. Lemma three is used in the proof of lemma four. Since the remainder of the proof will concern only committees considering three alternatives, the m subscript can be dropped from v^{nm}. Thus $v^n \equiv v^{n,3}$.

<u>Lemma 2</u>. Consider a committee with description $\langle I_1, S_3, v^1 \rangle$. A necessary condition for v^1 to be strategy proof is that it be either dictatorial or strong alternative excluding.

<u>Proof</u>. Let $\mathcal{V}^1$ be the collection of all possible voting procedures v^1 and let $\mathcal{V}^{1*} \subset \mathcal{V}^1$ be the collection all strategy proof $v^1 \in \mathcal{V}^1$. This proof's method consists of repeatedly partitioning $\mathcal{V}^1$ until all possible strategy proof voting procedures are identified. The partitioning process is based on the fact that every $v^1 \in \mathcal{V}^1$ can be written as

$$(5)\qquad v^1(B_1) = \begin{cases} s_1 & \text{if } B_1 = (x\ y\ z) \\ s_2 & \text{if } B_1 = (x\ z\ y) \\ s_3 & \text{if } B_1 = (y\ x\ z) \\ s_4 & \text{if } B_1 = (y\ z\ x) \\ s_5 & \text{if } B_1 = (z\ x\ y) \\ s_6 & \text{if } B_1 = (z\ y\ x) \end{cases}$$

where $s_i \in S_3 = \{x,y,z\}$, $i=1,\ldots,6.$[1] Partition $\mathcal{V}^1$ into

(6) $\mathcal{V}^1_X = \{v^1 | v^1 \in \mathcal{V}^1 \ \&\ v^1[(x\ y\ z)] = x\}$,

(7) $\mathcal{V}^1_Y = \{v^1 | v^1 \in \mathcal{V}^1 \ \&\ v^1[(x\ y\ z)] = y\}$, and

(8) $\mathcal{V}^1_Z = \{v^1 | v^1 \in \mathcal{V}^1 \ \&\ v^1[(x\ y\ z)] = z\}$.

Ignore temporarily $\mathcal{V}^1_Y$ and $\mathcal{V}^1_Z$. Partition $\mathcal{V}^1_X$ into

(9) $\mathcal{V}^1_{XX} = \{v^1 | v^1 \in \mathcal{V}^1_X \ \&\ v^1[(x\ z\ y)] = x\}$,

(10) $\mathcal{V}^1_{XY} = \{v^1 | v^1 \in \mathcal{V}^1_X \ \&\ v^1[(x\ z\ y)] = y\}$, and

(11) $\mathcal{V}^1_{XZ} = \{v^1 | v^1 \in \mathcal{V}^1_X \ \&\ v^1[(x\ z\ y)] = z\}$.

Consider $\mathcal{V}^1_{XY}$ and ask if $\mathcal{V}^1_{XY} \cap \mathcal{V}^{1*} = \emptyset$, where $\emptyset$ is the null set. The answer can be shown to be "yes" by supposing that a $v^1 \in \mathcal{V}^1_{XY}$ exists which is strategy proof. Let the committee member have preferences and sincere strategy $P_1 = (x\ z\ y)$. Equation (10) implies that $v^1(P_1) = y$. In contrast, if the committee member employs the sophisticated strategy $B_1 = (x\ y\ z)$, equation (6) implies that $v^1(B_1) = x$. Comparison of the two outcomes indicates that the committee member will prefer to employ his sophisticated strategy B_1 instead of his sincere strategy P_1. Therefore, contrary to assumption, no strategy proof $v^1 \in \mathcal{V}^1_{XY}$ exists. Thus $\mathcal{V}^1_{XY} \cap \mathcal{V}^{1*} = \emptyset$ as asserted. A parallel argument shows that

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[1]The notation $B_1 = (x\ y\ z)$ means that committee member one on his ballot ranks x highest, y second, and z lowest. More formally, $B_1 = (x\ y\ z)$ implies $x\ B_1\ y$, $x\ B_1\ z$, and $y\ B_1\ z$.

$\mathcal{V}^1_{XZ} \cap \mathcal{V}^{1*} = \emptyset$.

