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# AN INTEGRATION OF EQUILIBRIUM THEORY AND TURNPIKE THEORY *

by

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## 1) Introduction

This paper indicates one way to link equilibrium theory with capital theory and especially with turnpike theory. I consider a model with finitely many, infinitely lived consumers. Their utility functions are additively separable with respect to time and they discount future utility. There are finitely many, infinitely lived firms. Primary resources are necessary for production and their supply is constant over time. Technology and utility functions do not change over time either. The model is simply a general equilibrium model with infinitely many commodities. The infinity arises because the horizon is infinite and commodities are distinguished according to date.

I use results from a previous paper of my own [1972] in order to prove that the model has an equilibrium, for any strictly positive vector of initial resources. I also prove that the initial resources may be chosen so that there exists a stationary equilibrium. I prove the following analogue of Scheinkman's turnpike theorem [1976]. Suppose that all consumers discount future utility at the same rate. Then, the equilibrium allocation converges provided that the consumers' common rate of time preference is sufficiently close to zero. (The rate of time preference is the interest rate used in discounting future utility.)

Finally, I prove that if consumers do not all have equal rates of time preference, then the less patient consumers eventually consume nothing in equilibrium. The less patient consumers are those whose rates of time preference exceed the smallest rate among all consumers. In an equilibrium, the less patient mortgage all their future income beyond a certain date so as to consume more, earlier.

It is easy to relate the above results to capital theory. In capital theory, authors tend to use a reduced form, aggregate model in which a single utility function is defined directly on a single intertemporal production possibility set. The following is one of the many maximization problems studied in capital theory.

$$1.1) \qquad \max \{ \sum_{t=0}^{\infty} \delta^t u(k_t, k_{t+1}) \mid (k_t, k_{t+1}) \in D \ , \text{ for all } t \ ,$$

$$k_0 = \bar{k}_0 \} \ , \text{ where } \bar{k}_0 \text{ is given.}$$

In this problem, $t$ is the index for time. $\delta$ is the discount factor applied to future utility, where $0 < \delta < 1$ . $k_t$ is the vector of capital stocks at time $t$ . $u$ is the utility function, and $D$ is the intertemporal production possibility set.

If one makes appropriate assumptions, it is not hard to prove that (1.1) has a solution. A solution corresponds to an equilibrium in my model. Sutherland [1970] and Peleg and Ryder [1974] proved that one may choose $\bar{k}_0$ so that one solution to (1.1) is stationary. ( A solution is stationary if $k_t = \bar{k}_0$ , for all $t$ .) Such a stationary optimum is known as the modified golden rule. It corresponds to a stationary equilibrium in my case. Scheinkman [1976] proved that under appropriate conditions, any solution to (1.1) converges to a unique stationary optimum, provided that $\delta$ is sufficiently close to one.

The idea that less patient consumers eventually consume nothing may be found at the end of Ramsey's early paper [1928]. The same idea occurs in Rader [1971] and Becker [1978 and forthcoming].

I discuss the literature more thoroughly in section 6 .

There is a subtle difference between the turnpike theorem of capital theory and the one I prove. The turnpike theorem of capital theory asserts that optimum paths converge to a unique stationary optimum. Thus, the limit is independent of the initial conditions. In my case, the limit of an equilibrium is not necessarily a stationary equilibrium and the limit depends on the initial conditions. Also, stationary equilibrium is not necessarily unique. Stationary equilibrium is not necessarily unique for the same reason that equilibrium may not be unique for the Edgeworth box. The initial conditions affect the limit of an equilibrium because they affect the relative wealths of the consumers. Also, because conditions change over time, some consumers may borrow or lend early in time. For this reason, they may be paying or earning interest in the asymptotic state approached as the equilibrium converges. Thus, the limit is not necessarily a stationary equilibrium. The limit may, however, be labeled a stationary equilibrium with transfer payments. In such an equilibrium, consumers maximize utility on budget sets modified by lump-sum subsidies or taxes. The taxes should be thought of as interest payments.

It is easy to see why the turnpike theorem applies to equilibrium. I assume that all utility functions are concave. Hence, equilibrium maximizes a weighted sum of consumers' utility functions, the weights being the inverses of the marginal utilities of expenditure. Since I assume that all consumers discount future utility at the same rate, the maximand may be written as

$$1.2) \qquad \sum_{t=0}^{\infty} \delta^t \sum_i \Lambda_i^{-1} u_i(x_i^t) \quad .$$

In this expression, $i$ is the index for consumers, $\Lambda_i$ is the marginal utility of expenditure for consumer $i$, $u_i$ is his utility function, and $x_i^t$ is his consumption vector at time $t$. $\delta$ is the discount factor applied to future utility.

(1.2) looks much like the objective function in (1.1). Hence, a version of Scheinkman's theorem should imply that equilibrium converges.

In fact, I do not apply Scheinkman's theorem or any of the recent generalizations of it. Instead, I provide a direct proof of the convergence result. I do so for three reasons. 1) I do not want to make unnecessary assumptions. 2) I obtain exponential convergence, which is stronger than that of corresponding theorems in the literature. 3) My method of proof seems to improve on existing methods.

My proof is in many ways simply a modification of existing proofs. I use the value loss method. My main innovation is to use a one-sided value loss rather than a two-sided value loss. This value loss is easy to interpret and leads to many simplications.

Nevertheless, my proof is very long and complicated. The complications arise largely because I use a full general equilibrium model rather than the reduced form, aggregate model of capital theory.

If one assumes that there are one consumer and one firm, then my turnpike theorem becomes a turnpike theorem in the sense of capital theory can can be compared with theorems in the literature. In this case, my result is neither more general nor more special than existing ones. I elaborate in section 6.

I emphasize that my goal is only to link two distinct branches of economic theory. I do not claim that my model is realistic or that it justifies capital theory. The assumption of immortality is certainly not realistic. Also, prices in my model can be interpreted only as Arrow-Debreu prices of contracts for future delivery. Such prices seem especially unrealistic when there is an infinite horizon.

I emphasize that I cannot avoid interpreting prices as prices for forward contracts. I cannot interpret prices as spot prices and say that agents have perfect foresight. This last point of view is the one sometimes taken in capital theory. However in my model, consumers may borrow and lend, which means that there must be forward markets. In capital theory, there is only one consumer, and he owns the firm or firms. Hence, it is impossible for the consumer to borrow or lend.

By linking equilibrium theory and capital theory, I do give some insight into the nature of the assumptions that must be made in order to obtain the turnpike property. Dealing with a general equilibrium model obliges one to state assumptions only in terms of individual utility functions, endowments and production possibility sets. Assumptions in capital theory do not always have a concrete interpretation, since the models are aggregated.

I make strong assumptions. I assume that utility functions are strictly concave and that production possibility sets are strictly convex. The assumption about production possibility sets is especially strong, for it excludes constant returns to scale.

In order to exploit strict convexity, I must assure that price ratios exactly equal marginal rates of transformation in production. (The prices referred to are in the limit stationary equilibrium with transfer payments.) I assure equality by assuming that firms can use inputs and produce outputs efficiently in any ratios they like. This assumption excludes the fixed coefficients, linear production model.

There exists in capital theory a form of turnpike theorem which applies to models with constant returns to scale and fixed coefficients. If one allows constant returns to scale, then there can exist optimal programs which oscillate indefinitely, even if future utility is not discounted. (I give an example in section 5.) Therefore, the turnpike theorem is not valid if one insists on convergence to a point. However, one can ask that optimal programs converge only to a set of programs known as the von Neumann facet. The von Neumann facet includes the set of optimal programs which are stationary or oscillating. McKenzie [1963, 1968] has proved that if future utility is not discounted, then optimal programs do converge to the von Neumann facet. However, it is not known whether optimal programs converge to the facet when future utility is discounted. If such a turnpike theorem were valid, then it would be possible to eliminate my most objectionable assumptions. (McKenzie [1979] has proved that given $\varepsilon > 0$, optimal programs eventually stay within an $\varepsilon$-neighborhood of the facet if the discount factor is sufficiently close to one. I am insisting on convergence to that facet with a given discount factor.)

## 2) Definitions, Notation and the Model

### Commodities

There are $L$ types of commodities. $L_c \subset \{1,\ldots,L\}$ denotes the set of consumption goods. $L_o \subset \{1,\ldots,L\}$ denotes the set of primary commodities, such as land, labor and raw materials. $L_p = \{ k = 1,\ldots,L \mid k \notin L_o\}$ denotes the set of producible goods. Goods not in either $L_c$ or $L_o$ should be thought of as intermediate goods or goods in process.

### Vector Space Notation

$R^L$ denotes L-dimensional Euclidean space. A standard subspace of $R^L$ is one of the form $R^{L'} = \{ x \in R^L \mid x_k = 0 \text{ if } k \notin L'\}$, where $L' \subset \{1,\ldots,L\}$. $R^{L'}$ is said to be the subspace corresponding to $L'$. $R^{L_c}$, $R^{L_o}$ and

$R^{L_p}$ are the subspaces corresponding to $L_c$, $L_o$ and $L_p$, respectively. It is important to keep in mind that vectors in $R^{L_c}$, $R^{L_o}$ and $R^{L_p}$ are thought of as belonging to $R^L$.

$|\cdot|$ denotes the maximum norm of any Euclidean space. That is, $|z| = \max_k |z_k|$.

A vector $z = (z_1, \ldots, z_N)$ will often be written simply as $(z_n)$. The $z_n$ may themselves be vectors.

$\ell_{\infty,L}$ denotes $\{\mathbf{x} = (x^0, x^1, \ldots) \mid x^t \in R^L$, for all $t$, and $\sup_t |x^t| < \infty\}$.

$\ell_{\infty,L_c}$, $\ell_{\infty,L_p}$ and $\ell_{\infty,L_o}$ are the subspaces of $\ell_{\infty,L}$ corresponding $L_c$, $L_p$ and $L_o$, respectively. If $\mathbf{x} \in \ell_{\infty,L}$, $|\mathbf{x}|$ denotes $\sup_t |x^t|$.

$|\cdot|$ is called the supremum norm.

$\ell_{1,L}$ denotes $\{\mathbf{p} = (p^0, p^1, \ldots.) \mid p^t \in R^L$, for all $t$, and $\sum_{t=0}^{\infty} |p^t| < \infty\}$.

Infinite dimensional vectors are always written in bold face. However, components of infinite dimensional vectors are not written in bold face. Finite dimensional subvectors are not written in bold face either. Thus, $\mathbf{x} = (x^0, x^1, \ldots.) \in \ell_{\infty,L}$, where $x^t \in R^L$, for all $t$. $x_k^t$ is a component of $\mathbf{x}$. $\mathbf{0}$ denotes an infinite sequence of zeros.

If $\mathbf{x} \in \ell_{\infty,L}$ and $\mathbf{p} \in \ell_{1,L}$, then $\mathbf{p} \cdot \mathbf{x}$ denotes $\sum_{t=0}^{\infty} p^t \cdot x^t$.

If $x \in R^L$, then "$x \geqq 0$" means "$x_k \geqq 0$, for all k." "$x > 0$" means "$x \geqq 0$ and $x \neq 0$." "$x >> 0$" means "$x_k > 0$, for all k." $R^L_+$ denotes $\{x \in R^L | x \geqq 0\}$, and $R^L_-$ denotes $\{x \in R^L \mid x \leqq 0\}$.

Let $\mathbf{x} \in \ell_{\infty,L}$ or $\mathbf{x} \in \ell_{1,L}$. Then, "$\mathbf{x} \geqq 0$" means "$x_k^t \geqq 0$, for all $t$ and $k$ ." "$\underset{\sim}{x} > 0$" means "$\underset{\sim}{x} \geqq 0$ and $\underset{\sim}{x} \neq 0$ ."

