DISCUSSION PAPER NO. 419

A METHODOLOGY FOR GROUP DECISION MAKING

by

Jehoshua Eliashberg [1/]

and

Robert L. Winkler [2/]

February 1980

1/ Northwestern University

2/ Indiana University

ABSTRACT

In this paper, a methodology for modeling group decision-making problems is presented. As a result of a decision, a group will receive a payoff which must be divided among the members of the group. The ultimate payoff of interest is the vector of individual payoffs received by the members of the group, and preferences are quantified in terms of cardinal utility functions for such vectors of payoffs. Such utility functions can represent preferences concerning "equitable" and "inequitable" vectors of payoffs as well as attitudes toward risk. The individual utility functions are aggregated to form a group utility function for the vector of payoffs, and this latter function is, in turn, used to generate a group utility function for the overall group payoff and a sharing rule for dividing the group payoff into individual payoffs. The resulting group decisions are Pareto optimal. Properties of the sharing rule and the group utility function are investigated under different assumptions concerning the form of the group utility function.

1. INTRODUCTION

This paper considers the problem of a group facing a decision-making problem under uncertainty. The group must choose an action and will receive a payoff which depends on the action taken and on the outcomes of certain events or variables. After the group payoff is received, it must be divided among the members of the group. The ultimate interest of the group centers on the vector of payoffs to the individual members, not on the group payoff itself.

An example of the type of situation that is of interest here is a group of individuals who form a partnership and proceed to conduct business. Many decisions must be made by the partnership, and these decisions will affect the income of the partnership and hence of the partners themselves. Different partners may have different attitudes toward risk and different preferences concerning the division of the partnership's income. In this paper we present a group decision-making model that takes into account such attitudes and preferences.

The problem of group decision making has received a great deal of attention from researchers in many disciplines. The concern in this paper is with normative models of group decision making, as opposed to behavioral aspects involving interactions within the group. Much of the past research involving normative models of decision making has focused upon the area of social choice, and a dominant force in this area has been the early work of Arrow (1951). Arrow's "Impossibility Theorem" shows that given orderings of consequences by a number of individuals, no group ordering of these consequences exists that satisfies a set of seemingly reasonable behavioral assumptions. Good summaries of social choice theory are provided in the books of Sen (1970) and Fishburn (1973).

Working with cardinal utility functions rather than ordinal utility functions, Harsanyi (1955) presents conditions under which a group cardinal utility function can be expressed as a linear combination of the utility functions of the individuals comprising the group. Keeney (1976) provides alternate conditions leading to a group utility function. The interpersonal comparisons of utility necessary to arrive at a group utility function are of particular interest, and such comparisons are discussed in Harsanyi (1977).

Wilson (1968) uses Pareto optimality as a decision-making criterion and considers the existence and construction of a group utility function when each individual cares only about his own individual payoff. Raiffa (1968) discusses bargaining and arbitration as procedures for arriving at a single strategy from a Pareto-optimal set. These and other aspects of group decision making are covered by LaValle (1978), who illustrates these concepts quite lucidly with a series of examples. LaValle also proposes the use of an "allocation function" as an arbitration procedure for the selection of a Pareto-optimal solution.

The role of equity in group decision-making models has received particular attention, and models have been criticized on the groups of not allowing for equity. For example, see Kirkwood (1972), Keeney and Kirkwood (1975), and Keeney and Raiffa (1976). An important conflict between equity and Pareto optimality is discussed in Kirkwood (1979). However, this work has considered individual and group cardinal utility functions for the group payoff, or consequence, not for the payoffs to individual members.

The objective of this paper is to present a methodology for modeling group decision-making problems. Preferences are considered in terms of cardinal utilities for entire vectors of payoffs to individual members of the group. Thus, considerations such as equity can be taken into account. The individual utility functions are aggregated to form a group utility function for the vector of payoffs, and this latter function is, in turn, used to generate a group utility function for the group payoff and a sharing rule for dividing the group payoff into individual payoffs. Properties of the sharing rule and the group utility function are investigated under different assumptions concerning the form of the group utility function.

The general methodology of our group decision-making model is presented in Section 2. Cases in which the group utility function for the vector of individual payoffs is additive and multilinear are considered in Sections 3 and 4, respectively. Some general results are obtained for the additive and multilinear cases, and examples are given to illustrate these cases. A brief summary is presented in Section 5.

2. A GROUP DECISION-MAKING MODEL

Suppose that a group of n members faces a decision-making problem. The group must choose an action a from a set of alternatives A, and y represents the payoff to the group as a result of the action. The payoff y could simply be a monetary payoff, or it could be multidimensional in nature.

The group members are uncertain about the payoff that will be obtained, and for each $a \in A$, the uncertainty is represented in terms of a probability distribution for y given that a is chosen. In order to focus on utility-related aspects of the group decision-making problem, we assume for the purposes of this paper that the group members agree on the probability distributions of interest. In some situations, for instance, the probability mechanism generating the payoffs may be well known and easy to agree upon, as in the case of a group combining its resources to play a game of chance at a casino. When the group members have different information, they might pool all of their information and base their probabilities on the pooled information. Alternatively, the group members may have only limited knowledge upon which to base their probabilities and may therefore obtain probabilities from an outside expert. If each member's state of information is diffuse relative to the information provided by the expert, then the group might agree to use the expert's probabilities in the group decision-making problem.

Once a is chosen and y is obtained, y must be divided among the group members. For convenience, we assume that y is infinitely divisible, although constraints on the divisibility of y could easily be incorporated into the model. The payoff to member i is denoted by x_i, and $\underset{\sim}{x} = (x_1,\ldots,x_n)$ represents the vector of payoffs to the n group members. For a given payoff y, any vector satisfying $y = \sum_{i=1}^{n} x_i$ is attainable.

An important aspect of the model developed in this paper is the way in which the preferences of the individual group members are brought into the model. We assume that the group payoff y has no intrinsic "value" per se to the individuals comprising the group, but that its "value" is indirect in the sense that it can be divided into individual payoffs $x_1,\ldots,x_n$. Thus, the preferences of interest are preferences concerning $x_1,\ldots,x_n$, not preferences concerning y. Moreover, we are interested in each member's preferences concerning the entire vector $\underset{\sim}{x}$, not just the member's preferences for his or her own payoff. This enables each individual to express feelings about his or her own payoff in relation to the payoffs received by the other n-1 group members. Some members might strongly prefer an equal division of y, while others might feel that certain unequal divisions are "fairer" in some sense (or are preferred even if they are "unfair").

We assume that each group member assesses a cardinal utility function for $\underset{\sim}{x}$. Member i's utility function is denoted by $u_i(\underset{\sim}{x})$. These utility functions can be assessed by considering preferences among lotteries involving $\underset{\sim}{x}$. Since $\underset{\sim}{x}$ is multidimensional, the utility assessment procedures discussed in Keeney and Raiffa (1976) are relevant here. In particular, it is desirable to consider the applicability of various preferential assumptions (e.g., additive independence, mutual utility independence) that may simplify the form of the utility function. Examples involving such assumptions are presented in Sections 3 and 4.

