

Discussion Paper 411

THE COST OF CAPITAL IN NON-MARKETED FIRMS

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I. Introduction

All western economies are mixed in the sense that there is both a marketed and a nonmarketed sector of firms. A natural question then arises: Should unincorporated business use higher or lower discount rates than incorporated business? The same question has been a source of much controversy in public finance. Some authors, among them Arrow and Lind [1970] and Samuelson [1974], argue that public enterprises should use a lower discount rate because the government is better able to absorb risk and can more effectively diversify investments. Another group, which finds Hirschleifer [1966] and Diamond [1967] among its supporters, maintains that the discount rate should be the same for projects with comparable risk profiles, regardless of who undertakes them. As Diamond states:

by means of portfolio diversification on the part of consumers, all of the economy's averaging possibilities are brought to bear on each production decision. Thus, ..., large firms have no pooling advantages over small firms, nor does the government have an advantage over private business [1967, p. 771].

In an accompanying footnote, he further argues convincingly that this conclusion remains intact even when bankruptcy is considered.

Sandmo [1974] takes a consolidated view of the issue. He makes the important observation that market structure is essential for the discussion. If the government firms represent the same risk classes as the private ones and shares of the latter are traded on an efficient market, then Hirschleifer's and Diamond's view appears correct. On the other hand, if no market (or an imperfect market) for private shares exists, then this conclusion is not necessarily valid, though Sandmo does not go for an explicit comparison.

Recently, Stapleton and Subrahmanyam [1978] have provided such an explicit comparison in a highly simplified model, with the rather surprising (at least at first sight) conclusion that the government should use a higher discount rate! Their rationale sounds simple: in a perfect market risk is distributed more

efficiently than in government firms, where the tax system implies a somewhat arbitrary allocation of "shares". Hence, the market risk premium and, thereby, the cost of capital for marketed firms should be lower than for government. The purpose of this paper is to point out that Stapleton's and Subrahmanyam's result follows from an irrelevant comparison which favors the market and that the relevant comparison leads to no unambiguous conclusions about the cost of capital in the two sectors except under the assumption that risk classes in both sectors are identical or that sectors are totally independent.

To explicate: Stapleton and Subrahmanyam compare discount rates in a mixed economy with those in an identical but purely private economy. Why would that be of interest? The pure private economy is a hypothetical point of reference. It seems rather obvious that it will do better, since shares are allocated optimally there, whereas they are not in the mixed economy.¹⁾ In contrast, reality is a single mixed economy with marketed and non marketed firms living abreast. What should be done then is a contemporaneous comparison of how these two types of firms react to one and the same investment opportunity. This I will do in the paper, using for simplicity and comparison, essentially, the same model as Stapleton and Subrahmanyam.

The outcome of the analysis is that the cost of capital can be either higher or lower in the nonmarketed sector and this will depend on no simple characteristics that could easily be transformed to explicit rules. The reason is that, even if the capital market is perfect, an imperfect distribution of shares in the nonmarketed sector will induce distortions in share holdings in the marketed sector as investors try to balance their total portfolio. As a consequence, risk can no longer be measured in a single dimension and each sector will price dimensions differently. Furthermore, neither marketed nor

nonmarketed firms can look for appropriate discount rates in market data when shares are not correctly distributed, and value-maximization will be an inappropriate decision rule for traded firms.²⁾

In the public finance context, these conclusions complement Sandmo's qualifications of Hirschleifer's and Diamond's claim that the cost of capital is equal in the private and public sectors. Sandmo shows that for this to be true, it is essential that the private sector is perfect; this paper shows that it is also essential that all risk classes in the public sector should be represented in the private sector.

II. The Model

Consider an economy with two firms, A and B, and two investors indexed $i=1,2$. Both investors possess exponential utility functions over wealth:

$$(1) \quad U_i(z) = -\rho_i \cdot e^{\frac{-z}{\rho_i}}, \quad i=1,2,$$

where ρ_i is the risk tolerance of investor i . The framework is single-period.