"$\underset{\sim}{x} >> 0$" means "$x_k^t > 0$, for all $t$ and $k$ ." Finally, "$\underset{\sim}{x} >>> 0$" means "There exists a positive number $r$ such that $x_k^t > r$, for all $t$ and $k$." $\ell_{\infty,L}^{+}$ donotes $\{\underset{\sim}{x} \in \ell_{\infty,L} \mid \underset{\sim}{x} \geqq 0\}$. $\ell_{1,L}^{+}$ is defined similarly.

## Consumers

There are I consumers, where I is a positive integer. The utility function of consumer i for consumption in one period is $u_i : R_+^{L_c} \to (-\infty,\infty)$. Utility is additively separable with respect to time and consumer i discounts future utility by a factor $\delta_i$, where $0 < \delta_i < 1$. That is, if consumer i consumes the bundle $x_t \in R_+^{L_c}$ in period t, for $t = 0,1,2,\ldots,$ then his total utility from the point of view of period zero is

$$\sum_{t=0}^{\infty} \delta_i^t \; u_i(x^t).$$

The endowment of each consumer in each period is $\omega_i \in R_+^{L_o}$ . Notice that each consumer is endowed only with primary goods. This assumption is not necessary. It is made only for convenience.

## Firms

There are J firms, where J is a positive integer. A firm transforms inputs $y_0 \in R_-^L$ in one period into outputs $y_1 \in R_+^{L_p}$ in the succeeding period. Inputs carry a negative sign and outputs a positive sign. The production possibility set of firm $j$ is $Y_j \subset R_-^L \times R_+^{L_p}$. $y = (y_0, y_1)$ denotes a typical vector in $Y_j$, where $y_0 \in R_-^L$ and $y_1 \in R_+^{L_p}$.

Firms have an endowment of produced goods, available at time zero. These goods should be thought of as having been produced from inputs in period $-1$. The vector of goods available to firm $j$ is denoted by $y_{j1}^{-1} \in R_+^{L_p}$. $\sum_{j=1}^{J} y_{j1}^{-1}$ is the initial capital stock of the economy.

Firms are owned by consumers. Consumer $i$ owns a proportion $\theta_{ij}$ of firm $j$, where $i = 1,\dots,I$ and $j = 1,\dots,J$. $0 \leq \theta_{ij} \leq 1$, for all $i$ and $j$, and $\sum_i \theta_{ij} = 1$, for all $j$.

## The Economy

The economy is described by the list $\mathcal{E} = \{(u_i, \omega_i), (Y_j, y_{j1}^{-1}), \theta_{ij} :$ $i = 1,\dots,I$ and $j = 1,\dots,J\}$.

## Allocations

A consumption program for a particular consumer is of the form $\underset{\sim}{x} = (x^0, x^1, \dots)$, where $x^t \in R_+^{L_c}$, for all $t$ and $|\underset{\sim}{x}| < \infty$. That is, $\underset{\sim}{x} \in \ell^+_{\infty, L_c}$. $x^t$ is the consumption vector at time $t$. A consumption program is said to be stationary if $x^t = x^0$, for all $t$.

A production program is of the form $\underset{\sim}{y} = (y^0, y^1, \dots)$, where $y^t = (y_0^t, y_1^t) \in R_-^L \times R_+^{L_p}$, for all $t \geq 0$, and $|\underset{\sim}{y}| = \sup_t |y^t| < \infty$. The program is feasible for firm $j$ if $y^t \in Y_j$, for all $t \geq 0$. A production program is said to be stationary if $y^t = y^0$, for all $t$.

An allocation for the economy is of the form $((\underset{\sim}{x}_i), (\underset{\sim}{y}_j))$, where each $\underset{\sim}{x}_i = (x_i^0, x_i^1, \dots,)$ is a consumption program and each $\underset{\sim}{y}_j = (y_j^0, y_j^1, \dots)$ is a production program feasible for firm $j$. The allocation $((\underset{\sim}{x}_i), (\underset{\sim}{y}_j))$ is

<u>feasible</u> if $\sum_i x_i^t \leq \sum_i \omega_i + \sum_j (y_{j0}^t + y_{j1}^{t-1})$ , for all $t \geq 0$ . Notice that the feasibility of an allocation depends on the endowments $y_{j1}^{-1}$ of the firms. Also, the definition of feasibility implies free disposability.

$((\underset{\sim}{x}_i), (\underset{\sim}{y}_j))$ is said to be <u>stationary</u> if each of the programs $\underset{\sim}{x}_i$ and $\underset{\sim}{y}_j$ are stationary and if in addition $y_{j1}^t = y_{j1}^{-1}$ , for all $t$ .

The notation $((\underset{\sim}{x}_i), (\underset{\sim}{y}_j))$ should not be confused with the notation $((x_i^t), (y_j^t))$ , which is the vector of allocations at time $t$ .

## Pareto Optimality

A feasible allocation $((\underset{\sim}{x}_i), (\underset{\sim}{y}_j))$ is said to be Pareto optimal if there exists no feasible allocation $((\bar{\underset{\sim}{x}}_i), (\bar{\underset{\sim}{y}}_j))$ such that $\sum_{t=0}^{\infty} \delta_i^t u_i(\bar{x}_i^t) \geq \sum_{t=0}^{\infty} \delta_i^t u_i(x_i^t)$ , for all $i$ , with strict inequality for some $i$ .

## Prices

A <u>price system</u> is simply a non-zero vector $\underset{\sim}{p}$ in $\ell_{1,L}^{+}$ . $\underset{\sim}{p}$ is of the form $(p^0, p^1, \ldots)$ , where $p^t \in R_+^L$ is the vector of prices in period $t$ . $p_k^t$ is the price of commodity $k$ in period $t$. $\underset{\sim}{p}$ is said to be <u>stationary</u> if $p^t = \delta^t p^0$ , for all $t$ , where $0 < \delta < 1$ .

If $\underset{\sim}{x} \in \ell_{\infty,L}$ , then $\underset{\sim}{p} \cdot \underset{\sim}{x} = \sum_{t=0}^{\infty} p^t \cdot x^t$ .

## Profit Maximization

Given the price system $\underset{\sim}{p}$ , each firm chooses a program so as to maximize its profit. That is, the problem of firm $j$ is

$$\max \left\{ \sum_{t=0}^{\infty} (p^t \cdot y_0^t + p^{t+1} \cdot y_1^t) \;\middle|\; y \text{ is a production program feasible for firm } j \right\}.$$

$\eta_j(\underset{\sim}{p})$ denotes the set of solutions to this problem. $\pi_j(\underset{\sim}{p})$ denotes the maximum profit plus the value of the firm's initial endowment. That is,

$$\pi_j(\underset{\sim}{p}) = p^0 \cdot y_{j1}^{-1} + \sum_{t=0}^{\infty} (p^t \cdot y_0^t + p^{t+1} \cdot y_1^t) \quad ,$$

where $(y^0, y^1, \dots) \in \eta_j(\underset{\sim}{p})$. $\eta_j(\underset{\sim}{p})$ and $\pi_j(\underset{\sim}{p})$ may be empty.

## Utility Maximization

Given the price system $\underset{\sim}{p}$ , consumer i's budget set is $\beta_i(\underset{\sim}{p}) = \{\underset{\sim}{x} \in \ell^+_{\infty,L_c} \mid \underset{\sim}{p} \cdot \underset{\sim}{x} \leq \sum_{t=0}^{\infty} p^t \cdot \omega_i + \sum_{j=1}^{J} \theta_{ij}\pi_j(\underset{\sim}{p})\}$ . His maximization problem is

2.1) $$\max \{ \sum_{t=0}^{\infty} \delta_i^t u_i(x^t) \mid \underset{\sim}{x} \in \beta_i(\underset{\sim}{p})\} \quad .$$

$\xi_i(\underset{\sim}{p})$ denotes the set of solutions to this problem. $\xi_i(\underset{\sim}{p})$ may be empty.

## Equilibrium

An equilibrium consists of $((\underset{\sim}{x}_i), (\underset{\sim}{y}_j), \underset{\sim}{p})$ .

2.2) $((\underset{\sim}{x}_i), (\underset{\sim}{y}_j))$ is a feasible allocation,

2.3) $\underset{\sim}{p}$ is a price system and, for all t and k, $p_k^t = 0$ if $\sum_{i=1}^{I} x_k^t < \sum_{i=1}^{I} \omega_{ik} + \sum_{j=1}^{J} (y_{j0k}^t + y_{j1k}^{t-1})$ .

2.4) $\underset{\sim}{y}_j \in \eta_j(\underset{\sim}{p})$, for all j , and

2.5) $\underset{\sim}{x}_i \in \xi_i(\underset{\sim}{p})$ , for all i .

An equilibrium with transfer payments consists of $((\underset{\sim}{x}_i), (\underset{\sim}{y}_j), \underset{\sim}{p})$, where these satisfy conditions 2.2 - 2.4 and

2.6) for each i , $\underset{\sim}{x}_i$ solves the problem $\max \{ \sum_{t=0}^{\infty} \delta_i^t u_i(x^t) \mid \underset{\sim}{x} \in \ell^+_{\infty,L_c}$ and $\underset{\sim}{p} \cdot \underset{\sim}{x} \leq \underset{\sim}{p} \cdot \underset{\sim}{x}_i\}$ .

The transfer payment made by consumer $i$ is $\sum_{t=0}^{\infty} p^t \cdot \omega_i + \sum_{j=1}^{J} \theta_{ij} \pi_j(\underset{\sim}{p}) - \underset{\sim}{p} \cdot \underset{\sim}{x}_i$ .

An equilibrium is said to be stationary if the allocation $((\underset{\sim}{x}_i),(\underset{\sim}{y}_j))$ and the price system $\underset{\sim}{p}$ are all stationary.

## Notation for Stationary Sequences

I use the following notation for stationary allocation and price systems. If $((\underset{\sim}{x}_i), (\underset{\sim}{y}_j))$ is a stationary allocation, then $x_i \in R_+^{Lc}$ and $y_j \in Y_j$ denote the corresponding consumption bundles and input-output vectors at one moment of time. Thus, $\underset{\sim}{x}_i = (x_i, x_i, \ldots.)$ and $\underset{\sim}{y}_j = (y_j, y_j, \ldots)$ . Also, if $\underset{\sim}{p}$ is a stationary price system, then $p \in R_+^L$ denotes the corresponding vector of prices at time zero. Thus, $\underset{\sim}{p} = (p, \delta p, \delta^2 p, \ldots.)$ .

## Convergence of Allocations

Let $((\bar{\underset{\sim}{x}}_i),(\bar{\underset{\sim}{y}}_j))$ be a stationary allocation. An allocation $((\underset{\sim}{x}_i),(\underset{\sim}{y}_j))$ is said to converge to $((\bar{\underset{\sim}{x}}_i), (\bar{\underset{\sim}{y}}_j))$ if

$$\lim_{t\to\infty} \left| ((x_i^t), (y_j^t)) - ((\bar{x}_i), (\bar{y}_j)) \right| = 0 \quad .$$

The convergence is said to be exponential if there is $a > 0$ such that $0 < a < 1$ and $\left| ((x_i^t), (y_j^t)) - ((\bar{x}_i), (\bar{y}_j)) \right| < a^t$ , for all sufficiently large $t$ .

## Marginal Utilities of Expenditure

Corresponding to any equilibrium $((\underset{\sim}{x}_i), (\underset{\sim}{y}_j), \underset{\sim}{p})$ , there is a vector of marginal utilities of expenditure, $\Lambda = (\Lambda_1, \ldots, \Lambda_I)$. Each $\Lambda_i$ is simply the Lagrange multiplier corresponding to the budget constraint in consumer $i$'s utility maximization problem (2.1).

## 3) Assumptions

I list below the assumptions I use. Some have already been mentioned.

Non-Triviality

3.1) $L_c \cap L_o \neq \emptyset$ . $I \geqq 1$ and $J \geqq 1$.

Consumers

3.2) $\omega_i \in R_+^{L_o}$ , for all $i$ .

3.3) $u_i : R_+^{L_c} \to (-\infty,\infty)$ is twice continuously differentiable.

$Df$ and $D^2f$ denote the first and second derivatives, respectively, of the function $f$ .