It is impossible to show that $\mathcal{V}^1_{XX}$ and $\mathcal{V}^{1*}$ are disjoint. Therefore $\mathcal{V}^1_{XX}$ is itself partitioned into three subsets: $\mathcal{V}^1_{XXX}$, $\mathcal{V}^1_{XXY}$, and $\mathcal{V}^1_{XXZ}$. Of these three, only $\mathcal{V}^1_{XXZ}$ can be proved disjoint with $\mathcal{V}^{1*}$. The remaining two sets are each partitioned a fourth, fifth, and sixth time. This process results in four sets which can not be proved disjoint with $\mathcal{V}^{1*}$: $\mathcal{V}^1_{XXXXXX}$, $\mathcal{V}^1_{XXYYXY}$, $\mathcal{V}^1_{XXXZZZ}$, and $\mathcal{V}^1_{XXYYZZ}$. Each of these four sets consists of a single voting procedure which must be strategy proof because no counterexample can be constructed showing that it is not strategy proof. The four voting procedures are respectively:

(12) $v^1(B_1) = x = h^n_T(B_1)$,

(13) $v^1(B_1) = h^1_Z(B_1) = \begin{cases} x & \text{if} \quad x \ B_1 \ y \\ y & \text{if} \quad y \ B_1 \ x \end{cases}$,

(14) $v^1(B_1) = h^1_Y(B_1) = \begin{cases} x & \text{if} \quad x \ B_1 \ z \\ z & \text{if} \quad z \ B_1 \ x \end{cases}$, and

(15) $v^1(B_1) = f(B_1)$,

where $T = \{x\}$, $Z = \{x,y\}$, and $Y = \{x,z\}$.[1] Note that the first three are strong alternative excluding and the fourth is dictatorial. This proves that every element of $\mathcal{V}^1_X \cap \mathcal{V}^{1*}$ is either strong alternative excluding or dictatorial.

All that remains is to prove that same result for $(\mathcal{V}^1_Y \cup \mathcal{V}^1_Z) \cap \mathcal{V}^{1*}$. This is done by repeatedly partitioning $\mathcal{V}^1_Y$ and $\mathcal{V}^1_Z$ and at each level discarding those sets which are disjoint with $\mathcal{V}^{1*}$. The outcome is that $\mathcal{V}^1_Y$ and $\mathcal{V}^1_Z$ together contain only three strategy

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[1]Specifics of the partitioning process may be found in Satterthwaite [7].

proof voting procedures:

(16) $v^1(B_1) = y = h^1_T(B_1)$,

(17) $v^1(B_1) = h^1_X(B_1) = \begin{cases} y & \text{if } y\ B_1\ z \\ z & \text{if } z\ B_1\ y \end{cases}$, and

(18) $v^1(B_1) = z = h^1_U(B_1)$

where $T = \{y\}$, $X = \{y,z\}$, and $U = \{z\}$. Note that all three are strong alternative excluding. Thus identification of every strategy proof voting procedure $v^1 \in \mathcal{V}^1$ reveals that each one is either dictatorial or strong alternative excluding.||

The next lemma makes the problem of constructing strategy proof voting procedures for committees with n members analogous to the problem of constructing stragegy proof voting procedures for committees with one member.

Lemma 3. Consider a committee with description $\langle I_{n+1}, S_3, v^{n+1}\rangle$ where $n \geq 1$. Let $B^{n+1} = (B_1, B_2, \ldots, B_n)$. The voting procedure v^{n+1} may be written as

$$v^{n+1}(B^{n+1}, B_{n+1}) = \begin{cases} v^n_1(B^{n+1}) & \text{if } B_{n+1} = (x\ y\ z) \\ v^n_2(B^{n+1}) & \text{if } B_{n+1} = (x\ z\ y) \\ v^n_3(B^{n+1}) & \text{if } B_{n+1} = (y\ x\ z) \\ v^n_4(B^{n+1}) & \text{if } B_{n+1} = (y\ z\ x) \\ v^n_5(B^{n+1}) & \text{if } B_{n+1} = (z\ x\ y) \\ v^n_6(B^{n+1}) & \text{if } B_{n+1} = (z\ y\ x) \end{cases}$$

where $v^n_1, \ldots, v^n_6$ are voting procedures for committees with n members. A necessary and sufficient condition that the voting procedure v^{n+1} never gives any committee member i, where $i=1,\ldots,n$ (note that committee member n+1 is excluded), an incentive to look for a sophisticated strategy is that each of the six voting procedures $v_1^n, \ldots, v_5^n$, and v_6^n be strategy proof.

