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EXTREMAL PROCESSES, III

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1. INTRODUCTION.

In this paper we show the relation between extremal processes and their "inverses." Suppose that $Y(\ )$ is a real-valued, non-decreasing, right-continuous function defined on (a,b), $(-\infty < a < b < \infty)$. By the inverse of $Y(\ )$ we mean the function $Z(\ )$ defined by

$$Z(u) = \sup(t \mid Y(t) \leq u)$$

Thus, $Z(\ )$ is also a real-valued, non-decreasing, right-continuous function. It isnot hard to see that the inverse of $Z(\ )$ is the original Y. We will show that the inverses of extremal processes are certain additive processes, that is processes with independent (but not stationary) increments.

2. DISCRETE TIME CASE.

We first consider the case of successive maxima of a sequence of independent and identically distributed random variables. This case serves as a motivation for the limiting case of extremal processes (see Sec. 2 of [1]) and also has some real interest of its own. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with c.d.f. $F(\ )$. Define

$$M_n = \max(X_1, \ldots, X_n)$$

and

$$M(t) = M_n \text{ for } n - 1 \leq t < n,\ n = 1,2,\ldots$$

Thus the random function $M(\ )$ is with probability one a non-decreasing, right-continuous function. The structure of $M(\ )$ is completely determined by the sequence $M_1, M_2, \ldots$ which is a Markov chain with non-stationary transitions. The evolution of $M(\ )$ is as follows:

(a) If $M(n) = x$ then $M(\ )$ remains equal to x for a geometrically distributed period of time whose parameter depends on x. Specifically, if $W(n,x)$ is the amount of time that the process remains equal to x, we have,

$$P(W(n,x) = k) = [F(x)]^{k-1}[1 - F(x)],\ k = 1, 2, \ldots$$

(b) The process then jumps to a new height whose c.d.f. is given by

$$F_x(y) = \frac{F(y) - F(x)}{1 - F(x)} \quad x < y < -\infty$$

Notice that both the waiting time distributions in a state and the amount of the jump depend on x, the height of the process, but not on n, the time from which the evolution is being considered. Thus we can consider the inverse process $Z(\ )$ evolving as follows:

(a') Let $[A,B]$ be the smallest interval containing the total mass of the c.d.f. . That is

$$\lim_{x \to B} F(x) - \lim_{x \to A} F(x) = 1$$

and if $[A',B']$ is strictly contained in $[A,B]$ then the above limit, with A',B' replacing A, B, is less than one. (It may be that $A = -\infty$ or $B = \infty$.) We define the inverse process $Z(\ )$ by

$$Z(u) = \sup(t \mid M(t) \leq u),\ A < u < B \tag{2.1}$$

Thus, the values of $Z(\ )$ are non-negative integers.

(b') If $Z(u) = n$, then $Z(\ )$ remains equal to n for an additional amount of time $R(u,n)$ whose distribution depends on u, but not on n. Specifically,

$$P(R(u,n) > s) = \frac{1 - F(s)}{1 - F(u)} \tag{2.2}$$

This follows from (b) in the description of $M(\ )$ above since

$$P(Z(v) = n,\ u \leq v \leq s \mid Z(u) = n) = \frac{1 - F_x(s)}{1 - F_x(u)} = \frac{1 - F(s)}{1 - F(u)}$$

which does not depend on x whenever $x \leq u \leq s$.

(c') Given that $Z(u) = n$ and $Z(\ )$ remains equal to n until time s according to the distribution described in (b'), then $Z(\ )$ jumps to a new height $n + k$ with probability

$$[F(u)]^{k-1}[1 - F(u)],\ k = 1, 2, \ldots \tag{2.3}$$

It follows from the above description that $Z(\ )$ is a Markov process. In the next section we give a different description of this process.

3. L. F. PROCESSES.

Consider the probability generating function

$$G(s,\theta) = \frac{1 - F(s)}{1 - \theta F(s)} = \sum_0^\infty [\theta F(s)]^k\ [1 - F(s)]$$

for s in (A,B). We have that

$$H(s,t,\theta) = \frac{1 - F(t)}{1 - F(s)} \frac{1 - \theta F(s)}{1 - \theta F(t)} = G(s,\theta)/G(t,\theta) \tag{3.1}$$

is a probability generating function whenever $A \leq s \leq t < B$ since it is easy to verify that

$$H(s,t,\theta) = a + (1 - a)G(t,\theta)$$

which displays H as a convex combination of two probability generating functions with

$$a = \frac{1 - F(t)}{1 - F(s)} \frac{F(s)}{F(t)}$$

The distribution described by H has been called a linear fractional distribution. We now describe an additive process which we call a linear fractional (L.F.) process, $\hat{Z}(\ )$ as follows:

1) If $A < t_1 < t_2 < \dots < t_k < B$ then the joint distribution of

$$\hat{Z}(t_1),\ \hat{Z}(t_2) - \hat{Z}(t_1),\ \dots,\ \hat{Z}(t_k) - \hat{Z}(t_{k-1})$$

is that of k independent, non-negative, integer-valued random variables whose probability generating functions are

$$G(t_1,\theta), H(t_1,t_2,\theta),\ \dots,\ H(t_{k-1},t_k,\theta)$$

respectively.