3.4) (Strict Monotonicity) For all $i$ , $Du_i(x) >> 0$ , for all $x \in R_+^{L_c}$ .

3.5) (Strict Concavity) For all $i$ , $D^2 u_i(x)$ is negative definite, for all $x \in R_+^{L_c}$ .

Firms

3.6) $y_{j1}^{-1} \in R_+^{L_p}$ , for all $j$ .

I represent production in the following way. For each $j = 1,\ldots,J$, let $M_{j0}$ and $M_{j1}$ be subspaces of $R^L$ and $R^{L_p}$ , respectively. Let $M_{j0}^- = M_{j0} \cap R_-^L$ and let $M_{j1}^+ = M_{j1} \cap R_+^{L_p}$ . Finally, let $g_j : M_{j0}^- \times M_{j1}^+ \to R$.

3.7) $Y_j = \{y \in M_{j0}^- \times M_{j1}^+ \mid g_j(y) \leqq 0\}$, for all $j$ .

3.8) For all $j$ , $M_{j0}$ and $M_{j1}$ are standard subspaces of $R^L$ .

3.9) For all $j$ , $g_j$ is twice continuously differentiable.

3.10) For all $j$ , $Dg_j(y) >> 0$ , for all $y \in M_{j0}^- \times M_{j1}^+$ .

3.11) For all $j$ and for all $y \in M_{j0}^- \times M_{j1}^+$, $D^2 g_j(y)$ is positive definite on the subspace of $M_{j0} \times M_{j1}$ orthogonal to $Dg_j(y)$ .

This assumption says that production possibility frontiers have positive curvature. In other words, production possibility sets are differentiably strictly convex.

3.12) (Possibility of zero production) $g_j(0) = 0$ , for all j .

3.13) (Necessity of primary inputs) The following is true, for all j . Let $y = (y_0, y_1) \in M_{j0} \times M_{j1}$ . If $y_1 > 0$ and $y_{0k} = 0$ , for all $k \in L_o$ , then $g_j(y) > 0$.

<u>Adequacy</u>

The final assumptions guarantee that no consumer would have a zero income in equilibrium.

3.14) For each i , $\omega_{ik} > 0$ , for some $k \in L_c \cap L_o$ .

That is, every consumer is endowed with some primary good, such as labor, which is also a consumption good.

3.15) $\sum_{i=1}^{I} \omega_{ik} > 0$ , for all $k \in L_o$.

That is, there is a postive endowment of every primary good.

3.16) There are $\bar{\omega} \in R_+^{L_o}$ and $(y_{j0}, y_{j1}) \in Y_j$ , for $j = 1,\ldots,J$ , such that $\bar{\omega} + \sum_{j=1}^{J} (y_{j0} + y_{j1}) >> 0$.

That is, it is possible to produce some of every good in every period while using only primary inputs from outside the production system.

4) <u>Theorems</u>

I assume that assumptions 3.1 - 3.16 apply.

4.1) Theorem Suppose that $\sum_{j=1}^{J} y_{j1k}^{-1} > 0$, for all $k \in L_p$. Then there exists an equilibrium.

4.2) Theorem Suppose that $\delta_i = \delta$, for all i. If $\delta$ is sufficiently close to one, then $(y_{j1}^{-1})_{j=1}^{J}$ may be chosen so that an equilibrium exists which is stationary. The equilibrium price vector is of the form $\underset{\sim}{p} = (p, \delta p, \delta^2 p, \ldots)$ .

4.3) Theorem Any equilibrium allocation is Pareto optimal.

4.4) Theorem Let $((\underset{\sim}{x}_i), (\underset{\sim}{y}_j), \underset{\sim}{p})$ be a competitive equilibrium. If n is such that $\delta_n < \max_i \delta_i$ , then $x_n^t = 0$ , for t sufficiently large.

For the turnpike theorem, I need the following assumption.

## Interiority Assumption

There exists $\zeta > 0$ and $\underline{\delta}$ such that $0 < \underline{\delta} < 1$ and the following are true. If $\delta_i = \delta$, for all i, where $\underline{\delta} < \delta < 1$ and if $((\underset{\sim}{x}_i), (\underset{\sim}{y}_j), \underset{\sim}{p})$ is a stationary equilibrium with transfer payments, then $\sum_j y_{j1k}^{-1} > \zeta$ , for all $k \in L_p$.

4.5) Theorem (The turnpike property) Suppose that the interiority assumption is satisfied. Suppose also that $\sum_j y_{j1k}^{-1} > 0$, for all $k \varepsilon L_p$, and that $\delta_i = \delta$, for all i. If $\delta$ is sufficiently close to one, then the following true. If $((\underset{\sim}{x}_i),(\underset{\sim}{y}_j),\underset{\sim}{p})$ is a competitive equilibrium, then $((\underset{\sim}{x}_i), (\underset{\sim}{y}_j))$ converges exponentially to a stationary allocation $((\bar{\underset{\sim}{x}}_i), (\bar{\underset{\sim}{y}}_j))$.

5) Discussion of Assumptions

All my assumptions are more or less standard in equilibrium theory, except for assumption 3.8 (no fixed coefficients in production), assumption 3.10 (strict convexity in production) and the interiority assumption. I now discuss what is wrong with these assumptions. I then give an example, which indicates what can go wrong if one allows constant returns to scale.

Assumptions 3.10 excludes constant returns to scale. Constant returns to scale is a very natural assumption to make. Production possibility sets really describe production processes, not firms. There is no compelling reason to keep the number of firms fixed. In fact, one imagines that firms can be replicated. This possibility is one justification for assuming constant returns to scale. All these considerations are especially persuasive in the context of growth theory, where one thinks in terms of a very long run.

The interiority assumption is traditional in turnpike theory. It would be better to replace this assumption by assumptions about preferences and technology. It is, no doubt, possible to do so, but I have not found a convincing set of assumptions which do not lead to an excessively complicated proof.

Assumption 3.8 is especially awkward. It makes it impossible to represent the use of capital equipment in production. The conventional representation is as follows. One labels equipment according to age. A production process using a machine transforms the machine and other inputs into an older machine and other outputs. The process transforms one younger machine into one older machine. A fixed coefficient of one is unavoidable.

One way to overcome this problem might be to assume that machines are installed. That is, once one is used by a firm, it can be used by another firm only after an expensive transfer. However, this kind of phenomenon is hard to represent using a general equilibrium model.

I now give an example, which shows what may go wrong if one assumes constant returns to scale. There is one consumer and there are two firms. There is one primary good, good zero, and there are two produced goods, goods 1 and 2. The utility function of the consumer is $u(x_0,x_1,x_2) = \sqrt{x_0x_1x_2}$ . The consumer does <u>not</u> discount future utility. The initial endowment of the primary good is 3. The production possibility set of firm 1 is determined by the function $g_1(y_{00},y_{01},y_{02}\,;\,y_{11},y_{12}) = \sqrt{3}\ \sqrt{y_{11}^2 + 3y_{12}^2} - 3\sqrt{y_{00}y_{02}}$ . The corresponding function for firm 2 is $g_2(y_{00},y_{01},y_{02}\,;\,y_{11},y_{12}) = \sqrt{3}\ \sqrt{3y_{11}^2 + y_{12}^2} - 3\sqrt{y_{00}y_{02}}$.

This economy satisfies all of assumptions 3.1 - 3.16 except assumption 3.11. Production sets are convex, but not strictly convex. In fact, production satisfies constant return to scale.

It is easy to calculate that the following is a stationary equilibrium. The price of every good is one. The consumer consumes one unit of each good. The production vector of firm 1 is $(y_0, y_1) = (1,0,1,3/2,1/2)$. That of firm 2 is $(1,1,0,1/2,3/2)$.

Suppose that all prices are fixed at one. Then, each firm produces along a ray. The ray is the set of profit maximizing input-output combinations. When firms are restricted to these rays, the production system becomes a fixed coefficient linear production model. It is described by a vector $c = (2,2)$ and matrices $A = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$. $c_j$ is the quantity of good zero used when firm $j$ operates at rate one. $a_{ij}$ is the input of good $i$ into firm $j$ when it operates at rate one. $b_{ij}$ is the output of good $i$ by firm $j$ when it operates at rate one.

Now suppose that firms operate along these rays and that the initial capital stock at time zero is $K^0 = (1,3)$ and that firm 1 alone operates at time zero. Then, the consumer consumes one unit of each good. The input vector of firm 1 is $(2,0,2)$ and its output vector is $(3,1)$, which is the capital stock at time 1, call it $K^1$. At time 1, only firm 2 operates. The consumer again consumes one unit of each good. The input vector is $(2,2,0)$, and the output vector is $(1,3) = K^2 = K^0$. Thus, a cycle has been completed. The indefinite continuation of this cycle describes an optimum which may be interpreted as an equilibrium. Consumption is the same as in the golden rule, but production oscillates. Hence, the turnpike theorem does not apply. That is, optimal paths do not converge to a point. The cyclical path is, however, part of the von Neumann facet discussed by McKenzie [1963 and 1968].

The turnpike property cannot be recovered by discounting future utility. In fact, the undiscounted case is the most favorable to the turnpike theorem.

6) Relation to the Literature

This paper links Arrow-Debreu general equilibrium theory with capital theory. General equilibrium, of course, has a very long history. The proof that equilibrium exists is due to Arrow and Debreu [1954]. I treat prices as prices for contracts for future delivery. This interpretation was developed by Arrow [1953] and Debreu [1959] (Chapter 7). Arrow introduced contingent claims and Debreu extended this notion to the context of many periods.

In this paper, I make use of equilibrium theory for economies with infinitely many commodities. The extension of equilibrium theory to such economies was made by Debreu [1954], Peleg and Yaari [1970], myself [1972] and Stigum [1972, 1973]. Debreu proved that equilibria in such economies are Pareto optimal and that Pareto optima may be realized as equilibria with transfer payments. Peleg and Yaari, Stigum and I proved that equilibria exist. Peleg and Yaari and Stigum allowed the number of commodities to be only countably infinite. In my case, there may be a continuum of commodities. In the present paper, there are countably many commodities, so that I could have applied Stigum's results instead of my own. I could not have used the results of Peleg and Yaari, for they do not have production in their model.

Capital theory has almost as long a history as equilibrium theory. McKenzie has written an excellent, up to date survey of turnpike theory [1979]. Ramsey seems to have initiated the subject [1928].

The turnpike theory existing in the literature, deals with models which have only one utility function and one production possibility set.

I simply introduce many firms, many consumers and budget constraints for consumers. The turnpike theorem I prove is an analogue of that of Scheinkman [1976] (theorem 3, p. 28).

In my proof, I use a variation of the value loss method. I believe that this method traces back to the work of Radner [1961]. It has since been improved by Atsumi [1965], Brock [1970], McKenzie [1974, 1976], Cass and Shell [1976], Rockafellar [1976], Brock and Scheinkman [1976] and Magill [1977].

The most recent turnpike theorem of the type I prove is contained in McKenzie [1979] (theorem 10'). His proof builds on that of Scheinkman and uses methods developed in the list of papers just cited.

It is hard to compare McKenzie's theorem with my own, since our models are so different. In order to clarify the connection between his work and my own, I show how to derive from my model the reduced form used by McKenzie. This reduced form model is the one typically used in turnpike theory.

Suppose that in my model there is one consumer and one firm. The utility function of the consumer is $u: R_+^{L_c} \to (-\infty,\infty)$. His initial endowment is $\omega$. The production possibility set of the firm is $Y \subset R_-^L \times R_+^{L_p}$. Let $D = \{(K^0,K^1) \in R_+^{L_p} \times R_+^{L_p} \mid$ there exists $x \in R_+^{L_c}$ such that $(x - \omega - K^0,K^1) \in Y\}$. Let $v: D \to (-\infty,\infty)$ be defined by $v(K^0,K^1) = \max \{u(x) \mid x \in R_+^{L_c}$ and $(x - \omega - K^0,K^1) \in Y\}$. McKenzie's economy is defined simply by $D$ and $v$.