Given the n individual utility functions $u_1(\underset{\sim}{x}),\ldots,u_n(\underset{\sim}{x})$, the next step in the methodology is to form a group utility function $u_G(\underset{\sim}{x})$. Expressing u_G as a function of the individual utility functions necessitates the consideration of lotteries involving $u_1,\ldots,u_n$. Harsanyi (1955) shows that if certain conditions are satisfied, u_G can be expressed as a linear combination (with positive coefficients) of the individual utility functions. Keeney and Kirkwood (1975) and Keeney and Raiffa (1976) discuss alternate sets of conditions that lead to a group utility function that is a linear combination of $u_1,\ldots,u_n$ as well as conditions that lead to more general forms for the group utility function. Although our model does not place any restriction on the aggregation rule that is used to determine u_G from $u_1,\ldots,u_n$, we find the linear aggregation rule very appealing for reasons discussed later in this section.

The formation of $u_G(\underset{\sim}{x})$ provides the group with a utility function, but it is a function of $\underset{\sim}{x}$, whereas the payoff to the group from the group decision-making problem is a single value y. The choice of an action a from the set A can be thought of as the group's external decision-making problem, and the division of y into individual payoffs $x_1,\ldots,x_n$ can be thought of as the group's internal decision-making problem. Obviously, the group's external problem may be affected by the internal problem, because the desirability of an external payoff y depends on how the payoff is to be shared. In the model developed here, each group member's preferences for various sharing rules have been incorporated into the model in terms of the individual utility functions for $\underset{\sim}{x}$. Furthermore, the agreement of the group members concerning the probabilities of events or variables of interest removes the possibility of side bets involving these events or variables. Also, if the group has to make a number of decisions over time, considering the internal and external problems simultaneously is likely to lead to a complex, intractable model. As a result, it seems reasonable to separate the internal and external problems, and we treat the internal problem as a constrained maximization problem:

$$U_G(y) = \max \; u_G(\underset{\sim}{x}) \quad \text{subject to} \; \sum_{i=1}^{n} x_i = y. \tag{1}$$

Geometrically, for a given y we consider the hyperplane $\sum_{i=1}^{n} x_i = y$ in $(\underset{\sim}{x}, u_G)$-space and find the point on that hyperplane for which u_G is maximized. This procedure is illustrated in Figure 1 for a situation with n=2.

The $\underset{\sim}{x}$ at which u_G is maximized for a given y is denoted by $\underset{\sim}{x}^*(y)$, so that $U_G(y) = u_G(\underset{\sim}{x}^*(y))$. If $\underset{\sim}{x}^*(y)$ is considered as a function of y, it traces out the "optimal" sharing rule for all values of y. For example, if $\underset{\sim}{x}^*(y)$ follows the line for which $x_1=x_2=\cdots=x_n$, then the group will always divide the payoff equally; if it follows the x_i-axis, then member i will always receive the entire payoff; and so on. The sharing rule shown in Figure 2 for a case with n=2 provides perfect equity for y near zero but gives member 1 a larger proportion of y than member 2 for large absolute values of y. Geometrically, $\underset{\sim}{x}^*(y)$ represents a "ridge" in the graph of $u_G(\underset{\sim}{x})$ as a function of $\underset{\sim}{x}$.

The model developed in this paper, then, begins with a set of individual utility functions $u_i(\underset{\sim}{x})$ for the entire vector of individual payoffs $\underset{\sim}{x}$. These individual utility functions are aggregated to form a group utility function $u_G(\underset{\sim}{x})$ for $\underset{\sim}{x}$, and (1) is used to transform $u_G(\underset{\sim}{x})$ into a group utility function $U_G(y)$ for the group payoff y. The group utility function $U_G(y)$ is to be used for group decision-making purposes, with the group choosing an action a so as to maximize $E_{y|a}\{U_G(y)\}$.

At different steps in the process described here, two different types of tradeoffs are considered. First, in the initial assessment of $u_1,\ldots,u_n$, the group members must consider tradeoffs among the dimensions of $\underset{\sim}{x}$. Second, in the aggregation of $u_1,\ldots,u_n$ to form u_G, tradeoffs among the dimensions of $\underset{\sim}{u} = (u_1,\ldots,u_n)$ must be considered. In the second step, a linear aggregation rule,

$$u_G(\underset{\sim}{x}) = \sum_{i=1}^{n} \lambda_i u_i(\underset{\sim}{x}), \qquad (2)$$

with $\lambda_i > 0$ for $i=1,\ldots,n$, is very appealing in the sense that it is the only rule which guarantees that the resulting decision will be Pareto optimal, but linear rules have been criticized on the grounds that they do not take into account considerations of equity. For a discussion of this "conflict" between Pareto optimality and equity, see Kirkwood (1979). When the individual and group utility functions are functions of the group payoff y, as has generally been the case in the literature, the aggregation step provides the only opportunity for tradeoffs among different members of the group, and hence matters such as equity, to be considered. As a result, equity and Pareto optimality must both be considered in $\underset{\sim}{u}$-space.

Because the model developed here allows for the consideration of two types of tradeoffs, it is possible to avoid a conflict between Pareto optimality and equity. Pareto optimality involves $\underset{\sim}{u}$-space, and the use of the linear aggregation rule in (2) guarantees Pareto-optimal decisions. Therefore, although our general model does not restrict the aggregation rule, we invoke the assumption of a linear aggregation rule for the rest of this paper because of the desirability of Pareto optimality. As for equity, it seems to us that equity is best considered in terms of $\underset{\sim}{x}$-space rather than $\underset{\sim}{u}$-space , and the group members' feelings about equity should be reflected in $u_G(\underset{\sim}{x})$. An important advantage of assessing utilities in terms of $\underset{\sim}{x}$ is that it enables the group to take into account equity considerations without sacrificing Pareto optimality.

3. ADDITIVE GROUP UTILITY FUNCTIONS

As noted in Section 2, it is desirable to consider the applicability of various preferential assumptions that may simplify the form of the utility function (of an individual or of the group) for $\underset{\sim}{x}$. The most commonly encountered type of multiattribute utility function in decision analysis applications is an additive form, and additive group utility functions are studied in this section. In Section 4, multilinear group utility functions, which have the advantage of being able to reflect equity considerations, are investigated.