2) Since for $s < t < u$ we have that

$$H(s,t,\theta)H(t,u,\theta) = H(s,u,\theta)$$

it follows that a process with independent increments can be defined on (A,B) in accordance with the consistency requirements of Kolmogorov. The details are standard and are omitted. Notice that for any t in (A,B), $\hat{Z}(t)$ has the generating function $G(t,\theta)$.

3) We suppose, as we can, that we have a version of $\hat{Z}(\ )$ which is right-continuous.

<u>Theorem 3.1</u> The process $\hat{Z}(\ )$ defined in this section and the process $Z(\ )$ defined in Section 2 agree. This means that $\hat{Z}(t)$ and $Z(t)$ have the same distribution for any t in (A,B) and the increments

$$\hat{Z}(t_2) - \hat{Z}(t_1),\ldots,\hat{Z}(t_k) - \hat{Z}(t_{k-1}) \text{ and } Z(t_2) - Z(t_1),\ldots,Z(t_k) - Z(t_{k-1})$$

have the same joint distributions for

$$A < t_1 < t_2 < \ldots < t_k < B$$

<u>Proof</u> We see from the definition of $\hat{Z}(\ )$ that $P(\hat{Z}(t) = k) = [F(t)]^k[1 - F(t)]$, $k = 0,1,\ \ldots$. Let us check the distribution of $Z(u)$. From the definition of $Z(\ )$ as the inverse of $M(\ )$ we have that

$$P(Z(t) = 0) = P(X_1 > t) = 1 - F(t)$$

$$\begin{aligned} P(Z(t) = k) &= P(X_1 \le t,\ \ldots,\ X_k \le t,\ X_{k+1} > t) \\ &= [F(t)]^k[1 - F(t)],\ k = 1,2,\ \ldots \end{aligned}$$

Thus, the first requirement that $\hat{Z}(t)$ and $Z(t)$ have the same distributions, is satisfied. Next we check the second requirement that the increments of $\hat{Z}(\ )$ and of $Z(\ )$ have the same joint distributions. There are several different ways of doing this and it might be interesting to point out these different approaches.

Method 1 A straightforward calculation shows that

$$P(\hat{Z}(v) = n,\ u \leq v \leq s \mid \hat{Z}(u) = n) = \text{coefficient of } \theta^0 \text{ in } H(u,s,\theta)$$

$$= \frac{1 - F(s)}{1 - F(u)}$$

Thus the holding time distributions of $\hat{Z}(\ )$ and $Z(\ )$ coincide. Another straightforward computation shows that

$$\lim_{\varepsilon \downarrow 0} P(Z(u+\varepsilon) = k \mid Z(u+\varepsilon) > Z(u)) = [F(u)]^{k-1}[1 - F(u)], k = 1,2, \ldots$$

Hence the jump distributions for $\hat{Z}(\ )$ and $Z(\ )$ coincide and the proof is complete.

Method 2 It is not too hard to show that the increments of $\hat{Z}(\ )$ have the same joint distributions as do the increments of $Z(\ )$ by direct computation. For typographical ease we illustrate the necessary computation in a special case, leaving the general case to the reader. Suppose that $A < t_1 < t_2 < B$. Let $Z(t_1) = Z_1, Z(t_2) - Z(t_1) = Z_2$ and let r and k be positive integers. Then

$$P(Z_1 = r,\ Z_2 = k)$$

$$= P(X_1 \leq t_1, \ldots, X_r \leq t_1,\ t_1 < X_{r+1} \leq t_2, X_{r+2} \leq t_2, \ldots, X_{r+k} \leq t_2, X_{r+k+1} > t_2)$$

$$= [F(t_1)]^r [F(t_2) - F(t_1)][F(t_2)]^{k-1}[1 - F(t_2)]$$

Hence, since we have determined the distribution of Z_1 from the first part of the proof we have that

$$P(Z_2 = k \mid Z_1 = r) = \frac{1 - F(t_2)}{1 - F(t_1)}[F(t_2)]^{k-1}[F(t_2) - F(t_1)] \tag{3.2}$$

A similar computation shows that the same conditional probability applies when

$r = 0$. Hence Z_1 and Z_2 are independent. The quantity on the right side of (3.2) is the coefficient of θ^k in $H(t_1, t_2, \theta)$.