The key concavity assumption of McKenzie is stated in terms of the concavity of $v$. But the concavity of $v$ depends on the properties of both $u$ and $Y$

in a complicated way. Some of the long arguments in my proof of the turnpike theorem may be interpreted as proving that v is concave. Benhabib and Nishimura [1979] (section 3.3 , Remark 1) have already pointed out that the concavity of v requires very strong assumptions about underlying production relations. They work with a continuous time model.

I now return to the comparison of McKenzie's theorem with my own. My theorem is more general in that I prove exponential convergence and he does not. McKenzie's theorem is more general than mine in that he makes only a local strict concavity assumption. I assume strict concavity or convexity everywhere. McKenzie's assumption is that the Hessian of v is negative definite at $(\overline{K},\overline{K})$, where $(\overline{K},\overline{K})$ satisfies $v(\overline{K},\overline{K}) = \max\{v(K,K) | (K,K) \in D\}$. ($\overline{K}$ is the vector of golden rule capital stocks). The other differences between McKenzie's theorem and my own are of no great interest.

Araujo and Scheinkman [1977] prove a turnpike theorem with exponential convergence (theorem 3.2). They assume that a certain infinite dimensional matrix has the dominant diagonal property. I do not see that this condition necessarily applies in my case. For this reason, I did not use their result.

Remark Yano has generalized the turnpike theorem of this paper. (He also caught an error in an earlier version of my proof.) He assumed constant returns to scale. He also assumed that the von Neumann facet containing the golden rule input-output vector is a single ray. In this way, he avoided the problem illustrated by the example of section 5.

I now turn to my result that there exists a stationary equilibrium (theorem 4.2). When there is only one firm and one consumer, stationary equilibrium becomes what is known as the modified golden rule. Therefore, the proof of theorem 4.2 provides a new way to prove the existence of a modified golden rule.

Sutherland [1970] and Peleg and Ryder [1974] proved that a modified golden rule exists. They used fixed point arguments. I do so as well, but my argument is simply a modification of the usual argument which proves that a general equilibrium model has an equilibrium. Thus, I clarify the tie between general equilibrium theory and the work of Sutherland, Peleg and Ryder. My proof is a variation of one given in a previous paper [1979] of my own. In that paper, there is uncertainty and there is no discounting of future utility.

In section 1, I mentioned that the idea expressed by my theorem 4.4 has already appeared in the literature. This theorem asserts that less patient consumers eventually consume nothing. Ramsey [1928] (pp. 558-9) pointed out that in long-run or stationary equilibrium, those consumers with the highest rate of time preference would live at a subsistence level. Rader makes an argument similar to my own in chapter I of [1971]. A related idea appears in the last section of one of his recent papers [1979]. Finally, Becker [forthcoming] proved an assertion similar to Ramsey's.

Becker studies capital theory using a disaggregated model, just as I do. In [1979], he proves the existence of an equilibrium and in [forthcoming] he proves the existence of a stationary equilibrium. Becker's work differs from mine in that his model is one of temporary equilibrium, not of general equilibrium. Consumers can sell or accumulate capital but they cannot borrow. Becker also has only one commodity.

7) Boundedness

In this section, I prove that feasible allocations are uniformly bounded. The boundedness of feasible allocations is expressed by the following two lemmas. Their proofs are completely routine.

7.1) Lemma. Let $(y_{j1}^{-1})$ be given. There is $B > 0$ such that if $((\underset{\sim}{x}_i),(\underset{\sim}{y}_j))$ is a feasible allocation with initial resources $\sum_j y_{j1}^{-1}$, then $|((x_i^t),(y_j^t))| \leq B$, for all $t$.

7.2) Lemma. Let $(y_{j1}^{-1})$ be variable. There is $B > 0$ such that if $((\underset{\sim}{x}_i),(\underset{\sim}{y}_j))$ is a feasible stationary allocation, then $|((x_i),(y_j))| \leq B$ and $|\sum_j y_{j1}| \leq B$.

In proving these lemmas, I make use of the set $Y = \sum_j Y_j$.

7.3) $Y$ is closed.

This is so because each $Y_j$ is closed and contained in $R_-^L \times R_+^{L_p}$. (For a proof, see Debreu [1959], p. 23, statement (9).) Clearly, $Y$ is convex (see assumption 3.11) and contains the zero vector (see assumption 3.12).

In the following, $\omega$ denotes $\sum_{i=1}^{I} \omega_i$.

7.4) Lemma. There exists a positive number $B$ such that $|y_1| < |y_0|$ whenever the following are true: $(y_0,y_1) \in Y$, $|y_0| \geq B$ and $|y_{0k}| \leq |\omega|$, for all $k \in L_o$.

Proof Suppose that the lemma were false. Then for each $B = 1,2,\ldots,$ there would exist $(y_0^B, y_1^B) \in Y$ such that $|y_0^B| \geq B$ and $|y_{0k}^B| \leq |\omega|$, for all $k \in L_o$, and yet $|y_1^B| \geq |y_0^B|$. Because each function $g_j$ is non-decreasing (assumption 3.10) and because $y_1^B \geq 0$, it follows that $(y_0^B, \alpha y_1^B) \in Y$, for any $\alpha$ such that $0 \leq \alpha \leq 1$. Hence, I may assume that $|y_1^B| = |y_0^B|$. Let $k_B$ be such that $y_{1k_B} = |y_1^B|$. By passing to a subsequence, I may assume that $k_B = \underline{k}$, for all B.

Let $(\bar{y}_0^B, \bar{y}_1^B) = |y_0^B|^{-1}(y_0^B, y_1^B)$. $(\bar{y}_0^B, \bar{y}_1^B) \in Y$, since Y is convex and contains zero and since $|y_0^B| \geq B \geq 1$. Also, $|(\bar{y}_0, \bar{y}_1)| \leq 1$. Hence by passing to a subsequence, I may assume that $\lim_{B\to\infty} (\bar{y}_0^B, \bar{y}_1^B) = (\bar{y}_0, \bar{y}_1)$. $(\bar{y}_0, \bar{y}_1) \in Y$, since Y is closed (7.3). Since $|\bar{y}_{0k}^B| \leq |y_0^B|^{-1} |\omega|$, for all $k \in L_o$, it follows that $\bar{y}_{0k} = 0$, for all $k \in L_o$. Also, $y_{1\underline{k}} = 1$. These last two facts contradict the necessity of primary inputs (assumption 3.13).

Q.E.D.

7.5) Lemma An arbitrarily large number B may be chosen so that $|y_1| < B$ whenever the following are true: $(y_0, y_1) \in Y$, $|y_{0k}| \leq |\omega|$, for all $k \in L_o$, and $|y_0| \leq B$.

Proof. The proof of this lemma is entirely analogous to that of lemma 7.4.

Q.E.D.

Proof of lemma 7.1 Let $B$ be as in lemma 7.5 and such that $B > |\omega|$ and $B > |\sum_j y_{j1}^{-1}|$.

I show that it is enough to prove that $|\sum_j y_{j1}^t| < B$, for all $t \geqq -1$. By feasibility, $0 \leqq x_i^t \leqq \sum_{n=1}^{I} x_n^t \leqq \omega + \sum_j y_{j1}^{t-1}$, for all $i$. Hence, $x_{ik}^t \leqq |\omega| < B$, if $k \in L_o$ and $x_{ik}^t \leqq \sum_j y_{j1k}^{t-1} < B$, if $k \in L_p$, so that $|x_i^t| < B$. Similarly, since $0 \leqq -y_{j0}^t \leqq -\sum_{n=1}^{J} y_{n0}^t \leqq \omega + \sum_{n=1}^{J} y_{n1}^{t-1}$, for all $j$, it follows that $-y_{j0}^t < B$.

I now prove by induction on $t$ that $|\sum_j y_{j1}^t| < B$, for all $t$. It is true for $t = -1$, by assumption. Suppose that it is true for $t-1$. Then by feasibility, $-\sum_j y_{j0k}^t < |\omega|$, if $k \in L_o$, and $-\sum_j y_{j0k}^t < B$, if $k \in L_p$. $\sum_j (y_{j0}^t, y_{j1}^t) \in Y$. Hence by lemma 7.5, $|\sum_j y_{j1}^t| < B$. This completes the induction.

Q.E.D.

Proof of lemma 7.2 Let $B$ be as in lemmas 7.4 and 7.5 and such that $B > |\omega|$.

I first prove that

7.6) $|\sum_j y_{j0}| < B$, whenever $((x_i),(y_j))$ is a feasible stationary allocation.

Suppose that $((x_i),(y_j))$ is a feasible stationary allocation and that $|\sum_j y_{j0}| \geqq B$. By feasibility, $\sum_j y_{j0k} \leqq |\omega|$, for all $k \in L_o$. Hence by lemma 7.4, 7.7) $\quad \left| \sum_j y_{j0} \right| > \left| \sum_j y_{j1} \right|.$

Since $0 \leqq - \sum_j y_{j0k} \leqq |\omega| < B < \left| \sum_j y_{j0} \right|$, if $k \in L_o$, it follows that $\left| \sum_j y_{j0} \right| = \max_{k \in L_p} \left( - \sum_j y_{j0k} \right)$. But, $-\sum_j y_{j0k} \leqq \sum_j y_{j1k}$, if $k \in L_p$, so that $\left| \sum_j y_{j0} \right| \leqq \left| \sum_j y_{j1} \right|$. This contradicts (7.7) and so proves (7.6).

It now follows from lemma 7.5 that $\left| \sum_j y_{j1} \right| < B$.

Q.E.D.

## 8. The Economy $\mathcal{E}^*$

In proving theorem 4.1, it is easier to deal with an economy in which production possibility sets are cones. For this reason, I now modify the economy $\mathcal{E}$ in order to obtain an equivalent economy $\mathcal{E}^*$ in which all production possibility sets are cones. $\mathcal{E}^*$ will also be such that free disposability is incorporated in the production process rather than in the definition of feasibility.

I introduce one new factor of production for each firm. The $j^{th}$ such factor can be used only by firm $j$. This factor may be thought of as the entrepreneurial factor. McKenzie (1959) has suggested introducing such a factor in just the way I do.

More explicitly, I introduce $J$ new commodities, so that the list of commodities in $\mathcal{E}^*$ is $L \cup \{1,\ldots,J\}$. The commodity space of $\mathcal{E}^*$ is $R^L \times R^J$. An input vector for a firm may be written as $(y_0, y_0^*)$, where $y_0 \in R_-^L$ and $y_0^* \in R_-^J$. Let $e_j = (0,\ldots,0,1,0,\ldots,0)$ be the

$j^{th}$ standard basis vector of $R^J$. The production possibility set of the $j^{th}$ firm in $\mathcal{E}^*$ is $Y_j^* = \{ t(y_0, -e_j, y_1) : (y_0, y_1) \in Y_j, |(y_0, y_1)| \leqq B$ and $t \geqq 0\}$, where $B$ is as in lemma 7.1.

I let the endowment of the $i^{th}$ consumer be $\omega_i^* = (\omega_i, 0) + \sum_{j=1}^{J} \theta_{ij}(0, e_j) \in R_+^L \times R_+^J$. Notice that I have introduced one unit of factor $j$ into the economy, for each $j$. The consumption set of every consumer is still $R_+^{L_c}$ and the utility function of consumer $i$ is still $u_i: R_+^{L_c} \to (-\infty, \infty)$.

Finally, I introduce an extra firm, firm $J+1$, which disposes of goods. The production possibility set of firm $J+1$ is $Y_{J+1}^* = R_-^L \times R_-^J \times \{0\} \subset R^L \times R^J \times R^{L_p}$. Let $\theta_{i,J+1} = I^{-1}$, for all $i$. The initial endowment of firm $J+1$ is $y_{J+1,1}^{-1} = 0$.

In summary, the economy $\mathcal{E}^*$ is $\{(u_i, \omega_i^*), (y_{j1}^{-1}, Y_j^*), \theta_{ij}: i = 1,\ldots,I, j = 1,\ldots,J+1\}$.