The uncertain return of firm A is \tilde{x}_A ,

and that of B \tilde{x}_B . The distribution of the pair $(\tilde{x}_A, \tilde{x}_B)$ is multi-variate normal; the expected values are denoted μ_A, μ_B variances σ_A^2, σ_B^2 and the covariance σ_{AB} . Investor i is endowed with e_i units of money in the first period. He owns initially \bar{s}_{iA} shares in firm A and \bar{s}_{iB} shares in firm B. Shares of A he can trade on a perfect market where the only other asset is a riskless one, yielding a certain exogenously given rate of return r . Shares of B cannot be traded.

Investor i 's portfolio decision is to determine investment levels in firm A and the riskless asset. Let P be the price of firm A shares. Investor i will then choose s_{iA} , his purchase of shares in firm A, so as to maximize the certainly equivalent:³⁾

$$(2) \quad C_i = e_i(1+r) + \bar{s}_{iA}P(1+r) + s_{iA}(\mu_A - P(1+r)) + \bar{s}_{iB}\mu_B \\ - \frac{1}{2\rho_i} (s_{iA}^2\sigma_A^2 + \bar{s}_{iB}^2\sigma_B^2 + 2s_{iA}\bar{s}_{iB}\sigma_{AB}).$$

In (2), the investment in the riskless asset is included indirectly as a function of s_{iA} , determined by the budget constraint. Also, it is assumed investor i acts as a price taker. Maximizing (2) gives the solution:

$$(3) \quad s_{iA} = \frac{1}{\sigma_A^2} \{ \mu_A - P(1+r) - \sigma_{AB} \bar{s}_{iB} \}.$$

Since $s_{1A} + s_{2A} = 1$ in equilibrium, this implies that the equilibrium price is:

$$(4) \quad P = \frac{1}{1+r} \left\{ \mu_A - \frac{1}{\bar{\rho}} (\sigma_A^2 + \sigma_{AB}) \right\}.$$

Here $\bar{\rho} = \rho_1 + \rho_2$ is the risk tolerance for the syndicate of two.⁴⁾ Substituting (4) into (3) gives i 's purchase of shares in terms of given data:

$$(5) \quad s_{iA} = \frac{\rho_i}{\bar{\rho}} + \frac{\sigma_{AB}}{\sigma_A^2} \left(\frac{\rho_i}{\bar{\rho}} - \bar{s}_{iB} \right).$$

One notices that the equilibrium price in (4) is just the same as it would be if both firms were marketed. In contrast, share purchases will be different; in particular, the separation theorem does not hold. In a purely private market, agents would purchase shares $s_{iA} = s_{iB} = \rho_i / \bar{\rho}$, $i=1,2$. If the fixed allocation of shares in firm B differs from this ideal, investors will seek to counterbalance the bias by purchases in the market. That will be possible if $\sigma_{AB} \neq 0$. For instance, if $\sigma_{AB} > 0$ and investor 1's share is too high in the unincorporated firm ($\bar{s}_{1B} > \rho_1 / \bar{\rho}$), (5) says that he will buy correspondingly less than $\rho_1 / \bar{\rho}$ shares in firm A to avoid excess unsystematic risk. Looked upon in isolation then, the share distribution in the market will be imperfect as well.

A special case, that deserves to be mentioned, arises when there is perfect correlation (positive or negative) between the marketed and nonmarketed firm. That will enable the investor to fully offset his possibly biased allo-

cation of shares in the nonmarketed sector. For an illustration, let $\tilde{x}_A = b \cdot \tilde{x}_B$, b a constant. Investor i 's return from shares will then be by (5):

$$(6) \quad \bar{s}_{iB} \tilde{x}_B + \left[\frac{\rho_i}{\rho} + \frac{1}{b} \left(\frac{\rho_i}{\rho} - \bar{s}_{iB} \right) \right] b \tilde{x}_B = \frac{\rho_i}{\rho} \tilde{x}_B [1 + b]$$

which is the same return he would enjoy if both firms were marketed. The observation suggests that the imperfections in a mixed economy may be smaller than one would believe at first.

III. A Comparison of Cost of Capital

Let the mixed economy be in equilibrium as described in the previous section, that is, shares in firm A are given by (5). Next, introduce for both firms the same opportunity to invest in a new project. Which firm will require a higher expected return from the project before accepting it?