The utility function of member i for $\underset{\sim}{x}$ is said to be additive if it can be expressed in the form

$$u_i(\underset{\sim}{x}) = \sum_{j=1}^{n} k_{ij} u_{ij}(x_j), \qquad (3)$$

where u_{ij} is a conditional utility function of member i for x_j (which is assumed to be a monotonic, increasing function of x_j) and k_{ij} is a positive scaling constant for i, j=1,...,n. A utility function for $\underset{\sim}{x}$ is additive if and only if the elements of x are additive independent (Keeney and Raiffa, 1976), which means that preferences over lotteries involving $\underset{\sim}{x}$ depend only on their marginal probability distributions for the elements of $\underset{\sim}{x}$ and not on their joint probability distributions. For example, if $x_1,\ldots,x_n$ are additive independent, indifference is implied between (1) a lottery that yields $\underset{\sim}{x} = (z,\ldots,z)$ with probability 1/n and $\underset{\sim}{x} = (0,\ldots,0)$ with probability (n-1)/n, and (2) a lottery that yields, for i=1,...,n, x_i=z and x_j=0 for j≠i with probability 1/n. In the first lottery, either all group members receive a payoff z or none of them receive anything, whereas in the second lottery, one member receives z and the rest receive nothing. Thus, in an ex post sense, the first lottery is guaranteed to be equitable while the second lottery is sure to be inequitable. Additive independence leads to indifference between the two lotteries and therefore a lack of concern about the ex post equity of the payoffs. Of course, even when the outcomes of a lottery are inequitable ex post, the lottery may be equitable in an ex ante sense. For both of the lotteries in the example, each individual has a $1/n$ chance of receiving z and a $(n-1)/n$ chance of receiving nothing. Thus, when equity is discussed, it is important to distinguish between ex post equity and ex ante equity.

We would like to focus on the implications of group utility functions u_G that can be expressed in additive form:

$$u_G(\underset{\sim}{x}) = \sum_{j=1}^{n} k_{Gj} u_{Gj}(x_j), \tag{4}$$

with $k_{Gj} > 0$ for $j=1,\ldots,n$. Therefore, a few words about how such a u_G might arise from the individual utility functions $u_1,\ldots,u_n$ are in order. The most reasonable scenario leading to an additive u_G is one in which all of the individual utility functions are themselves additive. (Other scenarios could conceivably lead to an additive u_G but are highly unlikely because they would require the fortuitous circumstance that the non-additive terms from different individuals' utility functions cancel each other out!) One of Harsanyi's (1955) conditions leading to a linear aggregation rule is that if every individual is indifferent between certain lotteries, the group as a whole should be indifferent between the lotteries. Thus, indifference among lotteries with different joint probabilities but the same marginal probabilities carries over from the individuals to the group.

Formally, if u_i is given by (3) for $i=1,\ldots,n$ and if the linear aggregation rule in (2) is used to generate u_G, then

$$u_G(\underset{\sim}{x}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i k_{ij} u_{ij}(x_j) = \sum_{j=1}^{n} k_{Gj} u_{Gj}(x_j), \qquad (5)$$

where $k_{Gj} = \sum_{i=1}^{n} \lambda_i k_{ij}$ and $u_{Gj}(x_j) = \sum_{i=1}^{n} \lambda_i k_{ij} u_{ij}(x_j) / \sum_{i=1}^{n} \lambda_i k_{ij}$.

In order to be able to interpret (and thus to assess more easily) the utility functions of interest here, we impose, without loss of generality, some scaling requirements. For each j, we can choose x_j^o and x_j^{oo} such that x_j^{oo} is preferred to x_j^o. For instance, if the set of possible values of x_j is bounded, the best and worst possible values of x_j can be used. Then we scale the utility functions such that

$$u_{Gj}(x_j^o) = u_{ij}(x_j^o) = u_G(\underset{\sim}{x}^o) = u_i(\underset{\sim}{x}^o) = 0 \qquad (6)$$

and

$$u_{Gj}(x_j^{oo}) = u_{ij}(x_j^{oo}) = u_G(\underset{\sim}{x}^{oo}) = u_i(\underset{\sim}{x}^{oo}) = 1 \qquad (7)$$

for $i,j=1,\ldots,n$, where $\underset{\sim}{x}^o = (x_1^o,\ldots,x_n^o)$ and $\underset{\sim}{x}^{oo} = (x_1^{oo},\ldots,x_n^{oo})$. Now (6) and (7), together with (3) and (4), imply that $\sum_{j=1}^{n} k_{ij} = 1$ for $i=1,\ldots,n$ and $\sum_{j=1}^{n} k_{Gj} = 1$. The scaling constant k_{ij} equals $u_i(x_1^o,\ldots,x_{j-1}^o,x_j^{oo},x_{j+1}^o,\ldots,x_n^o)$, which can be assessed by determining the probability p that makes member i indifferent between $(x_1^o,\ldots,x_{j-1}^o,x_j^{oo},x_{j+1}^o,\ldots,x_n^o)$ and a lottery yielding $\underset{\sim}{x}^{oo}$ with p ability p and x^o with probability 1-p. Intuitively, k_{ij} might be considered to rep sent the power or importance of member j in the group, as judged by member i. The s interpretation in terms of group preferences can be given to k_{Gj}. Also, the scaling restrictions imply that the sum of the coefficients of the linear aggregation function must be one: $\sum_{i=1}^{n} \lambda_i = 1$.

Next, we will characterize the optimal sharing rule and the group utility function for y when u_G is additive. In order to look at the group's attitude toward risk, we assume that $u_{ij}(x_j)$ and $u_{Gj}(x_j)$ are twice differentiable for $j=1,\ldots,n$ and we define the Pratt-Arrow risk aversion functions (Pratt, 1964)

$$r_{ij}(x_j) = -u''_{ij}(x_j)/u'_{ij}(x_j), \tag{8}$$

$$r_{Gj}(x_j) = -u''_{Gj}(x_j)/u'_{Gj}(x_j), \tag{9}$$

and

$$r_G(y) = -U''_G(y)/U'_G(y), \tag{10}$$

where the primes denote differentiation. Here r_{ij} and r_{Gj} represent the risk aversion functions of member i and the group for the payoff to member j, and r_G represents the group's risk aversion function for the group payoff y. In the following two propositions, we assume positive risk aversion functions, which implies risk averse behavior.

<u>Proposition 3.1</u>. If u_G is additive with u_{Gj} twice differentiable and $r_{Gj} > 0$ for $j=1,\ldots,n$, then

$$\frac{\partial x_j^*(y)}{\partial y} = [r_{Gj}(x_j^*(y))]^{-1}\{\sum_{i=1}^{n}[r_{Gi}(x_i^*(y))]^{-1}\}^{-1} \tag{11}$$

and

$$r_G(y) = \{\sum_{i=1}^{n}[r_{Gi}(x_i^*(y))]^{-1}\}^{-1}. \tag{12}$$

A proof of the proposition is given in the Appendix. From (11), the rate of increase of member j's share of y increases as r_{Gj} decreases with r_{Gi} held constant for $j \neq i$. That is, as the group becomes less risk averse with respect to x_j, member j's "stake" in the group payoff y increases. Of course, as the examples presented later in this section will demonstrate, member j may have to pay for this increased "stake" in y by making side payments to other group members. The second result in Proposition 3.1 shows how the group's risk aversion for y can be related to the group's risk aversion for the individual payoffs $x_1,\ldots,x_n$. From (12), the group is less risk averse toward y than toward any individual x_i, and the following proposition provides even stronger statements about $r_G(y)$.