I now define an equilibrium for $\mathcal{E}^*$. A price system is still denoted by $\underset{\sim}{p}$. It is a non-zero vector in $\ell^+_{1,L\cup\{1,\ldots,J\}}$. That is, $\underset{\sim}{p} = (p^0, p^1, \ldots)$, where $p^t \in R_+^L \times R_+^J$ and $0 < \sum_{t=0}^{\infty} | p^t | < \infty$. The definitions of allocation, supply and demand are analogous to those made before. $\eta_j^*(\underset{\sim}{p})$, $\pi_j^*(\underset{\sim}{p})$ and $\xi_i^*(\underset{\sim}{p})$ denote, respectively, the supply and profit correspondence of firm $j$ and the demand correspondence of consumer $i$.

The major change comes in the definition of feasibility. An allocation $((\underset{\sim}{x}_i), (\underset{\sim}{y}_j))$ is feasible if $\sum_i x_i^t = \sum_i \omega_i^* + \sum_{j=1}^{J+1} (y_{j0}^t + y_{j1}^{t-1})$, for all $t \geqq 0$. Notice that there is an equality sign here, whereas before feasibility involved only inequality.

An equilibrium for $\mathcal{E}^*$ consists of $((\underset{\sim}{x}_i),(\underset{\sim}{y}_j), \underset{\sim}{p})$ which satisfy the following conditions.

8.1) $((\underset{\sim}{x}_i), (\underset{\sim}{y}_j))$ is a feasible allocation.

8.2) $\underset{\sim}{p}$ is a price system.

8.3) $\underset{\sim}{y}_j \in \eta_j^*(\underset{\sim}{p})$, for all $j$.

8.4) $\underset{\sim}{x}_i \in \xi_i^*(\underset{\sim}{p})$, for all $i$.

8.5) Lemma There is a one to one correspondence between equilibria for $\mathcal{E}$ and for $\mathcal{E}^*$.

Proof The proof is routine. I will prove only that to every equilibrium for $\mathcal{E}^*$ there corresponds an equilibrium for $\mathcal{E}$. The reverse statement should then be obvious.

Let $((\underset{\sim}{x}_i),(\underset{\sim}{y}_j),\underset{\sim}{p})$ be an equilibrium for $\mathcal{E}^*$. Then, $\underset{\sim}{x}_i = ((x_i^0,0),(x_i^1,0), \ldots)$, where $x_i^t \in R_+^{L_c}$, for all $t$. Similarly for all $j$, $\underset{\sim}{y}_j = ((y_{j0}^0,y_{j0}^{*0},y_{j1}^0),(y_{j0}^1,y_{j0}^{*1},y_{j1}^1),\ldots)$. Let $\underset{\sim}{\overline{x}}_i = (x_i^0,x_i^1,\ldots)$, for all $i$, and let $\underset{\sim}{\overline{y}}_j = ((y_{j0}^0,y_{j1}^0),(y_{j0}^1,y_{j1}^1),\ldots)$, for $j = 1,\ldots,J$. Write $\underset{\sim}{p}$ as $\underset{\sim}{p} = ((p^0,p^{*0}),(p^1,p^{*1}),\ldots)$, where $p^t \in R_-^L$ and $p^{*t} \in R^J$, for all $t$. Let $\underset{\sim}{\overline{p}} = (p^0,p^1,\ldots)$. I will prove that $((\underset{\sim}{\overline{x}}_i),(\underset{\sim}{\overline{y}}_j),\underset{\sim}{\overline{p}})$ is an equilibrium for $\mathcal{E}$.

First of all, I show that $((\underset{\sim}{\overline{x}}_i),(\underset{\sim}{\overline{y}}_j))$ is a feasible allocation for $\mathcal{E}$. Each $\bar{x}_i$ is clearly a consumption program. I now show that if $j = 1,\ldots,J$, then $\bar{y}_j$ is a feasible program for firm $j$ in $\mathcal{E}$. Since $((x_i),(y_j))$ is a feasible allocation for $\mathcal{E}^*$, it follows that $-1 \leqq y^{*t}_{j0j} \leqq 0$, for all $j = 1,\ldots,J$ and all $t$. From the definition of $Y^*_j$, it follows that

$(y^t_{j0}, y^t_{j1}) = -y^t_{j0j}(\hat{y}^t_{j0}, \hat{y}^t_{j1})$, where $(\hat{y}^t_{j0}, \hat{y}^t_{j1}) \in Y_j$. Since $0 \in Y_j$ and $Y_j$ is convex, it follows that $(y^t_{j0}, y^t_{j1}) \in Y_j$. Hence, $\bar{y}_j$ is a feasible program for firm $j$ in $\mathcal{E}$. By the definition of feasibility in $\mathcal{E}^*$, $\sum_i x^t_i = \sum_i \omega_i + \sum_{j=1}^{J} (y^t_{j0} + y^{t-1}_{j1}) + y^t_{J+1,0} \leqq \sum_i \omega_i + \sum_{j=1}^{J} (y^t_{j0} + y^{t-1}_{j1})$,

for all $t$. This completes the proof that $((\bar{x}_i),(\bar{y}_j))$ is a feasible allocation for $\mathcal{E}$.

Next observe that for $j = 1,\ldots,J$,

$$
\begin{aligned}
8.6) \qquad & p^t \cdot y^t_{j0} + p^{*t} \cdot y^{*t}_{j0} + p^{t+1} \cdot y^t_{j1} \\
& = \max \{p^t \cdot y_0 + p^{*t} \cdot y^*_0 + p^{t+1} \cdot y_1 \mid \\
& (y_0, y^*_0, y_1) \in Y^*_j \} = 0, \quad \text{for all } t.
\end{aligned}
$$

This follows from the facts that $y_j \in \eta^*_j(p)$ and that $Y^*_j$ is a cone with apex zero. (8.6) implies that

$$
8.7) \qquad p^t \cdot y^t_{j0} + p^{t+1} \cdot y^t_{j1} = -p^{*t}_j \cdot y^{*t}_{j0j} \; .
$$

I now show that $\bar{p} > 0$. Clearly, $\bar{p} \geq 0$, so that I must show that $\underset{\sim}{\bar{p}} \neq 0$. If $p^{*t} = 0$, for all $t$, then $\underset{\sim}{\bar{p}} \neq 0$ because $\underset{\sim}{p} \neq 0$. Suppose that $p_j^{*t} > 0$, for some $t$ and $j$. Then, it does not pay to dispose of the $j^{th}$ factor, so that $y_{j0j}^{*t} = -1$. Hence, $p^t \cdot y_{j0}^t + p^{t+1} \cdot y_{j1}^t = -p_j^{*t} y_{j0j}^{*t} = p_j^{*t} > 0$. That is, $p^{t+1} \neq 0$ and so $\underset{\sim}{\bar{p}} \neq 0$.

I next show that $\underset{\sim}{\bar{y}}_j \in \eta_j(\underset{\sim}{\bar{p}})$ for $j = 1,\ldots,J$. It is sufficient to prove that

$$8.8) \qquad p^t \cdot y_{j0}^t + p^{t+1} \cdot y_{j1}^t = \max \{p^t \cdot y_0 + p^{t+1} \cdot y_1 \mid (y_0\ y_1) \in Y_j\},$$

for every $t$ and for $j = 1,\ldots,J$.

Suppose that $y_{j0j}^{*t} = -1$. Then by (8.6) and the definition of $Y_j^*$, $p^t \cdot y_{j0}^t + p^{t+1} \cdot y_{j1}^t \geq p^t \cdot y_0 + p^{t+1} \cdot y_1$, for all $(y_0,y_1) \in Y_j$ such that $|(y_0,y_1)| \leq B$. By lemma 7.1, $|(y_{j0}^t,y_{j1}^t)| < B$. Since $Y_j$ is convex, it follows that $p^t \cdot y_{j0}^t + p^{t+1} \cdot y_{j1}^t \geq p^t \cdot y_0 + p^{t+1} \cdot y_1$, for all $(y_0,y_1) \in Y_j$. Now suppose that $y_{j0j}^{*t} > -1$. Then, $p_j^{*t} = 0$. Hence by (8.7), $p^t \cdot y_{j0}^t + p^{t+1} \cdot y_{j1}^t = 0$. Suppose that there were $(y_0,y_1) \in Y_j$ such that $p^t \cdot y_0 + p^{t+1} \cdot y_1 > 0$. Since $Y_j$ is convex and contains zero, I may suppose that $|(y_0,y_1)| \leq B$. Then, $(y_0,-e_j,y_1) \in Y_j^*$ and $p^t \cdot y_0 - p^{*t} \cdot e_j + p^{t+1} \cdot y_1 > 0$, which contradicts (8.6). This completes the proof that $\underset{\sim}{\bar{y}}_j \in \eta_j(\underset{\sim}{\bar{p}})$.

I now prove that $\underset{\sim}{\bar{x}} \in \xi_i(\underset{\sim}{\bar{p}})$, for all $i$. Clearly, it is sufficient to prove that the income of each consumer in $\mathcal{E}$ is the same as his income in $\mathcal{E}^*$. By (8.6), $\pi_j^*(\underset{\sim}{p}) = p^0 \cdot y_{j1}^{-1}$. The value of consumer $i$'s initial endowment in $\mathcal{E}^*$ is $\sum_{t=0}^{\infty} p^t \cdot \omega_i + \sum_{j=1}^{J} \theta_{ij} \sum_{t=0}^{\infty} p_j^{*t}$. Hence,

consumer i's income in $\mathcal{E}$ is $\sum_{t=0}^{\infty} p^t \cdot \omega_i + \sum_{j=1}^{J} \theta_{ij}(p^0 \cdot y_{j1}^{-1} + \sum_{t=0}^{\infty} p_j^{*t})$.
Thus, it is sufficient to prove that

$\pi_j(\underset{\sim}{p}) = p^0 \cdot y_{j1}^{-1} + \sum_{t=0}^{\infty} p_j^{*t}$, for all j. But this follows immediately from (8.7) and (8.8) and the fact that for all j and t, $p_j^{*t} = 0$ if $y_{j0j}^{*t} = -1$. This completes the proof that $\underset{\sim}{x}_i \in \xi_i(\underset{\sim}{p})$.

I have now verified that $((\bar{\underset{\sim}{x}}_i),(\bar{\underset{\sim}{y}}_j),\bar{\underset{\sim}{p}})$ satisfies conditions (2.2) - (2.4) of the definition of an equilibrium for $\mathcal{E}$.

Q.E.D.

9) Proof of Theorem 4.1

I prove this theorem by applying results from a previous paper [1972] on the existence of equilibrium when there are infinitely many commodities. The appropriate economy with infinitely many commodities is $\mathcal{E}^{**}$, defined as follows. The commodity space is $\ell_{\infty, L \cup \{1,\dots,J\}}$. $\underset{\sim}{x} \in \ell_{\infty, L \cup \{1,\dots,J\}}$ is written as $(x^0, x^1, \dots)$,

where $x^t \in R^L \times R^J$, for all t. Throughout the rest of this section, I write $\ell_\infty$ for $\ell_{\infty, L \cup \{1,\dots,J\}}$.

The consumption set of each consumer, is $X = \{\underset{\sim}{x} \in \ell_\infty^+ \mid x_k^t = 0,$ if $K \notin L_c\}$. The utility function of the $i^{th}$ consumer is $U_i(\underset{\sim}{x}) = \sum_{t=0}^{\infty} \delta_i^t u_i(x^t)$. The initial endowment of the $i^{th}$ consumer, $\underset{\sim}{\omega}_i^{**} = (\omega_i^{**0}, \omega_i^{**1}, \dots)$, is defined by $\omega_i^{**0} = \omega_i^* + \sum_{j=1}^{J} \theta_{ij}(y_{j1}^{-1}, 0)$

$\in R^L \times R^J$ and $\omega_i^{**t} = \omega_i^*$, for $t > 0$. Notice that the firms' initial endowments have been transferred to the consumers.