For convenience, assume that the new project yields constant returns to scale, \tilde{y} per unit of investment, and that \tilde{y} is normally distributed with mean μ_y , variance σ_y^2 and covariances $\sigma_{Ay} = \text{cov}(\tilde{x}_A, \tilde{y})$, $\sigma_{By} = \text{cov}(\tilde{x}_B, \tilde{y})$. Thus, if firm A invests an amount I in the project, its return changes to $\tilde{x}_A + I\tilde{y}$ and correspondingly for firm B.

Because shares are not held so as to minimize risk in the mixed economy, value-maximization cannot be considered the appropriate objective for the marketed firm. Instead, I will use the Pareto criterion for judging investments in both firms. With exponential utilities the Pareto criterion becomes particularly simple: it is just the maximization of the sum of the investors' certainty equivalents in (2).

Investment in the Marketed Firm

When firm A invests I in the new project price will change to $P(I)$, share purchases to $s_{iA}(I)$, the return of firm A to $\mu_A(I)$, its variance to $\sigma_A^2(I)$ and the covariance with B to $\sigma_{AB}(I)$. No other variables will change, since \bar{s}_{iB} 's were fixed and r was exogenously given. The certainty equivalent of

i then becomes:

$$(7) \quad C_i(I) = e_i(1+r) + s_{iA} P(I)(1+r) + s_{iA}(I)(\mu_A(I) - P(I)(1+r)) + \\ - \bar{s}_{iB} \mu_B - \frac{1}{2\rho_i} \{s_{iA}^2(I)\sigma_A^2(I) + s_{iB}^2(I)\sigma_B^2 + 2s_{iA}(I)\bar{s}_{iB}\sigma_{AB}(I)\},$$

where s_{iA} is given by (5). The welfare criterion recommends investment (in some positive amount) if:

$$(8) \quad \left. \frac{d}{dI} (C_1(I) + C_2(I)) \right|_{I=0} > 0.$$

By definition $s_{iA} = s_{iA}(0)$. Hence, using the first-order condition for optimal portfolio choice, which gives $s_{iA}(I)$, one gets:

$$(9) \quad \left. \frac{dC_i}{dI} \right|_{I=0} = s_{iA}\mu_y - \frac{1}{2\rho_i} (s_{iA}^2 2\sigma_{Ay} + 2\sigma_{By} s_{iA} \bar{s}_{iB}).$$

Here are used the facts that:

$$(10) \quad \left. \frac{d\mu_A(I)}{dI} \right|_{I=0} = \left. \frac{dE(\tilde{x}_A + I\tilde{y})}{dI} \right|_{I=0} = \mu_y,$$

$$(11) \quad \left. \frac{d\sigma_A^2(I)}{dI} \right|_{I=0} = \left. \frac{dE(\tilde{x}_A + I\tilde{y} - \mu_A - I\mu_y)^2}{dI} \right|_{I=0} = 2\sigma_{Ay},$$

$$(12) \quad \left. \frac{d\sigma_{AB}(I)}{dI} \right|_{I=0} = \left. \frac{dE(\tilde{x}_A + I\tilde{y} - \mu_A - I\mu_y)(\tilde{x}_B - \mu_B)}{dI} \right|_{I=0} = \sigma_{By},$$

where E stands for the expectation operator. One should notice that the variance of \tilde{y} does not enter the expressions. On the margin, when I is small, its effect is negligible. Nor does the change in price affect the certainty equivalent, because it was assumed that we started from equilibrium asset holdings. Placing (9) in (8) gives the Pareto improvement criterion:

$$(13) \quad \mu_y - \left(\frac{s_{1A}^2}{\rho_1} + \frac{s_{2A}^2}{\rho_2} \right) \sigma_{Ay} - \left(\frac{s_{1A} \bar{s}_{1B}}{\rho_1} + \frac{s_{2A} \bar{s}_{2B}}{\rho_2} \right) \sigma_{By} > 0.$$