Proposition 3.2. If u_i is additive with u_{ij} twice differentiable and $r_{ij} > 0$ for $i,j=1,\ldots,n$, then

$$n^{-1} \min_j\{r_{Gj}(x_j^*(y))\} \leq r_G(y) \leq n^{-1} \max_j\{r_{Gj}(x_j^*(y))\} \tag{13}$$

and

$$n^{-1} \min_{i,j}\{r_{ij}(x_j^*(y))\} \leq r_G(y) \leq n^{-1} \max_{i,j}\{r_{ij}(x_j^*(y))\}. \tag{14}$$

A proof is given in the Appendix. Propositon 3.2 indicates that not only is the group less risk averse toward y than toward any individual x_i, but its risk aversion function for y is less than the largest $r_{Gj}(x_j^*(y))$ by a factor of $1/n$. This suggests that groups consisting of very large numbers of risk-averse members should be approximately risk neutral, which can be attributed to the group members sharing the risk. In the special case of equally risk-averse members, with $r_{Gj} \equiv r$ for all j, we have $r_G(y) = r(x_j^*(y))/n$, which clearly goes to zero as n increases.

To illustrate group decision making with additive group utility functions, two examples with $n=2$ will be considered. In the first example, the members' utility functions are identical with the possible exception of differences in scaling constants. Both conditional utility functions are exponential, implying constant risk aversion:

$$u_i(\underset{\sim}{x}) = k_{i1}(1 - e^{-c_1x_1}) + k_{i2}(1 - e^{-c_2x_2})$$ for $i=1,2$, where $c_1 > 0$, $c_2 > 0$, $x_1 > 0$, and $x_2 > 0$. The group utility function for $\underset{\sim}{x}$ is of the form

$$u_G(\underset{\sim}{x}) = k_{G1}(1 - e^{-c_1x_1}) + k_{G2}(1 - e^{-c_2x_2}),$$

with $k_{Gi} = \lambda_1 k_{1i} + \lambda_2 k_{2i}$ for $i = 1,2$. Maximizing $u_G(\underset{\sim}{x})$ under the constraint $x_1 + x_2 = y$ yields

$$x_1^*(y) = [c_2/(c_1 + c_2)]y + (c_1 + c_2)^{-1}\ln(k_{G1}c_1/k_{G2}c_2) = qy + s$$

and

$$x_2^*(y) = [c_1/(c_1 + c_2)]y - (c_1 + c_2)^{-1}\ln(k_{G1}c_1/k_{G2}c_2) = (1 - q)y - s.$$

Here $q = c_2/(c_1 + c_2)$ represents the proportion of y that member 1 receives, and $s = (c_1 + c_2)^{-1}\ln(k_{G1}c_1/k_{G2}c_2)$ represents a side payment from member 2 to member 1. Note that the proportional division of y depends only on the two risk aversion measures $r_{G1}(x_1^*(y)) = c_1$ and $r_{G2}(x_2^*(y)) = c_2$, not on the scaling constants. From (11), the rate of increase of x_1^* as y increases should be $c_1^{-1}\{(c_1 + c_2)/c_1c_2\}^{-1} = c_2/(c_1 + c_2)$, which is consistent with the sharing rule just derived. The more risk averse member receives a smaller share of y but will be compensated by receiving a positive side payment unless the scaling constant for the less risk averse member is sufficiently large to overcome the difference in risk aversion coefficients. The side payment depends on the scaling constants as well as on the risk aversion coefficients, and the scaling constants might be thought of as an indication of the relative power of the group members.

The group utility function for y in this example can be found by substituting $x_1^*(y)$ and $x_2^*(y)$ for x_1 and x_2 in $u_G(\underline{x})$, and the result is

$$U_G(y) = 1 - (k_{G1}e^{-c_1 s} + k_{G2}e^{c_2 s})e^{-c_1c_2y/(c_1 + c_2)}.$$

Thus, the group utility function for y is exponential with constant risk aversion coefficient $r_G(y) = c_1c_2/(c_1 + c_2)$, as we could have determined by using (12) to find $r_G(y)$. Also, from (13) or (14), we have $\min\{c_1,c_2\}/2 \leq c_1c_2/(c_1 + c_2) \leq \max\{c_1,c_2\}/2$, which reduces in this case to the result that the more risk averse member receives less than half of y. Note that when $c_1 = c_2$, $r_G(y)$ is exactly one-half the common risk aversion coefficient. When $c_1 \neq c_2$, however, $r_G(y)$ is less than the average of $c_1/2$ and $c_2/2$ because the less risk averse member is taking on a larger share of the risk than is the more risk averse member.

In our second example involving an additive group utility function, the members' utility functions differ in terms of the conditional utility functions as well as the scaling constants. Each member is risk averse with respect to his own payoff and risk neutral with respect to the other member's payoff:

$$u_1(\underline{x}) = k_{11}[a_1x_1 - (a_1 - 1)x_1^2] + k_{12}x_2$$

and

$$u_2(\underline{x}) = k_{21}x_1 + k_{22}[a_2x_2 - (a_2 - 1)x_2^2],$$

where $1 < a_i < 2$ and $0 \leq x_i \leq 1$ for $i=1,2$. The risk aversion functions corresponding to the conditional utility functions are $r_{11}(x_1) = [a_1(a_1 - 1)^{-1}/2 - x_1]^{-1}$, $r_{12}(x_2) = r_{21}(x_1) = 0$, and $r_{22}(x_2) = [a_2(a_2 - 1)^{-1}/2 - x_2]^{-1}$. The condition that $1 < a_i < 2$ guarantees that $u_{ii}(x_i)$ is increasing and risk averse on the unit interval.

The group utility function for x in this example is of the form $u_G(\underline{x}) = (\lambda_1k_{11}a_1 + \lambda_2k_{21})x_1 - \lambda_1k_{11}(a_1 - 1)x_1^2 + (\lambda_1k_{12} + \lambda_2k_{22}a_2)x_2 - \lambda_2k_{22}(a_2 - 1)x_2^2$. Maximizing $u_G(x)$ under the constraint $x_1 + x_2 = y$ yields

$$x_1^*(y) = \frac{\lambda_2k_{22}(a_2 - 1)y}{\lambda_1k_{11}(a_1 - 1) + \lambda_2k_{22}(a_2 - 1)} + \frac{\lambda_1k_{12} + \lambda_2k_{22}a_2 - \lambda_2k_{21} - \lambda_1k_{11}a_1}{2[\lambda_1k_{11}(1 - a_1) + \lambda_2k_{22}(1 - a_2)]}$$

$$= qy + s$$

and

$$x_2^*(y) = \frac{\lambda_1k_{11}(a_1 - 1)y}{\lambda_1k_{11}(a_1 - 1) + \lambda_2k_{22}(a_2 - 1)} - \frac{\lambda_1k_{12} + \lambda_2k_{22}a_2 - \lambda_2k_{21} - \lambda_1k_{11}a_1}{2[\lambda_1k_{11}(1 - a_1) + \lambda_2k_{22}(1 - a_2)]}$$

$$= (1 - q)y - s,$$

where q and s once again represent the proportion of y that member 1 receives and a side payment from member 2 to member 1. Here the division of y depends not just on a_1 and a_2, which are parameters of u_{11} and u_{22} (and hence of r_{11} and r_{22} as well), but also on the scaling constants k_{11} and k_{22} and on the coefficients λ_1 and λ_2 of the linear aggregation function. The proportion q increases as λ_2 increases relative to λ_1, as k_{22} increases relative to k_{11}, and as a_2 increases relative to a_1. When λ_2 and k_{22} increase relative to λ_1 and k_{11}, respectively, more weight is placed on u_{22} relative to u_{11}, and this greater weight on risk aversion with respect to x_2 leads to a smaller proportion of y for member 2. Similarly, as a_2 increases relative to a_1, u_{22} becomes more risk averse relative to u_{11}.