The production possibility set of firm j is $Y_j^{**} = \{\underset{\sim}{y} = (y^0, y^1, \dots)$ $\in \ell_\infty \mid$ there exist $(z_0^t, z_1^t) \in Y_j^*$, for $t = 0,1,\dots$ , such that

$y^0 = z_0^0$ and $y^t = z_0^t + z_1^{t-1}$ , for $t > 0\}$ .

In summary, the economy $\mathcal{E}^{**}$ is $\{ (X, U_i, \omega_i^{**}), \; Y_j^{**} , \; \theta_{ij} \mid$ $i = 1, .., I \; ; \; j = 1, \ldots, J+1\}$ .

Prices in $\mathcal{E}^{**}$ are non-zero vectors in $\ell^+_{1,L \cup \{1,\ldots,J\}}$ . $\underset{\sim}{p} \in \ell^+_{1, L \cup \{1,\ldots,J\}}$ is written as $\underset{\sim}{p} = (p^0, p^1, \ldots.)$, where $p^t \in R^L_+ \times R^J_+$, for all t. Throughout the rest of this section, I write $\ell_1$ for $\ell_{1,L \cup \{1,\ldots,J\}}$ .

An equilibrium for $\mathcal{E}^{**}$ is defined in the obvious way. It should be clear that on equilibrium for $\mathcal{E}^{**}$ may be interpreted as an equilibrium for $\mathcal{E}^*$ . Hence by lemma 8.5, it is sufficient to prove that $\mathcal{E}^{**}$ has an equilibrium.

Key assumptions in my paper of [1972] are that X and the $Y_j^{**}$ are Mackey closed and that the $U_i$ are Mackey continuous. (The Mackey topology is defined below.) X is clearly Mackey closed. It is proved in Appendix II of [1972] that the $U_i$ are Mackey continuous. It is not obvious that the sets $Y_j^{**}$ are Mackey closed. However, it is not necessary to prove that the entire set $Y_j^{**}$ is Mackey closed. It is sufficient to prove that $\{ y \in Y_j^{**} \mid |\underset{\sim}{y}| \leq b\}$ is Mackey closed, for $b > 0$ . (This may be seen by examining the proof of theorem 1 in [1972].) On such norm bounded sets, the Mackey topology is the same as the product topology or the topology of componentwise convergence.

The proof of this last assertion proceeds as follows. A base of Mackey neighborhoods of zero consists of sets of the form $V = \{\underset{\sim}{x} \in \ell_\infty \mid |p \cdot x| < 1$ , for all $\underset{\sim}{p} \in C\}$ , where C is a weakly compact, convex, circled subset of $\ell_1$ . (See Kelley and Namioka [1963], p. 173, or Schaefer [1971], p. 131.) C is weakly compact only if $\sup_{\underset{\sim}{p} \in C} \sum_{t=0}^{\infty} |p^t| < \infty$

and $\lim_{T\to\infty} \sup_{p\in C} \sum_{t=T}^{\infty} |p^t| = 0$ . (This is problem 3 on p. 338 of Dunford and Schwartz [1957].) It follows at once that if $\underset{\sim}{x}_n$ is a sequence in $\ell_\infty$ such that $\sup_n |\underset{\sim}{x}_n| < \infty$ and $\lim_{n\to\infty} x_n^t = 0$ , for all $t$ , then $\underset{\sim}{x}_n \in V$ , for $n$ sufficiently large.

I now prove that

9.1) <u>Lemma</u> For each $j$ , $\{\underset{\sim}{y} \in Y_j^{**} \mid |\underset{\sim}{y}| \leq b\}$ is closed in the product topology for any $b > 0$ .

<u>Proof</u> $Y_{J+1}^{**}$ is simply $-\ell_\infty^+$ , which clearly is closed. Hence, I may suppose that $j \leq J$.

It should be clear from the definition of $Y_j^*$ that it is a closed subset of $R^L \times R^J \times R^{L_p}$ . It follows by an easy compactness argument that it is enough to prove the following.

9.2) $m_t < \infty$ , for all $t$ , where $m_t = \max\{|z^t| \mid$ there is $\underset{\sim}{y} \in Y_j^{**}$ such that $|y| \leq b$ and $\underset{\sim}{y} = (z_0^0, z_0^1 + z_1^0, \ldots.)$ , where $z^n \in Y_j^*$ , for all $n\}$.

I prove (9.2) by induction on $t$ . Clearly, $m_0 \leq b$ . Suppose that $m_{t-1} < \infty$ and let $\underset{\sim}{y} = (z_0^0, z_0^1 + z_1^0, \ldots) \in Y_j^{**}$ be such that $|\underset{\sim}{y}| \leq b$ . Then, $|(z_0^t + z_1^{t-1})| \leq b$ , so that $|z_0^t| \leq m_{t-1} + b$ . I now use the following fact.

9.3) For any $c > 0$, $\max\{|z_1| \mid (z_0, z_1) \in Y_j^*$ and $|z_0| < c\} < \infty$ .

This follows from the closedness of $Y_j^*$ and the necessity of primary inputs. (The proof is similar to that of lemma 7.4.)

(9.3) implies that $|y_1^t|$ is bounded, so that $m_t < \infty$ . This completes the proof of lemma 9.1.

Q.E.D.

The equilibrium existence theorem of [1972] makes use of what I called an "adequacy assumption." This says that for i=1, ..,I , there exists $y_i \in \sum_{j=1}^{J+1} Y_j^{**}$ such that $\underset{\sim}{y}_i + \underset{\sim}{\omega}_i^{**} >>> \underset{\sim}{0}$. Here, I have only the weaker adequacy property stated in the following lemma.

9.4) <u>Lemma</u> There exists $\underset{\sim}{y}_j \in Y_j^{**}$ , for j = 1,..., J+1, such that

$$\sum_{j=1}^{J+1} \underset{\sim}{y}_j + \sum_i \underset{\sim}{\omega}_i^{**} >>> \underset{\sim}{0} .$$

<u>Proof</u> By assumption 3.16, there are $\bar{\omega} \in R_+^{L_o}$ and $(y_{j0}, y_{j1}) \in Y_j$, for j = 1,...., J, such that $\bar{\omega} + \sum_{j=1}^{J} (y_{j0} + y_{j1}) >> 0$ . By assumption 3.15, $\sum_i \omega_{ik} > 0$ , for all $k \in L_o$ . Also by the hypothesis of theorem 4.1, $\sum_{j=1}^{J} y_{j1k}^{-1} > 0$ , for all $k \in L_p$ . Hence, by multiplying $\bar{\omega}$ and the $(y_{j0}, y_{j1})$ by a small positive constant, if necessary, I may obtain that $\bar{\omega} \leqq \sum_i \omega_i$ and that $\sum_{j=1}^{J} (y_{j0k} + y_{j1k}^{-1}) \geqq 0$ , for all $k \in L_p$ . Let $\underset{\sim}{y}_j = \frac{1}{2}((y_{j0}, -e_j), (y_{j0} + y_{j1}, -e_j), (y_{j0} + y_{j1}, -e_j), \ldots)$, for j = 1,....,J . Let $\underset{\sim}{y}_{J+1} = 0 \in \ell_\infty$ . It is easy to see that the $\underset{\sim}{y}_j$ satisfy the conditions of the lemma.

Q.E.D.

In [1972], I also made a boundedness assumption. What is required is that the set of feasible allocations of $\mathcal{E}^{**}$ be bounded in the supremum norm $|\cdot|$ . That this is so follows directily from lemma 7.1.

I now apply the proof of theorem 1 of [ 1972]. I cannot apply the theorem itself since the adequacy assumption here is weaker than that made there. If one follows the proof carefully, one finds that all but the last three lines apply and what one obtains is a quasi equilibrium with a price system in ba . ba is the set of bounded additive set functions defined on subsets of the positive integers. It is also the set of linear functionals on $\ell_\infty$ which are continous with respect to the supremum norm.

More precisely, I obtain $((\underset{\sim}{x}_i), (\underset{\sim}{y}_j), \underset{\sim}{p})$ which satisfies the following.

9.5) $((\underset{\sim}{x}_i), (\underset{\sim}{y}_j))$ is a feasible allocation for $\mathcal{E}^{**}$.

9.6) $\underset{\sim}{p} > 0$ and $\underset{\sim}{p} \in ba$.

9.7) For all $j$, $0 = \underset{\sim}{p} \cdot \underset{\sim}{y}_j \geq \underset{\sim}{p} \cdot \underset{\sim}{y}$, for all $\underset{\sim}{y} \in Y_j^{**}$.

9.8) For all $i$, $\underset{\sim}{p} \cdot \underset{\sim}{x} \geq \underset{\sim}{p} \cdot \underset{\sim}{\omega}_i^{**} + \sum_{j=1}^{J+1} \theta_{ij}\, \underset{\sim}{p} \cdot \underset{\sim}{y}_j = \underset{\sim}{p} \cdot \underset{\sim}{\omega}_i^{**} =$

$\underset{\sim}{p} \cdot \underset{\sim}{x}_i$, for all $\underset{\sim}{x} \in X$ such that $U_i(\underset{\sim}{x}) \geq U_i(\underset{\sim}{x}_i)$.

It is sufficient to show that

9.9) $\underset{\sim}{p} \cdot \underset{\sim}{\omega}_i^{**} > 0$, for all $i$.

It will then follow by a standard argument that

9.10) for all $i$, $U_i(\underset{\sim}{x}_i) \geq U_i(\underset{\sim}{x})$, for all $\underset{\sim}{x} \in X$ such that $\underset{\sim}{p} \cdot \underset{\sim}{x}$

$\leq \underset{\sim}{p} \cdot \underset{\sim}{\omega}_i^{**} + \sum_{j=1}^{J+1} \theta_{ij}\, \underset{\sim}{p} \cdot \underset{\sim}{y}_j$.

(The standard argument referred to is given on page 69 of Debreu [1959].) (9.5) - (9.7) and (9.10) say that $((\underset{\sim}{x}_i), (\underset{\sim}{y}_j), \underset{\sim}{p})$ is an equilibrium with prices in ba . I may then apply theorem 3 of [1972] in order to prove that there is $\bar{\underset{\sim}{p}} \in \ell_1$ such that $((\underset{\sim}{x}_i), (\underset{\sim}{y}_j), \bar{\underset{\sim}{p}})$ is an equilibrium for $\mathcal{E}^{**}$.

(In applying theorem 3, I make use of the fact that the production possibility sets $Y_j^{**}$ are cones. This is why I introduced the economy $\mathcal{E}^*$ .)

In order to prove (9.9), I first prove that

$$9.11) \qquad \underset{\sim}{p} \cdot (\sum_i \underset{\sim}{\omega}_i^{**}) > 0.$$

By lemma 9.4, there exist $\bar{\underset{\sim}{y}}_j$, for $j = 1,\ldots,J+1$, such that $\sum_i \omega_i^{**} + \sum_{j=1}^{J+1} \bar{y}_j >> 0$. But by (9.7), $\underset{\sim}{p} \cdot (\sum_i \underset{\sim}{\omega}_i^{**}) = \underset{\sim}{p} \cdot (\sum_i \underset{\sim}{\omega}_i^{**} + \sum_{j=1}^{J+1} \underset{\sim}{y}_j)$ $\geq \underset{\sim}{p} \cdot (\sum_i \underset{\sim}{\omega}_i^{**} + \sum_{j=1}^{J+1} \bar{\underset{\sim}{y}}_j) > 0$. This proves (9.11).

By (9.11), $\underset{\sim}{p} \cdot \underset{\sim}{\omega}_i^{**} > 0$, for some $i$, say for $i = 1$. But then by the standard argument, $U_1(\underset{\sim}{x}_1) \geq U_1(\underset{\sim}{x})$, for all $\underset{\sim}{x} \in X$ such that $\underset{\sim}{p} \cdot \underset{\sim}{x} \leq \underset{\sim}{p} \cdot \underset{\sim}{\omega}_1^{**}$ .