4. ANOTHER REPRESENTATION FOR $\hat{Z}(\ )$.

Let $-\infty < C < D \leq \infty$. We want to describe a process on (C,D) which is a generalization of compound Poisson processes. Let $U(\ )$ be a Poisson process with stationary, independent increments over the time interval (C,D) and parameter λ. That is, if $C \leq t_1 < t_2 < t_3 < D$, then $U(t_2) - U(t_1)$ and $U(t_3) - U(t_2)$ are independent random variables which are Poisson distributed with parameters $\lambda(t_2 - t_1)$ and $\lambda(t_3 - t_2)$ respectively. Let $K(\theta, t)$ be a characteristic function of a probability distribution which depends on t and is well-defined for every t in (C,D). That is

$$K(\theta, t) = \int_{-\infty}^{\infty} (\exp i\theta x)\, dH_t(x)$$

where H is a c.d.f. for every t in (C,D). Now define a process $Y(\ )$ as follows:

If t is a jump point of $U(\ )$ then $Y(\ )$ jumps by an amount which has the c.d.f. H_t. That is, given that $T_1 < T_2 < \ldots$ are the jump points of $U(\ )$ let $J_1, J_2, \ldots$ be a sequence of random variables which are independent and have c.d.f.'s $H_{T_1}, H_{T_2}, \ldots$ respectively. Now define $Y(\ )$ as follows:

$$Y(t) = \begin{cases} 0, & C < t < T_1, \\ J_1, & T_1 \leq t < T_2, \\ J_1 + J_2, & T_2 \leq t < T_3 \\ \vdots & \end{cases}$$

If there are no jumps of $U(\ )$ in (C,D) then $Y(t) = 0$ for all t in (C,D). If there are only a finite number N of jumps in (C,D) then $Y(t) = J_1 + \ldots + J_N$ for t in (T_N, D).

Lemma 4.1 $Y(\ )$ is a process with independent increments and

$$E(\exp i\theta Y(t)) = \exp - \lambda(t-C) - \lambda \int_C^t K(\theta,s)ds = \exp - \lambda(\int_C^t (1-K(\theta,s))ds \quad (4.1)$$

Proof That $Y(\ )$ has independent increments is evident from the fact that $U(\ )$ has independent increments. To evaluate the characteristic function of $Y(t)$ we use the fact that given that there are n jump points in (C,t), their locations are distributed the same as that of n points independently selected at random in that interval. Hence

$$E(\exp i\theta Y(t)) =$$

$$\sum_{n=0}^{\infty} \left[n! \int_{C < u_1 < \ldots u_n < t} \cdots \int \frac{K(\theta,u_1) \ldots K(\theta,u_n)du_1 \ldots du_n}{(t-C)^n} \right] [\lambda(t-C)]^n \, e^{-\lambda(t-C)} / n!$$

(The integral is understood to be 1 for $n = 0$.) We use the fact that

$$n! \int_{C < u_1 < \ldots < u_n < t} \cdots \int K(\theta,u_1) \ldots K(\theta,u_n)du_1 \ldots du_n = \left[\int_C^t K(\theta,u)du \right]^n$$

Hence the assertion about the characteristic function of $Y(t)$ follows immediately.

If the jumps happen to be non-negative integer-valued random variables we will find it more convenient to deal with the probability generating function of $Y(t)$. In this case if we understand $K(\theta,u)$ to be the probability generating function of a jump at time u then $E\theta^{Y(t)}$ equals the right side of (4.1) with no further changes, $(|\theta| \leq 1)$. Similarly, if the jumps are positive it will be convenient for $K(\theta,t)$ to be the Laplace transform of a jump at time u.

We point out that if $K(\theta,s) = K(\theta)$ does not depend on s then $Y(\ )$ is a conventional compound Poisson process. We shall refer to $Y(\ )$ in general as an integrated compound Poisson (ICP process).

We now will show that under certain circumstances there exists a function $D(\ )$ on $(0,\infty)$ such that $Z(D(u))$ is a process of the type described in Lemma 4.1, namely an ICP process.