Investment should be undertaken if and only if (13) holds. In comparison, the value-maximizing criterion reads:

$$(14) \quad \mu_y - \frac{1}{\bar{\rho}} (\sigma_{Ay} + \sigma_{By}) > 0.$$

Notice that if $s_{1A} = \bar{s}_{1B} = \rho_1/\bar{\rho}$ so that $s_{2A} = \bar{s}_{2B} = \rho_2/\bar{\rho}$, then (13) and (14) coincide.⁵⁾ Another case of equivalence arises if $\tilde{x}_A = b\tilde{x}_B$, b a constant; cf (6).⁶⁾ The rationale for both cases is that they enable investors to reach exactly the same position as they would, were both firms marketed. Except under these special circumstances, however, (13) indicates that, in contrast to standard market models, there is no single measure for project risk, nor a market price by which the cost of risk can be evaluated.⁷⁾ Instead, the risk is measured in as many dimensions as there are firms, each component being charged with a unique price. For these component prices one has:

$$(15) \quad \frac{s_{1A}^2}{\rho_1} + \frac{s_{2A}^2}{\rho_2} \geq \frac{1}{\bar{\rho}},$$

$$(16) \quad \frac{s_{1A}\bar{s}_{1B}}{\rho_1} + \frac{s_{2A}\bar{s}_{2B}}{\rho_2} \leq \frac{1}{\bar{\rho}}, \text{ if } \sigma_{AB} > 0,$$

$$(17) \quad \frac{s_{1A}\bar{s}_{1B}}{\rho_1} + \frac{s_{2A}\bar{s}_{2B}}{\rho_2} \geq \frac{1}{\bar{\rho}}, \text{ if } \sigma_{AB} < 0;$$

Equalities hold only if $s_{1A} = \bar{s}_{1B} = \rho_1/\bar{\rho}$. (15) is proved by noticing that $s_{2A} = 1 - s_{1A}$, and that the minimum of the left hand side is achieved for $s_{1A} = \rho_1/\bar{\rho}$. (16) and (17) are proved in a similar way; the left hand sides can be written as quadratic functions of \bar{s}_{1B} using market clearing conditions and (5), and optimization yields the result.

Since σ_{AB} , σ_{Ay} and σ_{By} need not carry any particular relationship with each other - they can be chosen independently if desired - (13)-(17) show that the Pareto criterion may be either stricter or leaner than the value-maximization criterion. In particular, if $\sigma_{AB} > 0$, $\sigma_{Ay} > 0$ and $\sigma_{By} = 0$,

then market data gives a too low value for the cost of capital, whereas the reverse is true when $\sigma_{AB} > 0$, $\sigma_{By} > 0$ and $\sigma_{Ay} = 0$.

Investment in the Nonmarketed Firm

The analysis for the same investment in the nonmarketed firm B, proceeds identically. The acceptance criterion corresponding to (13) becomes:

$$(18) \quad \mu_y - \left(\frac{s_{1A}s_{1B}}{\rho_1} + \frac{s_{2A}s_{2B}}{\rho_2} \right) \sigma_{Ay} - \left(\frac{s_{1B}}{\rho_1} + \frac{s_{2B}}{\rho_2} \right) \sigma_{By} > 0.$$

Thus, risk is measured in two dimensions $(\sigma_{Ay}, \sigma_{By})$ as before, but prices are changed. Analogously to (15),

$$(19) \quad \frac{s_{1B}}{\rho_1} + \frac{s_{2B}}{\rho_2} \geq \frac{1}{\rho},$$

with equality only when shares are efficiently allocated.

The interesting comparison is between (18), which gives the cost of capital for the nonmarketed firm, and (13), which gives the cost of capital for the marketed firm. We can immediately see the main conclusion that the cost of capital could in general be either higher or lower in the marketed sector than in the nonmarketed sector contrary to the result of Stapleton and Subrahmanyam [1978]. To be a bit more specific, the following sample relationships can be recorded:

Proposition: Assume $\sigma_{AB} > 0$, $\bar{s}_{1B} \neq \rho_1/\bar{\rho}$. Then

- a. $\sigma_{Ay} \leq 0, \sigma_{By} > 0 \Rightarrow CoC_A < CoC_B$,
- b. $\sigma_{By} \leq 0, \sigma_{Ay} > 0 \Rightarrow CoC_B < CoC_A$,
- c. for any σ_{Ay} , there exists a $\bar{\sigma}_{By}$ such that $\sigma_{By} (\overset{>}{<}) \bar{\sigma}_{By} \Rightarrow CoC_A (\lesseqgtr) CoC_B$,

- d. for any σ_{By} , there exists a $\bar{\sigma}_{Ay}$ such that $\sigma_{Ay} (\gtrless) \bar{\sigma}_{Ay} = CoC_A (\gtrless) CoC_B$,
- e. if $\sigma_{Ay} > \sigma_{By} > 0$, $\sigma_A^2 > \sigma_{AB}$, then $CoC_A < CoC_B$.