When $x_1^*(y)$ and $x_2^*(y)$ are substituted into $u_G(\underset{\sim}{x})$, the resulting group utility function for y is quadratic:

$$U_G(y) = ay^2 + by + c,$$

where

$$a = -k_{11}(a_1 - 1)q^2 - k_{22}(a_2 - 1)(1 - q)^2$$

and

$$b = k_{11}a_1 - 2k_{11}(a_1 - 1)qs + k_{12}(1 - q) + k_{21}q + k_{22}a_2(1 - q) + 2k_{22}(a_2 - 1)(1 - q)s$$

(c is irrelevant for decision-making purposes). The risk aversion function corresponding to $U_G(y)$ is $r_G(y) = [-(b/2a) - y]^{-1}$, as compared with $r_{Gi}(x_i) = (-d_i - x_i)^{-1}$ for i=1,2, where $d_1 = (\lambda_1 k_{11}a_1 + \lambda_2 k_{21})/\lambda_1 k_{11}(1 - a_1)$ and $d_2 = (\lambda_1 k_{12} + \lambda_2 k_{22}a_2)/\lambda_2 k_{22}(1 - a_2)$.

The scaling constants and aggregation coefficients play an important role in this example, but they make the results somewhat difficult to interpret. Suppose that equal weights are used everywhere, so that λ_1, λ_2, k_{11}, k_{12}, k_{21}, and k_{22} all equal one-half. Then $q = (a_2 - 1)/(a_1 + a_2 - 2)$ and $s = (a_1 - a_2)/2(a_1 + a_2 - 2)$, implying that the more risk averse member receives less than half of the group payoff y but receives a positive side payment from the other member. The group risk aversion function for y is $r_G(y) = [(a_1a_2 - 1)/(a_1 - 1)(a_2 - 1) - y]^{-1}$.

4. MULTILINEAR GROUP UTILITY FUNCTIONS

Additive group utility functions are convenient to work with but are not able to reflect preferences in favor of or opposed to ex post equity of the individual payoffs. One of the advantages of the approach presented in this paper is the ability of the model to take such preferences into account without sacrificing Pareto optimality. In this section, we study a class of group utility functions, multilinear group utility functions, that are able to reflect preferences concerning ex post equity. To simplify the discussion, we restrict our attention to groups with two members, although generalizations to groups with more than two members certainly are possible [e.g., see the discussion of multilinear utility functions in Chapter 6 of Keeney and Raiffa (1976)].

The group utility function for $\underset{\sim}{x}$ is said to be multilinear if it can be expressed in the form

$$u_G(\underset{\sim}{x}) = k_{G1}u_{G1}(x_1) + k_{G2}u_{G2}(x_2) + k_G k_{G1} k_{G2} u_{G1}(x_1) u_{G2}(x_2), \qquad (15)$$

where for $i=1,2$, u_{Gi} is a conditional utility function which is assumed to be a monotonic, increasing function of x_i; k_{G1} and k_{G2} are positive scaling constants; and k_G is a scaling constant not restricted in sign. A utility function for $\underset{\sim}{x}$ is multilinear if x_1 and x_2 are mutually utility independent, which means that conditional preferences for lotteries on x_1 given x_2 do not depend on the level of x_2 and conditional preferences for lotteries on x_2 given x_1 do not depend on the level of x_1 (Keeney and Raiffa, 1976). While the definition of mutual utility independence may not appear at first glance to take equity considerations into account, we shall see that the multilinear utility function can indeed reflect such considerations.

In order to be able to interpret the multilinear utility function given by (13), we impose, without loss of generality, some scaling requirements. As in Section 3, we choose x_i^o and x_i^{oo} for i=1,2 such that x_i^{oo} is preferred to x_i^o. Then we scale the utility functions such that $u_G(\underset{\sim}{x}^o) = u_{Gi}(x_i^o) = 0$ and $u_G(\underset{\sim}{x}^{oo}) = u_{Gi}(x_i^{oo}) = 1$ for i=1,2. Thus, $k_{Gi} = u_G(x_i^{oo}, x_j^o)$ for i=1,2 and j≠i. Also, from $u_G(\underset{\sim}{x}^{oo}) = k_{G1} + k_{G2} + k_G k_{G1} k_{G2}$, we get $k_G = (1 - k_{G1} - k_{G2})/k_{G1}k_{G2}$.

The group's preferences regarding equity are indicated by the sign of k_G. If $k_G = 0$, the multilinear function reduces to an additive function, and we have seen that ex post equity considerations are irrelevant when the utility function is additive. If k_G is positive, the group prefers ex post equity, whereas if k_G is negative, the group prefers ex post inequity. For example, consider a choice between (1) a lottery that yields $\underset{\sim}{x}^{oo}$ with probability 1/2 and $\underset{\sim}{x}^o$ with probability 1/2, and (2) a lottery that yields (x_1^{oo}, x_2^o) with probability 1/2 and (x_1^o, x_2^{oo}) with probability 1/2. The expected utility of the first lottery is 1/2, and the expected utility of the second lottery is $(k_{G1} + k_{G2})/2 = (1 - k_G k_{G1} k_{G2})/2$. Indifference between the two lotteries implies indifference about equity considerations and leads to $k_G = 0$; the marginal distributions of x_1 and x_2 are the same in the two lotteries. A preference for the first lottery suggests a preference for ex post equity and implies that $(1 - k_G k_{G1} k_{G2})/2 < 1/2$, which means that k_G must be positive. Finally, a preference for the second lottery is a preference for ex post inequity that reverses the above inequality and requires that k_G be negative. The contemplation of lotteries such as these, with probabilities varied to find indifference points, can be useful in the assessment of the scaling constants.