Let $p^0 \in R_+^L$ be the vector of prices of goods sold in period zero. By the strong monotonicity of $U_1$ (assumption 3.4),

$$9.12) \qquad p_k^0 > 0, \quad \text{for all} \quad k \in L_c .$$

For if $p_k^0 = 0$, for some $k \in L_c$, consumer 1 would want to buy an infinite amount of good $k$ in period zero and $x_1^0$ would not be well-defined.

By assumption 3.14, for each $i$, there exists $k \in L_c \cap L_o$ such that $\omega_{ik} > 0$. But then by (9.12), $\underset{\sim}{p} \cdot \underset{\sim}{\omega}_i^{**} \geq p_k^0 \omega_{ik} > 0$. This proves (9.9).

Q.E.D.

10) Proof of Theorem 4.2

I prove the existence of what I call a $\delta$-equilibrium for a two period economy, $\mathcal{E}^o$, where $\delta$ is the discount factor applied to future utility. It will be easy to see that a $\delta$-equilibrium for $\mathcal{E}^o$ corresponds to a stationary equilibrium for $\mathcal{E}$. In $\mathcal{E}^o$, consumption takes place in the first period. Firms $j = 1,\ldots,J$ use inputs in the first period in order to produce outputs in the second period. An artificial firm, firm zero, transfers goods from the second period back to the first. Firms $j$, for $j = 1,\ldots,J$, are subject to a sales tax of $1 - \delta$ times the value of their output. This tax is paid to consumers according to the shares $\theta_{ij}$. Firm zero pays no tax. The tax embodies the distortion caused by discounting future utility.

I now define $\mathcal{E}^o$ precisely. The commodity space of $\mathcal{E}^o$ is $R^L \times R^{L_p}$. The production set of firm zero is $Y_0 = \{(y_0,y_1) \in R^L \times R^{L_p} \mid y_0 = -y_1\}$. The production set of firm $j$, for $j = 1,\ldots,J$, is simply $Y_j$. The consumption set of each consumer is $X = R_+^{L_c} \times \{0\} \subset R^L \times R^{L_p}$. The utility function of consumer $i$ is $u_i^o(x,0) = u_i(x)$. His initial endowment is $\omega_i^o = (\omega_i,0) \in R^L \times R^{L_p}$. The profit shares, $\theta_{ij}$, are as before, for $j = 1,\ldots,J$. $\theta_{i0} = I^{-1}$, for all $i$. Formally,
$\mathcal{E}^o = \{(X,u_i^o,\omega_i^o),Y_j,\theta_{ij} : i = 1,\ldots,I;\ j = 0,1,\ldots,J\}$.

An allocation for $\mathcal{E}^o$ will be written as $((x_i^o),(y_j))$, where $x_i^o = (x_i,0) \in X$, for all $i$, and $y_j = (y_{j0},y_{j1}) \in Y_j$, for $j = 0,1,\ldots,J$. $((x_i^o),(y_j))$ is feasible if $\sum_i x_i^o \leq \sum_i \omega_i^o + \sum_{j=0}^{J} y_j$.

Price systems for $\mathcal{E}^o$ belong to
$\Delta = \{p = (p_0,p_1) \in R_+^L \times R_+^{L'} \mid \sum_{k\in L} p_{0k} + \sum_{k\in L_p} p_{ik} = 1\}$. If $p \in \Delta$, then $p_0 \in R^L$ and

$p_1 \in R^{L_p}$ always denote the component vectors of p.

Given $p \in \Delta$, the maximization problem of firm zero is simply

$$\max \{p_0 \cdot y_0 + p_1 \cdot y_1 \mid (y_0, y_1) \in Y_0\}$$

$\eta_0^o(p)$ denotes the set of solutions of this problem. For $j = 1,\ldots,J$, the maximization problem of firm j is

$$\max \{p_0 \cdot y_0 + \delta p_1 \cdot y_1 \mid (y_0, y_1) \in Y_j\} \quad .$$

$\eta_j^o(p)$ denotes the set of solutions of this problem. Notice that for $j \geq 1$, firm j maximizes his after tax profits, the tax being $(1 - \delta)\, p_1 \cdot y_1$.

If $j \geq 1$, the tax paid by firm j to consumer i is $\theta_{ij}(1 - \delta)p_1 \cdot y_1$, where $(y_0, y_1) \in \eta_j^o(p)$. Hence, the income of consumer i, given $p \in \Delta$, is $w_i(p) = p_0 \cdot \omega_i + \sum_{j=0}^{J} \theta_{ij}(p_0 \cdot y_{j0} + p_1 \cdot y_{j1})$, where $(y_{j0}, y_{j1}) \in \eta_j^o(p_1)$, for $j = 0,1,\ldots,J$. $w_i(p)$ is well-defined provided that $p_0 \cdot y_{j0} + p_1 \cdot y_{j1}$ is independent of $(y_{j0}, y_{j1}) \in \eta_j^o(p)$, for $j = 1,\ldots,J$.

The maximization problem of consumer i, given $p \in \Delta$, is

$$\max \{u_i^o(x^o) \mid x^o \in X \text{ and } p \cdot x^o \leq w_i(p)\} .$$

$\xi_i^o(p)$ denotes the set of solutions to this problem, for $i = 1,\ldots,I$.

A $\delta$-equilibrium for $\mathcal{E}^o$ is of the form $((x_i^o),(y_j),p)$, where

$((x_i^o),(y_j))$ is a feasible allocation for $\mathcal{E}^o$;

$p \in \Delta, p_{0k} = 0$ if $\sum_i x_{ik} < \sum_i \omega_{ik} + \sum_{j=0}^{J} y_{j0k}$, and

$p_{1k} = 0$ if $0 < \sum_{j=0}^{J} y_{j1k}$;

$y_j \in \eta_j^o(p)$, for $j = 0,1,\ldots,J$; and

$x_i^o \in \xi_i^o(p)$, for all $i$.

I now show that a stationary equilibrium for $\mathcal{E}$ corresponds to every $\delta$-equilibrium for $\mathcal{E}^o$. Let $((x_i^o),(y_j),p)$ be a $\delta$-equilibrium for $\mathcal{E}^o$, where $x_i^o = (x_i,0)$ and $y_j = (y_{j0},y_{j1})$, for all $i$ and $j$. Let $\underset{\sim}{x}_i = (x_i,x_i,\ldots)$ and let $\underset{\sim}{y}_j = (y_j,y_j,\ldots)$, for all $i$ and $j$. Finally, let $\underset{\sim}{p} = (p_0, \delta p_0, \delta^2 p_0,\ldots)$. I claim that

10.1) $\quad ((\underset{\sim}{x}_i)_{i=1}^{I},(\underset{\sim}{y}_j)_{j=1}^{J},\underset{\sim}{p})$ is a stationary equilibrium for $\mathcal{E}$,

provided that $y_{j1}^{-1} = y_{j1}$, for $j = 1,\ldots,J$.

In order to see that (10.1) is true, observe first of all that

10.2) $\quad p_{0k} = p_{1k}$, for all $k \in L_p$ .

This follows from the fact that $p$ is such that $\eta_0^o(p)$ is well-defined. Next, I claim that

10.3) $\sum_i x_i \leqq \sum_i \omega_i + \sum_{j=1} (y_{j0} + y_{j1})$ .

By the feasibility of $((x_i^o),(y_j))$, I know that

10.4) $\sum_i (x_i, 0) \leqq \sum_i (\omega_i, 0) + \sum_{j=0}^{J} (y_{j0}, y_{j1})$.

Also by the definition of $Y_0$,

10.5) $y_{00} = - y_{01}$.

(10.4) implies that $- y_{01} \leqq \sum_{j=1}^{J} y_{j1}$. Hence by (10.4) and (10.5), $\sum_i x_i \leqq \sum_i \omega_i + \sum_{j=0}^{J} y_{j0} \leqq \sum_i \omega_i + \sum_{j=1}^{J} (y_{j0} + y_{j1})$, which is (10.3).

(10.3) implies that $((x_i),(y_j))$ is a feasible stationary allocation for $\mathcal{E}$. That $((\tilde{x}_i),(\tilde{y}_j),\tilde{p})$ is an equilibrium for $\mathcal{E}$ follows from (10.2) by an easy argument, which I omit. This completes the proof of (10.1).

By (10.5), it is sufficient to prove that $\mathcal{E}^o$ has an equilibrium. The proof is standard. I imitate the argument of chapter 5 of Debreu [1959].

First, I truncate $\mathcal{E}^o$ so as to obtain compact consumption and production sets. Let $B > 0$ be as in lemma 7.2. Let $\hat{\mathcal{E}}^o$ be $\mathcal{E}^o$, truncated at B. That is, $\hat{\mathcal{E}}^o = \{(\hat{X}, \hat{u}_i^o, \omega_i), \hat{Y}_j, \theta_{ij}: i = 1,\dots,I;\ j = 0,1,\dots,J\}$, where $\hat{X} = \{(x,0) \in X \mid |x| \leqq B\}$; $\hat{u}_i^o$ is the restriction of $u_i^o$ to $\hat{X}$, and $\hat{Y}_j = \{ y \in Y_j \mid |y| \leqq B\}$.

A $\delta$-equilibrium for $\hat{\mathcal{E}}^o$ is defined in the obvious way. If $((x_i^o),(y_j),p)$ is a $\delta$-equilibrium for $\hat{\mathcal{E}}^o$, then by lemma 7.2, $x_i^o$ belongs to the interior of $\hat{X}$ relative to $X$, for all $i$. Similarly, $y_j$ belongs to the interior of $\hat{Y}_j$ relative to $Y_j$, for all $j$. It follows easily that $((x_i^o),(y_j),p)$ is a $\delta$-equilibrium for $\mathcal{E}^o$. Hence, it is sufficient to prove that $\hat{\mathcal{E}}^o$ has an equilibrium.

For $p \in \Delta$, let $\hat{w}_i(p)$. $\hat{\xi}_i^o(p)$ and $\hat{\eta}_j^o(p)$ be, respectively, the income and demand of consumer $i$ and the supply of firm $j$ in the economy $\hat{\mathcal{E}}^o$. These are defined in the obvious way. Because $\hat{Y}_j$ is compact, $\hat{\eta}_j^o(p)$ is non-empty, for all $j$ and $p$. Similarly, $\hat{\xi}_i^o(p)$ is well-defined and non-empty, provided that $\hat{w}_i(p)$ is well-defined, $\hat{w}_i(p)$ is well-defined if for each $j = 1,\ldots,J$, $p_0 \cdot y_0 + p_1 \cdot y_1$ does not depend on the choice of $(y_0,y_1) \in \hat{\eta}_j^o(p)$. Let $(\bar{y}_0,\bar{y}_1) \in \hat{\eta}_j^o(p)$, where $j \geq 1$. By the strict convexity of $Y_j$ (assumption 3.11), if $p_{0k} > 0$, then $y_{0k} = \bar{y}_{0k}$, for all $(y_0,y_1) \in \hat{\eta}_j^o(p)$. Similarly, if $p_{1k} > 0$, then $y_{1k} = \bar{y}_{1k}$, for all $(y_0,y_1) \in \hat{\eta}_j^o(p)$. Hence, $p_0 \cdot y_0 + p_1 \cdot y_1 = p_0 \cdot \bar{y}_0 + p_1 \cdot \bar{y}_1$, for all $(y_0,y_1) \in \hat{\eta}_j^o(p)$. This proves that $\hat{w}_i(p)$ is well-defined, for all $i$. Hence, $\hat{\xi}_i^o(p)$ is well-defined and non-empty, for all $p$ and $i$.

Now, let $\hat{\hat{\xi}}_i^o(p)$ be defined as follows. $\hat{\hat{\xi}}_i^o(p) = \hat{\xi}_i^o(p)$, if $\hat{\omega}_i(p) > 0$. Otherwise, $\hat{\hat{\xi}}_i^o(p) = \{x^o \in \hat{X} \mid p \cdot x^o = 0\}$.