Proof: Considering the necessary part first, suppose that v_1^n is not strategy proof for some committee member i, $1 \le i \le n$, but that v^{n+1} is strategy proof for all committee members j, $1 \le j \le n$.[1] Since v_1^n is not strategy proof for committee member i there exists a sincere strategy and preferences $P_i \in \Pi_3$, a sophisticated strategy $B_i \in \Pi_3$, and a set of ballots $B^i = (B_1, \ldots, B_{i-1}, B_{i+1}, \ldots, B_n) \in \Pi_3^{n-1}$ such that

(20) $$v_1^n(B_i, B^i) \; P_i \; v_1^n(P_i, B^i),$$

i.e., there exists a situation where a sophisticated strategy is the best strategy for committee member i.

Now let committee member n+1 cast ballot $B_{n+1} = (x\ y\ z)$. This implies, based on the manner in which v^{n+1} is written in lemma three, that

(21) $$v^{n+1}(B_i, B^i, B_{n+1}) = v_1^n(B_i, B^i) \text{ and}$$

(22) $$v^{n+1}(P_i, B^i, B_{n+1}) = v_1^n(P_i, B^i).$$

Substitution into (20) gives

(23) $$v^{n+1}(B_i, B^i, B_{n+1}) \; P_i \; v^{n+1}(P_i, B^i, B_{n+1})$$

which is proof that v^{n+1}, contrary to assumption, is not strategy proof. Therefore a necessary condition that v^{n+1} be strategy proof is that $v_1^n, \ldots, v_6^n$ be strategy proof.

Now considering the sufficient part, suppose that $v_1^n, \ldots, v_6^n$ are strategy proof for every committee member $j=1, \ldots, n$, but that

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[1] The choice of v_1^n as being not strategy proof is arbitrary. The proof would follow identically if v_2^n had been chosen.

v^{n+1} is not strategy proof for some committee member i, $1 \le i \le n$. This means there exists a sincere strategy and preferences $P_i \in \Pi_3$, a sophisticated strategy $B_i \in \Pi_3$, and a set of ballots $(B^i, B_{n+1}) = (B_1,\ldots,B_{i-1},B_{i+1},\ldots,B_n,B_{n+1}) \in \Pi_3^n$ such that

(24) $\quad v^{n+1}(B_i,B^i,B_{n+1})\ P_i\ v^{n+1}(P_i,B^i,B_{n+1})$.

Arbitrarily assume that $B_{n+1} = (x\ y\ z\ )$. Equations (21) and (22) hold and therefore v_1^n may be substituted into relation (24):

(25) $\quad v_1^n(B_i,B^i)\ P_i\ v_1^n(P_i,B^i)$.

Thus, contrary to assumption, v_1^n can not be strategy proof. Therefore if $v_1^n,\ldots,v_6^n$ are strategy proof, then v^{n+1} must be strategy proof against all committee members j, $1 \le j \le n$.||

Lemma three does not imply that a voting procedure v^{n+1} is strategy proof it it is constructed of strategy proof voting procedures $v_1^n,\ldots,v_6^n$. Depending on how v^{n+1} is constructed, committee member n+1 may in specific situations find that his best strategy is a sophisticated strategy. This is the possibility with which limma four's proof deals.

Lemma 4. Consider a committee with description $\langle I_{n+1}, S_3, v^{n+1}\rangle$ where $n \ge 1$. Assume that every strategy proof voting procedure v^n for committees with n members is either dictatorial or strong alternative excluding. A necessary condition for v^{n+1} to be strategy proof is then that it be either dictatorial or strong alternative excluding.