We are interested in this section in cases in which $u_G(\underset{\sim}{x})$ is of the multilinear form. Although we are not focusing on the individual utility functions $u_i(\underline{x})$, we are still interested in types of individual utility functions over $\underline{x}$ that lead to multilinear group utility functions. Unfortunately it is not sufficient for both of the individual utility functions to be multilinear themselves. Of course, if they are multilinear and identical, then the group utility function will equal the individual utility functions. They need not be completely identical, however. For example, suppose that they agree on the conditional utility function of x_1 and their other point of agreement is that $k_1k_{11} = k_2k_{21}$, so that

$$u_1(\underline{x}) = k_{11}u_{11}(x_1) + k_{12}u_{12}(x_2) + k_1k_{11}k_{12}u_{11}(x_1)u_{12}(x_2)$$

and

$$u_2(\underline{x}) = k_{21}u_{21}(x_1) + k_{22}u_{22}(x_2) + k_2k_{21}k_{22}u_{21}(x_1)u_{22}(x_2),$$

with $k_1k_{11} = k_2k_{21}$ and $u_{11}(x_1) = u_{21}(x_1)$. Then $u_G(\underline{x})$ is multilinear with $u_{G1} = u_{11}$, $u_{G2} = (\lambda_1k_{12}u_{12} + \lambda_2k_{22}u_{22})/(\lambda_1k_{12} + \lambda_2k_{22})$, $k_{G1} = \lambda_1k_{11} + \lambda_2k_{21}$, $k_{G2} = \lambda_1k_{12} + \lambda_2k_{22}$, and $k_G = k_1k_{11}/(\lambda_1k_{11} + \lambda_2k_{21})$. Alternatively, of course, we could assume that the group bypasses the assessment of individual utility functions (or assesses them but does not use a mechanical aggregation procedure) and meets to assess a group utility function. In a group meeting, equity considerations would be hard to ignore, and a multilinear group utility function might be quite appealing.

However the group utility function is determined, the next step is to solve (1) given that u_G is multilinear. The case with $k_G = 0$ was treated in Section 3, and the case with $k_G < 0$ (a preference for inequity) seems unlikely to arise often in practice. Therefore, to guarantee the existence of a maximum in the general case, we require that k_G be positive in the following proposition.

<u>Proposition 4.1.</u> If u_G is multilinear with $k_G > 0$ and with u_{Gj} twice differentiable and $r_{Gj} \geq 0$ for $j = 1,2$, then

$$\partial x_j^*(y)/\partial y = [r_{G1}(x_1^*(y)) + t(x_1^*(y))]/[r_{G1}(x_1^*(y)) + r_{G2}(x_2^*(y)) + 2t(x_1^*(y))] \quad (15)$$

and

$$r_G(y) < \frac{1}{2}\,[\max\,\{r_{G1}(x_1^*(y)),\ r_{G2}(x_2^*(y))\} - t(x_1^*(y))], \tag{17}$$

where

$$t(x_1^*(y)) = k_G k_{G1} k_{G2} u'_{G1}(x_1^*(y))/[k_{G2} + k_G k_{G1} u_{G1}(x_1^*(y))],$$

$j=1,2$, $i\neq j$, and the prime denotes differentiation. A proof is given in the Appendix. The implications of the optimal sharing rule can best be understood by comparing it with the optimal sharing rule derived in Section 3 for additive utility functions. The rule given by (16) can be interpreted as a two-step division of the joint payoff, as follows:

$$\frac{\partial x_1^*(y)}{\partial y} = \frac{r_{G2}(x_2^*(y))}{r_{G1}(x_1^*(y)) + r_{G2}(x_2^*(y)) + 2t(x_1^*(y))} + \frac{t(x_1^*(y))}{r_{G1}(x_1^*(y)) + r_{G2}(x_2^*(y)) + 2t(x_1^*(y))}$$

and

$$\frac{\partial x_2^*(y)}{\partial y} = \frac{r_{G1}(x_1^*(y))}{r_{G1}(x_1^*(y)) + r_{G2}(x_2^*(y)) + 2t(x_1^*(y))} + \frac{t(x_1^*(y))}{r_{G1}(x_1^*(y)) + r_{G2}(x_2^*(y)) + 2t(x_1^*(y))}.$$

When y increases, a portion of the increase is divided equally between the two members, and this is represented by the second term on the right-hand side of each equation. The remainder of the increase is not necessarily divided equally, with the division depending on r_{G1} and r_{G2}. In constrast, the entire increase in y is allocated on the basis of r_{G1} and r_{G2} when u_G is additive. The additional first-step equal division of part of the increase when u_G is multilinear with $k_G > 0$ reflects the preference for some degree of ex post equity.

As for the second part of Proposition 3.1, the upper bound for $r_G(y)$

given by (17) is lower than the upper bound given by (13) for the additive case. Although this is just an upper bound, it does suggest that the group may tend to be less risk averse in the multilinear case than in the additive case. Perhaps the knowledge that the payoffs will be somewhat equitable makes the members willing to assume more risk in their joint decisions.

An example involving exponential conditional utility functions will be presented to illustrate group decision making with multilinear group utility functions. The group utility function for $\underline{x}$ is of the form

$$u_G(\underline{x}) = k_{G1}(1 - e^{-cx_1}) + k_{G2}(1 - e^{-cx_2}) + k_G k_{G1} k_{G2}(1 - e^{-cx_1})(1 - e^{-cx_2}),$$

where k_{G1}, k_{G2}, k_G, c, x_1, and x_2 are positive. The optimal sharing rule is

$$x_1^*(y) = (y/2) + (1/2c)\ln[(k_{G1} + k_G k_{G1} k_{G2})/(k_{G2} + k_G k_{G1} k_{G2})] = (y/2) + s$$

and

$$x_2^*(y) = (y/2) - (1/2c)\ln[(k_{G1} + k_G k_{G1} k_{G2})/(k_{G2} + k_G k_{G1} k_{G2})] = (y/2) - s.$$

If the group utility function were additive with the same conditional utility functions for x_1 and x_2, y would still be divided equally but the side payment would be $s = (1/2c)\ln(k_{G1}/k_{G2})$. The side payment is smaller in the multilinear case, and $|x_1^*(y) - x_2^*(y)|$ is smaller as a result.

The group utility function for y in this example is not exponential, but is a weighted average of two exponential utility functions:

$$U_G(y) = 1 + ae^{-cy} - be^{-cy/2}$$

where $a = k_G k_{G1} k_{G2}$ and $b = k_{G1}e^{-cs} + k_{G2}e^{cs} + k_G k_{G1} k_{G2}(e^{-cs} + e^{cs})$. A weighted average of exponential functions does not have constant risk aversion, and the group risk aversion function for y is

$$r_G(y) = \frac{c}{2}[1 - \frac{2ae^{-cy}}{be^{-cy/2} - 2ae^{-cy}}].$$

But $e^{-cs} + e^{cs} \geq 2$, implying that $b > 2a$ and hence that $be^{-cy/2} - 2ae^{-cy} > 0$. Thus

$r_G(y) < c/2$, which is the value of $r_G(y)$ in the additive case with the same conditional utility functions. The group is less risk averse in the multilinear case.

5. SUMMARY

The methodology developed in this paper deals with group decision-making problems. As a result of a decision, a group will receive a payoff which must be divided among the members of the group. Thus, the group faces both an external problem and an internal problem. The external problem involves the choice of an action to be taken by the group, and the internal problem involves the distribution of the group payoff among the members. Obviously, the internal and external problems are intertwined.