Proof: Straightforward calculations in (13) and (18), using (5).

Statements a and b show that diversification is an essential consideration when there is a nonmarketed sector. In standard versions of the capital asset pricing model (see for instance Mossin (1973)), this is not the case, since firms do not face a portfolio choice problem at all. In such models, the acceptance criterion for an investment is independent of which firm undertakes it. This leads Diamond (1967) to conclude that the cost of capital should be the same for government as for private firms, but as we see, explicit consideration of market structure and the type of risk present in different sectors may alter the conclusion. Notice in particular that even if the nonmarketed firm B contemplates an investment in the same risk class as firm A (so that $\sigma_{Ay} \propto \sigma_A^2$, $\sigma_{By} \propto \sigma_{AB}$), it cannot read the price of risk from (4), nor will it use the same criteria as firm A would use in considering the same project.

This shows that for Diamond's result to be true, it is essential that all risk classes in the public sector should be represented in the private (marketed) sector as well.

Statements c and d show monotonicity properties. The higher σ_{Ay} ceteris paribus the more likely the cost of capital for firm A will be higher than for firm B. The reverse is true for increases in σ_{By} . Statement e is included to show that the size of firms also plays a role. If the marketed sector is large, $\sigma_{AB} < \sigma_A^2$, $\sigma_{By} < \sigma_{Ay}$, and this makes the cost of capital in that sector relatively lower.

Finally, one may note that if the return of the new project is uncorrelated with both sectors (i.e., $\sigma_{Ay} = \sigma_{By} = 0$), then the acceptance criterion becomes the same in marketed and unmarketed firms. It is for this case that Arrow and Lind argued that the public discount rate should be lower, because they assumed that risk is shared inefficiently in the private sector.

In the preceding, I have used the Pareto criterion as a yard stick for accepting or rejecting projects in both sectors of the economy. Results will be slightly different, of course, if the private sector uses market value maximization. However, the main conclusion remains intact: no unambiguous statements can in general be put forward as a comparison of (14) and (18) quickly reveals.

IV. Concluding Remarks

The main demonstration of the paper has been to show that no unambiguous comparisons of the cost of capital between marketed and nonmarketed firms seem possible when the two sectors are imperfectly correlated. This contrasts with recent results of Stapleton and Subrahmanyam [1978]. Since the conclusions are negative already in a highly stylized example, one should not expect any unambiguous statements in more complex situations either. The explicit comparison that was possible in the example shows that the determinants of differences in the cost of capital are many and hardly easy to use for conditional judgement. Ultimately, this is due to the fact that all firms will face a portfolio problem when a nonmarketed sector is present.

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Footnotes

1. This disregards the important possibility that government can incorporate nonlinear sharing rules not available in the private market. In Stapleton's and Subrahmanyam's model, it is of no consequence, since they work with exponential utilities.
2. Stiglitz [1972] has shown that even with a competitive capital market in a mean-variance framework, market value maximization will yield a slightly different result than the Pareto optimal decision rule prescribed.
3. The certainty equivalent of a normally distributed variable with mean μ and variance σ^2 is $\mu - \frac{1}{2\rho}\sigma^2$ when utility is exponential.
4. General versions of (3) and (4) are given in Mayers [1976].
5. This is so because the variance of \tilde{y} dropped out. Otherwise, a difference remained as Stiglitz [1972] observes.
6. Sandmo's assumption about same risk classes in both sectors corresponds to this special case.
7. This statement does not, of course, apply to the evaluation of firms in equilibrium; see (4).
8. Extensions to the case with many firms in each sector is straightforward. A model where firm scales are determined endogenously could be developed paralleling Sandmo, but is likely to yield less transparent results.