Proof. Let $\mathcal{V}^{n+1}$ be the collection of all voting procedures v^{n+1} for committees with n+1 members. Let $W^n \subset \mathcal{V}^n$ be the collection of all voting procedures $v^n \in \mathcal{V}^n$ that are dictatorial or strong alternative excluding. Let $\mathcal{W}^{n+1} \subset \mathcal{V}^{n+1}$ be the collection of all voting procedures $v^{n+1} \in \mathcal{V}^{n+1}$ that are constructed from voting procedures $v^n \in W^n$, i.e., $v^{n+1} \in \mathcal{W}^{n+1}$ if and only if v^{n+1} can be written as

$$(26) \quad v^{n+1}(B^{n+1}, B_{n+1}) = \begin{cases} v_1^n(B^{n+1}) & \text{if } B_{n+1} = (x\ y\ z) \\ v_2^n(B^{n+1}) & \text{if } B_{n+1} = (x\ z\ y) \\ v_3^n(B^{n+1}) & \text{if } B_{n+1} = (y\ x\ z) \\ v_4^n(B^{n+1}) & \text{if } B_{n+1} = (y\ z\ x) \\ v_5^n(B^{n+1}) & \text{if } B_{n+1} = (z\ x\ y) \\ v_6^n(B^{n+1}) & \text{if } B_{n+1} = (z\ y\ x) \end{cases}$$

where $B^{n+1} = (B_1, \ldots, B_n)$ and $v_1^n, \ldots, v_6^n \in W^n$. Finally let $\mathcal{V}^{n*}$, $\mathcal{V}^{n+1*}$, W^{n*}, and $\mathcal{W}^{n+1*}$ be the collection of all strategy proof voting procedures contained respectively in the sets $\mathcal{V}^n$, $\mathcal{V}^{n+1}$, W^n, and $\mathcal{W}^{n+1}$.

Assume that $\mathcal{V}^{n*} \subset W^n$. Lemma three therefore implies $\mathcal{V}^{n+1*} \subset \mathcal{W}^{n+1}$. Consequently every strategy proof voting procedure $v^{n+1} \in \mathcal{V}^{n+1*}$ can be identified by repeatedly partitioning $\mathcal{W}^{n+1}$ and at each level discarding those subsets which are disjoint with $\mathcal{V}^{n+1*}$. This partitioning of $\mathcal{W}^{n+1}$ depends on the fact that W^n contains seven classes of distatorial and strong alternative excluding voting procedures:

$$(27) \quad v^n(B^{n+1}) = f(B_i) \text{ where } 1 \leq i \leq n,$$

$$(28) \quad v^n(B^{n+1}) = h_T^n(B^{n+1}) = x,$$

$$(29) \quad v^n(B^{n+1}) = h_U^n(B^{n+1}) = y,$$

$$(30) \quad v^n(B^{n+1}) = h_{\mathcal{U}}^n(B^{n+1}) = z,$$

$$(31) \quad v^n(B^{n+1}) = h_X^n(B^{n+1}),$$

$$(32) \quad v^n(B^{n+1}) = h_Y^n(B^{n+1}), \text{ and}$$

$$(33) \quad v^n(B^{n+1}) = h_Z^n(B^{n+1})$$

where $B^{n+1} = (B_1,\ldots,B_n)$, $T = \{x\}$, $U = \{y\}$, $\mathcal{U} = \{z\}$, $X = \{y,z\}$, $Y = \{x,z\}$, and $Z = \{x,y\}$. Type (27) clearly represents every possible dictatorial voting procedure for a committee with n members. Types (28) through (33) exhaustively represent every possible strong alternative excluding voting procedure because $\{T, U, \mathcal{U}, X, Y, Z\}$ is the collection of all possible proper non-empty subsets of $S_3 = \{x,y,z\}$.

The set $\mathcal{W}^{n+1}$ can be partitioned into seven subsets:

(34) $$\mathcal{W}_1^{n+1} = \{v^{n+1} | v^{n+1} \in \mathcal{W}^{n+1} \ \&\ v^{n+1}[B^{n+1},(x\ y\ z)] = f(B_i)$$
$$\text{where } 1 \le i \le n\},$$

(35) $$\mathcal{W}_2^{n+1} = \{v^{n+1} | v^{n+1} \in \mathcal{W}^{n+1} \ \&\ v^{n+1}[B^{n+1},(x\ y\ z)] = h_T^n(B^{n+1})\},\ldots,$$

. . .