The ultimate payoff in the group decision-making problem is not the group payoff, but the vector of individual payoffs received by the members. Thus, we assume that each individual assesses a cardinal utility function for this vector of payoffs. Such utility functions can represent an individual's preferences concerning "equitable" and "inequitable" vectors as well as attitudes toward risk. Interpersonal comparisons at this stage involve comparisons of payoffs, not comparisons of utilities.

The next stage, the aggregation of individual utility functions, does require interpersonal comparisons of utilities, and we utilize previous results from the literature to arrive at a linear aggregation rule. This aggregation rule guarantees that the group decision will be Pareto optimal. A major advantage of the approach developed here is that it results in Pareto optimal decisions without sacrificing equity considerations.

The conversion from a group utility function for the vector of individual payoffs to a group utility function for the overall group payoff is achieved by solving a constrained maximization problem. This step determines a rule for dividing the group payoff (the internal problem). Furthermore, the substitution of this sharing rule in the group utility function for the vector of individual payoffs yields a group utility function for the group payoff. The latter function can be used to make the group's decisions (the external problem).

Conceptually, the methodology presented here is not difficult. The actual application of this methodology, on the other hand, may not be a simple matter because multiattribute utility functions (with the individual payoffs representing the attributes) are involved. Thus, any implementation of the methodology must rely heavily on procedures available for the assessment of multiattribute utility functions. In particular, the analysis is simplified considerably if certain preferential assumptions can be invoked to permit the decomposition of the multiattribute utility function into some function of single-attribute utility functions. For extensive discussions of such decompositions, which can simplify both the assessment of the utility functions and the analysis of these functions using the methodology in Section 2, see Keeney and Raiffa (1976).

The results obtained in Sections 3 and 4 for additive and multilinear group utility functions provide information about some implications of our group decision-making model. In Section 3, the optimal sharing rule and the group utility function are characterized in the additive case. As the group becomes less risk averse with respect to a particular member's payoff, that member's "stake" in the group payoff increases, although side payments may be necessary to pay for this increased stake. Also, as would be expected, the group is less risk averse toward the group payoff than toward any individual payoffs, and large groups of risk-averse members might be expected to be approximately risk neutral.

Unlike additive group utility functions, multilinear group utility functions are capable of reflecting preferences regarding ex post equity of payoffs. A comparison of the results of Section 4 with those of Section 3 demonstrates the implications of including ex post equity considerations by adding a multiplicative term to the additive utility function. The resulting multilinear utility function appears to lead to smaller ex post differences in individual payoffs and smaller side payments. Also, an upper bound derived for the group risk aversion function in the multilinear case suggests that the group may tend to be less risk averse in the multilinear case than in the additive case.

The discussion here has been couched in terms of a group making a decision. The methodology also applies to a single decision maker representing a group, as long as the payoff will be distributed among the members of the group. The decision maker might have the group members assess their individual utility functions, or, especially in the case of a large group, might directly assess a "group" utility function for the vector of payoffs, possibly with some guidance from the group members. Thus, the methodology may have potential implications not just for situations such as decisions made by a partnership or other small group, but also for situations such as decisions made by a public official with payoffs (monetary or otherwise) that will be received by members of the general public.

REFERENCES

Arrow, K.J., Social Choice and Individual Values. New York: Wiley, 1951.

Fishburn, P.C., The Theory of Social Choice. Princeton, N.J.: Princeton University Press, 1973.

Harsanyi, J.C., "Cardinal Welfare, Individualistic Ethics, and Interpersonal Comparisons of Utility," Journal of Political Economy, 63 (1955), 309-321.

Harsanyi, J.C., Rational Behavior and Bargaining Equilibrium in Games and Social Situations. Cambridge: Cambridge University Press, 1977.

Keeney, R.L., "A Group Preference Axiomatization with Cardinal Utility," Management Science, 23 (1976), 140-145.

Keeney, R.L., and Kirkwood, C.W., "Group Decision Making Using Cardinal Social Welfare Functions," Management Science, 22 (1975), 430-437.

Keeney, R.L., and Raiffa, H., Decisions with Multiple Objectives: Preferences and Value Tradeoffs. New York: Wiley, 1976.

Kirkwood, C.W., "Decision Analysis Incorporating Preferences of Groups," Technical Report No. 74, O.R. Center, M.I.T., Cambridge, Mass., 1972.

Kirkwood, C.W., "Pareto Optimality and Equity in Social Decision Analysis," IEEE Transactions on Systems, Man, and Cybernetics, 9 (1979), 89-91.

LaValle, I.H., Fundamentals of Decision Analysis. New York: Holt, Rinehart and Winston, 1978.

Pratt, J.W., "Risk Aversion in the Small and in the Large," Econometrica, 32 (1964), 122-136.

Raiffa, H., Decision Analysis. Reading, Mass.: Addison-Wesley, 1968.

Sen, A.K., Collective Choice and Social Welfare. San Francisco: Holden-Day, 1970.

Wilson, R., "The Theory of Syndicates," Econometrica, 36 (1968), 119-132.

Proof of Proposition 3.1

The first-order condition for optimality is $k_{Gi}u'_{Gi}(x^*_i) = \lambda$, $i=1,\ldots,n$, where the prime denotes differentiation, the argument y of x^*_i is omitted for notational simplicity, and λ is a Lagrange multiplier such that $\lambda = U'_G(y)$. Thus, $k_{Gi}u'_{Gi}(x^*_i) = U'_G(y)$. Differentiating w.r.t. y yields $k_{Gi}u''_{Gi}(x^*_i)(\partial x^*_i/\partial y) = U''_G$ and dividing the new equation by the preceding equation gives us $(\partial x^*_i/\partial y)r_{Gi}(x^*_i) = r_G(y)$, or $r_G(y)/r_{Gi}(x^*_i) = (\partial x^*_i/\partial y)$. Summing over i yields $r_G(y)\sum_{i=1}^{n}[r_{Gi}(x^*_i)]^{-1} = 1$, which proves (12). Substituting (12) in $r_G(y)/r_{Gi}(x^*_i) = (\partial x^*_i/\partial y)$, we then get (11). The second-order condition assuring that (11) is a maximum is $k_{Gi}u''_{Gi}(x^*_i) < 0$, $i=1,\ldots,n$, which follows from the concavity of u_{Gi}, $i=1,\ldots,n$.