Suppose that F is continuous in (A,B). Then there exists a monotone non-decreasing function R on $(0,1)$ such that

$$F(R(u)) = u, \ u \text{ in } (0,1).$$

If F is strictly increasing in (A,B) then $R = F^{-1}$, the inverse function of F. If F is continuous but not strictly increasing then R is not uniquely determined but we suppose we choose one particular version of R once and for all. We now define a function D by

$$D(u) = R\left(\frac{u}{1+u}\right), \ u \text{ in } (0,\infty).$$

Theorem 4.1 Suppose that F is continous. Then $Z(D(u))$ for u in $(0,\infty)$ defines an ICP process with jump distribution determined by

$$K(\theta,s) = E\theta^J = (s + (1-s)\theta)/(1 + s - \theta s)$$

$$= \frac{s}{1+s} + \sum_{k=1}^{\infty} \frac{s^{k-1}}{(1+s)^k} \theta^k$$

and $\lambda = 1$.

Proof We know that $Z(\ )$ is a process with independent increments so the same is true of $Z(D(\ ))$. We have that

$$E\theta^{Z(D(u))} = G(D(u),\theta) = \frac{1 - F(D(u))}{1 - \theta F(D(u))} = \frac{1}{1 + u - \theta u}$$

Let us now compute $E\theta^{Y(u)}$ for the ICP process with $\lambda = 1$ and the above indicated K. An easy computation shows that

$$\int_0^t K(s,\theta)ds = t - \log(1 + t(1-\theta))$$

Hence, for all positive u,

$$E\theta^{Y(u)} = \frac{1}{1 + u - \theta u}$$

This completes the proof.

We are now able to show that the number of times that M_n rises to new heights between height a and height b is Poisson distributed with parameter $\log[(1 - F(a))/(1 - F(b))]$. Let us describe this more carefully. We say that a record is set at time n and at height x if

$$x = M_n > \max(X_1,\ldots,X_{n-1}),\ n > 1$$

We agree that a record is automatically set at time $n = 1$ at height X_1.

Theorem 4.2 Let $V(t)$ equal the number of records set with heights no greater than t. If F is continuous then $V(\ )$ is a process with independent increments which are Poisson distributed. Specifically, if $A < a < b < B$ then $V(b) - V(a)$ is Poisson distributed with parameter

$$\log [(1 - F(a))/(1 - F(b))].$$

Proof By Theorem 4.1, the process $Z(D(\ ))$ is an ICP process. The process $U(\ )$ which counts the jumps of this ICP process is a Poisson process. Hence $V(\ )$, the process which counts the jumps of $Z(\ )$ also has independent increments which are Poisson distributed. It remains to determine the parameters of these Poisson distributions. We have that

$$P(V(b) - V(a) = 0) = H(a,b,0) = \frac{1 - F(b)}{1 - F(a)} = \exp\text{-}\log [(1\text{-}F(a))/(1\text{-}F(b))]$$

which completes the proof. (For a direct proof, see Theorem 2.2 of [1].)

5. EXTREMAL PROCESS.

We now consider the counterpart of the above results for extremal processes. (See [1], [2] for the definition and a discussion of these processes.) These processes are defined in terms of a monotone, non-increasing, non-negative function Q whose total variation is concentrated on $[a,b]$ where $-\infty \le a < b \le \infty$. A representation of an extremal process $Y(t), 0 < t < \infty$ is given in terms of Q in Section 4 of [2] as follows: Under the condition that $Y(t) = x$, (x in (a,b)), $Y(\ )$ remains equal to x for a random amount of time $W_1/Q(x)$. The process then jumps to a height Z_1 greater than x where Z_1 has the c.d.f. R_x defined by

$$R_x(u) \begin{cases} 0, & u \leq y, \\ 1 - Q(u)/Q(x), & u > y \end{cases}$$

The process then remains at height Z_1 a random amount of time $W_2/Q(Z_1)$, and so forth. The random variables $W_1, W_2, \ldots$ are independent, exponential (parameter 1), independent of all other random variables under consideration.

We now define the inverse $Z(\ )$ to the extremal process $Y(\ )$, where

$$Z(u) = \sup(t \mid Y(t) \leq u),\ u \text{ in } (a,b),\ (\text{c.f. } (2.1)) \tag{5.1}$$