(36) $$\mathcal{W}_7^{n+1} = \{v^{n+1} | v^{n+1} \in \mathcal{W}^{n+1} \ \&\ v^{n+1}[B^{n+1},(x\ y\ z)] = h_Z^n(B^{n+1})\}.$$

Each of these seven subsets can itself be partitioned into seven subsets: $\mathcal{W}_{11}^{n+1},\ldots,\mathcal{W}_{17}^{n+1}$, $\mathcal{W}_{21}^{n+1},\ldots,\mathcal{W}_{77}^{n+1}$.

Most of these subsets are easily proved to be disjoint with $\mathcal{V}^{n+1*}$. For example, consider

(37) $$\mathcal{W}_{27}^{n+1} = \{v^{n+1} | v^{n+1} \in \mathcal{W}_2^{n+1} \ \&\ v^{n+1}[B^{n\ 1},(x\ z\ y)] = h_Z^n(B^{n+1})\}.$$

Let committee member n+1 have preferences and sincere strategy $P_{n+1} = (x\ z\ y)$ and let the other n committee members cast identical ballots $B_1 = B_2 = \ldots = B_n = (z\ y\ x)$. The definition of $\mathcal{W}_{27}^{n+1}$, h_Z^n, and condition L jointly imply that $v^{n+1}[B^{n+1},(x\ z\ y)] = h_Z^n(B^{n+1}) = y$. This is the least preferable outcome for committee member n+1. He can improve the outcome relative to his own preferences by employing the sophisticated strategy $B_{n+1} = (x\ y\ z)$ because $v^{n+1}[B^{n+1},(x\ y\ z)] = h_T^n(B^{n+1}) = x$. Therefore every $v^{n+1} \in \mathcal{W}_{27}^{n+1}$ is not strategy proof, i.e., $\mathcal{W}_{27}^{n+1} \cap \mathcal{V}^{n+1*} = \emptyset$.

This process of elimination is continued through six levels with the result that seventeen subsets of $\mathcal{V}^{n+1}$ are identified that are not disjoint with $\mathcal{V}^{n+1*}$; each subset contains a single strategy proof voting procedure v^{n+1}. The collection of these seventeen subsets constitutes $\mathcal{V}^{n+1*}$. Inspection of each $v^{n+1} \in \mathcal{V}^{n+1*}$ reveals that each is either dictatorial or strong alternative excluding, i.e., $\mathcal{V}^{n+1*} \subset W^{n+1}$. || [1]

Lemma two and four together constitute a proof by mathematical induction of lemma five. Lemma five is just a special case of the theorem where the alternative set is limited to three elements.

<u>Lemma 5</u>. Consider a committee with description $\langle I_n, S_3, v^n \rangle$ where $n \geq 1$. A necessary condition for v^n to be strategy proof is that it be either dictatorial or strong alternative excluding.

Note that since every strong alternative excluding voting procedure is also weak alternative excluding, lemma five could be stated in terms of weak alternative excluding voting procedures.

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[1] Specifics of this partitioning process are contained in Satterthwaite [1].

5. CONDLUDING COMMENTS

Theorem one can be generalized in several directions. The most obvious extension is to allow committee members' preferences and ballots to be weak preference relations instead of being restricted to strict preference relations. This would allow committee members to express indifference between elements of the alternative set. A second generalization is to allow committee members to cast ballots that are partial preference relations. This would conform to the reality of many widely used voting procedures. General elections for public officials in the United States is a good example of this. The voter does not rank all the candidates; neglecting strategic considerations, he merely votes for the candidate he most prefers. Both of these generalizations may be found in Satterthwaite [7]. Neither changes the substance of the theorem: every strategy proof voting procedure is either dictatorial or weak alternative excluding.

This result, like so many others in the field of social choice and voting theory, is negative; viz., no reasonably "democratic" strategy proof voting procedure exists for committees considering three or more alternatives. Nevertheless one means of circumventing this result immediately suggests itself. Perhaps a "democratic" voting procedure exists that is almost strategy proof. Almost strategy proof means that situations where a committee member's best strategy is sophisticated are very rare. They might be so rare that it would not be worthwhile for any committee member to look for them. Then each committee member would employ his sincere strategy as a matter of course and neglect the remote possibility that he could do better employing a sophisticated strategy.