Proof of Proposition 3.2

From the proof of Proposition 3.1, $r_G(y) = r_{Gj}(x^*_j)(\partial x^*_j/\partial y)$, and summing over j yields $nr_G(y) = \sum_{j=1}^{n} r_{Gj}(x^*_j)(\partial x^*_j/\partial y)$. Since $0 < \partial x^*_j/\partial y < 1$ and $\sum_{j=1}^{n}(\partial x^*_j/\partial y) = 1$, $nr_G(y)$ is a convex combination of $r_{Gj}(x^*_j)$, $j=1,\ldots,n$
(13) follows directly. Next, we can write $u_{Gj}(x_j) = \sum_{i=1}^{n} b_i u_{ij}(x_j)$, where $b_i = \lambda_i k_{ij}/\sum_{i=1}^{n}\lambda_i k_{ij}$. Differentiating twice w.r.t. x_j, we get $u'_{Gj}(x_j) = \sum_{i=1}^{m} b_i u'_{ij}(x_j)$ and $u''_{Gj}(x_j) = \sum_{i=1}^{n} b_i u''_{ij}(x_j)$, and dividing the former equation by the latter yields $r_{Gj}(x_j) = \sum_{i=1}^{n} a_i r_{ij}(x_j)$, where $a_i = b_i u'_{ij}(x_j)/\sum_{i=1}^{n}$
$r_{Gj}(x_j)$ is also a convex combination of $r_{1j}(x_j),\ldots,r_{nj}(x_j)$, which implies that

$$\min_i\{r_{ij}(x^*_j)\} \le r_{Gj}(x^*_j) \le \max_i\{r_{ij}(x^*_j)\}.$$

Combining this result with (13) yields (14).

Proof of Proposition 4.1

The first-order condition for optimality can be written in the form $u'_{G1}(x^*_1)[k_{G1} + k_G k_{G1}k_{G2}u_{G2}(x^*_2)] = u'_{G2}(x^*_2)[k_{G2} + k_G k_{G1}k_{G2}u_{G1}(x^*_1)]$. Differentiating both sides w.r.t. y, dividing the new equation by the old equation, and simplifying yields

$$-r_{G1}(x_1^*)(\partial x_1^*/\partial y) + t(x_1^*)[1 - (\partial x_1^*/\partial y)] = -r_{G2}(x_2^*)[1 - (\partial x_1^*/\partial y)] + t(x_1^*)(\partial x_1^*/\partial y).$$

Solving for $\partial x_1^*/\partial y$ yields (16) for j=1, and (16) for j=2 follows from $\partial x_2^*/\partial y = 1 - (\partial x_1^*/\partial y)$. The second-order condition indicates that the solution is a maximum.

Differentiating both sides of the first-order condition [each of which equals $U_G'(y)$] w.r.t. y, dividing by the first-order condition, and simplifying gives us

$$r_G(y) = (\partial x_1^*/\partial y)r_{G1}(x_1^*) - t(x_1^*)[1 - (\partial x_1^*/\partial y)] = [1 - (\partial x_1^*/\partial y)]r_{G2}(x_2^*) - t(x_1^*)(\partial x_1^*/\partial y$$

or

$$2r_G(y) = (\partial x_1^*/\partial y)r_{G1}(x_1^*) + [1 - (\partial x_1^*/\partial y)]r_{G2}(x_2^*) - t(x_1^*).$$

But this equation, together with the inequality

$$(\partial x_1^*/\partial y)r_{G1}(x_1^*) + [1 - (\partial x_1^*/\partial y)]r_{G2}(x_2^*) \leq \max\{r_{G1}(x_1^*), r_{G2}(x_2^*)\},$$ yields (17).

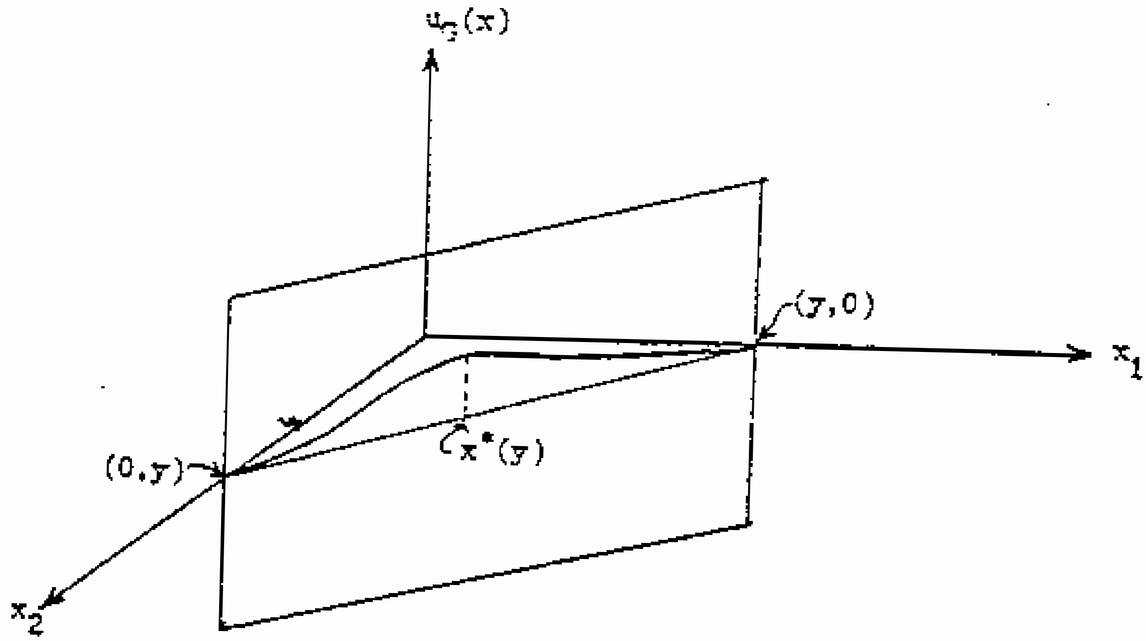

Figure 1

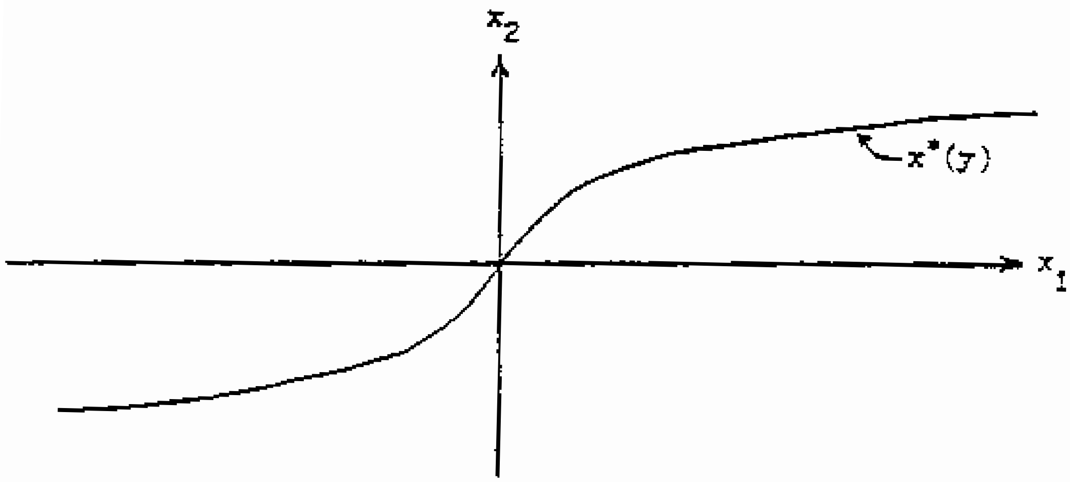

Figure 2