In accordance with the above representation of $Y(\ )$ we can give the following representation for $Z(\ )$: If $Z(u) = z$ then $Z(\ )$ remains equal to z for an additional amount of time $R(u,z)$ where

$$P(R(u,z) > s\ ) = Q(u)/Q(s) \tag{5.2}$$

Given that $Z(u) = z$ and $Z(\ )$ remains equal to z until time s according to the distribution described by (5.2), then $Z(\ )$ jumps to a new height $z + J$ where J is exponentially distributed with parameter $Q(z)$. That is

$$P(J > v) = \exp - vQ(z),\ v > 0 \tag{5.3}$$

Notice the formal similarity between (5.1),(5.2),(5.3) and (2.1),(2.2), (2.3). We refer to the fact that for $t > 0$,

$$P(Y(t) < u) = \exp - tQ(u) \quad (\text{see } [\ 2\ ])$$

Hence,

$$P(Z(u) > t) = P(Y(t) < u) = \exp - tQ(u)$$

and we have the Laplace transform,

$$E \exp - \theta Z(u) = \frac{Q(u)}{Q(u) + \theta}, \ u \text{ in } (a,b), \ \theta > 0 \tag{5.4}$$

In a manner which is wholly analogous to that of Section 3, we can now construct an additive process $\hat{Z}(\ )$ with the following properties:

(a) $Z(u)$ is exponentially distributed with Laplace transform given by (5.4).

(b) If $a < t_1 < t_2 < \ldots < t_k < b$ then the increments $\hat{Z}(t_1), \hat{Z}(t_2) - \hat{Z}(t_1), \ldots, \hat{Z}(t_k) - \hat{Z}(t_{k-1})$ are mutually independent and $\hat{Z}(t_i) - \hat{Z}(t_{i+1})$ has Laplace transform

$$\frac{Q(t_2)}{Q(t_1)} \left[\frac{Q(t_1) + \theta}{Q(t_2) + \theta}\right]$$

(This transform is analogous to the generating function (3.1).)
The following theorem is comparable to Theorem 3.1.

<u>Theorem 5.1</u> The processes $\hat{Z}(\ )$ and $Z(\ )$ agree, in the sense that $\hat{Z}(t_1), \hat{Z}(t_2) - \hat{Z}(t_1), \ldots, \hat{Z}(t_k) - \hat{Z}(t_{k-1})$ and $Z(t_1), Z(t_2) - Z(t_1), \ldots, Z(t_k) - Z(t_{k-1})$ have the same joint distributions.

<u>Proof</u> A proof can be modelled after the proofs of Theorem 3.1 and we leave the details in that direction to the reader. For another proof we offer the following: According to Theorem 9.2 of [2] it is evident that

that $\hat{Z}(\ )$ has independent increments, as does $Z(\ )$. Hence to prove the theorem it need only be shown that for any t in (a,b) $\hat{Z}(t)$ and $Z(t)$ are identically distributed. But this is true from the definition of $\hat{Z}(\ )$ and from (5.4).

We now prove a counterpart of Theorem 4.1. First we point out that if Q is continuous on (a,b) then there exists a function R such that

$$Q(R(u)) = u, \quad u \text{ in } (0,\infty).$$

We now define a function D by

$$D(u) = R\left(\frac{1}{u+1} \right)$$

Note that D is a monotone non-decreasing function of u.

<u>Theorem 5.2</u> Suppose that Q is continuous in (a,b). Then the process $Z(D(\ ))$ is distributed like $Y_0 + Y(\ )$ where $Y(\ )$ is an ICP process on the time interval $(0,\infty)$ with jump distribution whose Laplace transform is

$$K(\theta,s) = E\exp{-\theta J} = \frac{1 + \theta u}{1 + \theta(u+1)}, \quad \theta \text{ and } u \text{ in } (0,\infty)$$

and $\lambda = 1$, and Y_0 is an exponential (parameter 1) random variable independent of the process $Y(\ )$.

<u>Proof</u> Since $Z(\ )$ has independent increments the same is true of $Z(D(\ ))$. We have that

$$\text{Eexp}-\theta Z(D(u)) = \frac{Q(R(\frac{1}{u+1}))}{Q(R(\frac{1}{u+1}))+\theta} = \frac{1}{1 + (u+1)\ \theta}$$

Let us now compute $\text{Eexp}-\theta Y(u)$ for the ICP process $Y(\ )$ with $\lambda = 1$ and the above K.

$$\int_0^u (1 - K(\theta,s))ds = \log \frac{1 + \theta(u+1)}{1 + \theta}$$

Hence

$$\text{Eexp}-\theta Y(u) = \frac{1 + \theta}{1 + \theta(u+1)}$$

and

$$\text{Eexp}-\theta(Y_0 + Y(u)) = \frac{1}{1 + \theta(u+1)}$$

Since $Z(D(\ ))$ and $Y_0 + Y(\ )$ are both processes with independent increments, this completes the proof.

From Theorem 5.2 one can easily prove the counterpart of Theorem 4.2. Since this appears in [2] as Theorem 5.1, (b), (c) we leave the details to the reader.