Let $\zeta(p) = \sum_{i=1}^{I} (\hat{\hat{\xi}}_i^o(p) - \omega_i^o) - \sum_{j=0}^{J} \hat{\eta}_j^o(p)$, for $p \in \Delta$. It is easy to see that the correspondence $\zeta$ has closed graph and maps into a compact set. Also, $p \cdot z \leq 0$, for all $z \in \zeta(p)$. Therefore by

(1) on page 82 of Debreu [1959], there exists $\hat{p} \in \Delta$ and $\hat{z} \in \zeta(\hat{p})$ such that $\hat{z} \leqq 0$. Let $\hat{x}_i^0 = (\hat{x}_i, 0) \in \hat{\xi}_i^o(\hat{p})$, for $i = 1,\dots,I$ and $\hat{y}_j \in \hat{\eta}_j^o(\hat{p})$, for $j = 0,1,\dots,J$ be such that $\sum_{i=1}^{I} (\hat{x}_i^o - \omega_i^o) - \sum_{j=0}^{J} \hat{y}_j = \hat{z}$.

It should be clear that $((\hat{x}_i^o),(\hat{y}_j^0),\hat{p})$ is a quasi $\delta$-equilibrium for $\hat{\mathcal{E}}^o$. That is, $((\hat{x}_i^o),(\hat{y}_j))$ is a feasible allocation for $\hat{\mathcal{E}}^0$; $\hat{p}_{0k} = 0$ if $\sum_{i=1}^{I} \hat{x}_{ik} < \sum_{i=1}^{I} \omega_{ik} + \sum_{j=0}^{J} \hat{y}_{j0k}$ and $\hat{p}_{1k} = 0$ if $0 < \sum_{j=0}^{J} \hat{y}_{j0k}$; $\hat{y}_j \in \hat{\eta}_j^o(\hat{p})$, for all $j$; and $\hat{x}_i^o \in \hat{\xi}_i^o(\hat{p})$, whenever $\hat{w}_i(p) > 0$. Hence, in order to prove that $((\hat{x}_i^o),(\hat{y}_j'),\hat{p})$ is a $\delta$-equilibrium for $\hat{\mathcal{E}}^o$, it is sufficient to prove that

10.6) $\hat{w}_i(p) > 0$, for all $i$.

At this point, it is necessary to introduce a restriction on $\delta$. By assumptions 3.15 and 3.16, there exist $(y_{j0}, y_{j1}) \in Y_j$, for $j = 1,\dots,J$, such that $\sum_{i=1}^{I} \omega_i + \sum_{j=1}^{J} (y_{j0} + y_{j1}) >> 0$. Clearly, there exists $\bar{\delta}$ such that $0 < \bar{\delta} < 1$ and

10.7) $$\sum_{i=1}^{I} \omega_i + \sum_{j=1}^{J} (y_{j0} + \bar{\delta}\, y_{j1}) >> 0.$$

I now assume that $\delta$ is such that $\bar{\delta} \leq \delta < 1$.

In order to prove (10.6), I first prove that

10.8) $$\sum_{i=1}^{I} \hat{w}_i(p) > 0.$$

Let $(y_{j0}, y_{j1})$ be as in 10.7, for $j = 1,\ldots,J$, and let

$$(y_{00}, y_{01}) = (\delta \sum_{j=1}^{J} y_{j1} - \epsilon e, -\delta \sum_{j=1}^{J} y_{j1} + \epsilon e), \text{ where } e = (1,\ldots,1) \in R^{L_p}$$

and $\epsilon > 0$ is so small that the following is true.

10.9) $$\sum_{i=1}^{I} (\omega_i, 0) + (y_{00}, y_{01}) + \sum_{j=1}^{J} (y_{j0}, \delta y_{j1}) >> 0.$$

By lemma 7.2, $|(y_{j0}, y_{j1})| < B$, for $j = 0,1,\ldots,J$. Hence

$$\sum_{i=1}^{I} \hat{w}_i^o(p) = \sum_{i=1}^{I} \hat{p}_0 \cdot \omega_i + (\hat{p}_0 \cdot \hat{y}_{00} + \hat{p}_1 \cdot \hat{y}_{01}) + \sum_{j=1}^{J} (\hat{p}_0 \cdot \hat{y}_{j0} + \hat{p}_1 \cdot \hat{y}_{j1})$$

$$\geqq \sum_{i=1}^{I} \hat{p}_0 \cdot \omega_i + (\hat{p}_0 \cdot \hat{y}_{00} + \hat{p}_1 \cdot \hat{y}_{01}) + \sum_{j=1}^{J} (p_0 \cdot y_{j0} + \delta \hat{p}_1 \cdot \hat{y}_{j1})$$

$$\geqq \sum_{j=1}^{I} \hat{p}_0 \cdot \omega_i + (\hat{p}_0 \cdot y_{00} + \hat{p}_1 \cdot y_{01}) + \sum_{j=1}^{J} (\hat{p}_0 \cdot y_{j0} + \delta \hat{p}_1 \cdot y_{j1})$$

$$= (\hat{p}_0, \hat{p}_1) \cdot \left[ \sum_{i=1}^{I} (\omega_i, 0) + (y_{00}, y_{01}) + \sum_{j=1}^{J} (y_{j0}, \delta y_{j1}) \right] > 0.$$ The first inequality above follows from the fact that $p_1 \cdot y_{j1} > 0$, for all $j$. The second inequality follows from the fact that $(\hat{y}_{j0}\ \hat{y}_{j1}) \in \hat{\eta}_j^o(\hat{p})$, for $j = 0,1,\ldots,J$. The last inequality follows from (10.9). This proves (10.8).

As in the proof of theorem 4.1, it now follows that $p_{0k} > 0$, for all $k \in L_c$. (Here, I use the monotonicity of preferences and the fact that each $\hat{x}_i$ lies in the relative interior of $\hat{X}$.) Hence by assumption 3.14, $\hat{w}_i(p) \geq \hat{p}_0 \cdot \omega_i > 0$, for all $i$. This proves that $((\hat{x}_i),(\hat{y}_j),\hat{p})$ is a $\delta$-equilibrium for $\hat{\mathcal{E}}^o$.

Q.E.D.

11) Proof of Theorem 4.3

I do not give a detailed proof of this theorem, since the proof is completely routine. One approach is as follows. To any equilibrium for $\mathcal{E}$, there corresponds an equilibrium for the economy $\mathcal{E}^{**}$ defined in section 9. (The commodity space of $\mathcal{E}^{**}$ was $\ell_{\infty, L \cup \{1,\ldots,J\}}$.) A theorem of Debreu implies that the $\mathcal{E}^{**}$ - equilibrium allocation is Pareto optimal among feasible allocations for $\mathcal{E}^{**}$. (See Debreu [1954], theorem 1, p. 589.) It follows at once that the corresponding equilibrium allocation for $\mathcal{E}$ is Pareto optimal.

12) Proof of Theorem 4.4

Let $((\underset{\sim}{x}_i),(\underset{\sim}{y}_j),\underset{\sim}{p})$ be an equilibrium and let $\Lambda_i > 0$ be the marginal utility of expenditure for consumer $i$ in the equilibrium. Suppose that $p$ is so normalized that $\sum_i \Lambda_i = 1$. Let $B$ be as in lemma 7.1. Then, $|x_i^t| \leq B$, for all $i$ and $t$. By assumptions 3.3 and 3.4, $Du_i(x)$ is a continuous function of $x$ whose components are positive. Therefore, there exist positive numbers $a$ and $b$ such that $a \leq \frac{\partial u_i(x)}{\partial x_k} \leq b$, for all $i$ and $k$, if $|x| \leq B$. Let

$\delta = \max_i \delta_i$.

Let $i$ be such that $\delta_i = \delta$ . Then, for any $t$ and $k$ ,

$$\delta^{-t} p_k^t \geqq \Lambda_i \delta^{-t} p_k^t \geqq \frac{\partial u_i(x_i^t)}{\partial x_k} \geqq a \quad .$$

Now suppose that $i$ is such that $\delta_i < \delta$ . If $x_{ik}^t > 0$, then

$$b \geqq \frac{\partial u_i(x_i^t)}{\partial x_k} = \Lambda_i \delta_i^{-t} p_k^t \geqq \Lambda_i (\delta_i^{-1}\delta)^t a.$$ Let $T$ be such that $\Lambda_i(\delta_i^{-1}\delta)^T a > b$ .

Then if $t \geqq T$ , it must be that $x_i^t = 0$ .

Q.E.D.

13) <u>Proof of Theorem 4.5</u>

Let $((\hat{x}_i), (\hat{y}_j), \hat{p})$ be a competitive equilibrium for $\mathcal{E}$ . I first define the allocation to which $((\hat{x}_i), (\hat{y}_j))$ converges. For each $i$ , let $\Lambda_i > 0$ be the marginal utility of expenditure for consumer $i$ in the equilibrium $((\hat{x}_i), (\hat{y}_j), \hat{p})$ . This marginal utility was defined in section 2. I assume that $\hat{p}$ is so normalized that $\sum_i \Lambda_i = 1$. Let $U: R_+^{L_c} \to (-\infty, \infty)$ be defined by $U(x) = \max \{ \sum_i \Lambda_i^{-1} u_i(x_i) \mid x_i \in R_+^{L_c}$, for all and $\sum_i x_i = x \}$ . Let $\mathcal{E}'$ be the economy obtained from $\mathcal{E}$ by replacing all the consumers with a single consumer whose utility function is $U$ and whose intiial endowment is $\omega = \sum_i \omega_i$ . By a slight modification of theorem 4.2, $\mathcal{E}'$ has a stationary equilibrium $(\bar{x}, (\bar{y}_j), \bar{p})$ . I will assume that $\bar{p}$ is so normalized that the marginal utility of expenditure of the single consumer is one. Let $(\bar{x}_i)$ be such that $\bar{x} = \sum_i \bar{x}_i$ and $U(\bar{x}) = \sum_i \Lambda_i^{-1} u(\bar{x}_i)$ . It is easy to see that $((\bar{x}_i), (\bar{y}_j), \bar{p})$ is a stationary equilibrium for $\mathcal{E}$ with transfer payments. In this equilibrium, the marginal utility of expenditure for each consumer $i$ is $\Lambda_i$. $((\bar{x}_i),(\bar{y}_j))$

is the stationary allocation to which $((\hat{x}_i), (\hat{y}_j))$ converges.

The proof that $((\hat{x}_i),(\hat{y}_j))$ converges to $((\bar{x}_i),(\bar{y}_j))$ uses the fact that these allocations solve related maximization problems. The set of feasible allocations for $\mathcal{E}$ depends on the initial holdings, $\sum_j y_{j1}^{-1}$, of produced goods in the economy. Think of these initial stocks as a variable. This variable is denoted by K, where $K \in R_+^{L_p}$. Let $\hat{K}^0$ be the vector of initial resources associated with the given equilibrium $((\hat{x}_i), (\hat{y}_j), \hat{p})$. For each value of $K$, let $\mathcal{J}(K)$ be the set of feasible allocations $((x_i), (y_j))$ for $\mathcal{E}$ such that $\sum_i x_i^0 \leq \sum_i \omega_i + \sum_j y_{j0}^0 + K$. The relevant maximization problem is the following.

$$13.1) \quad \max \{ \sum_{t=0}^{\infty} \delta^t \sum_i \Lambda_i^{-1} u_i(x_i^t) \mid ((x_i), (y_j)) \in \mathcal{J}(K)\} .$$

The stationary allocation $((\bar{x}_i), (\bar{y}_j))$ solves this problem with initial resources $\bar{K} = \sum_j \bar{y}_{j1}$. The allocation $((\hat{x}_i), (\hat{y}_j))$ solves this problem with initial resources $\hat{K}^0$. These assertions may be proved as follows. Because $((\bar{x}_i), (\bar{y}_j), \bar{p})$ and $((\hat{x}_i), (\hat{y}_j), \hat{p})$ are both equilibria with marginal utilities of expenditure $\Lambda_1,\ldots,\Lambda_I$, they solve the first order conditions for solutions of (13.1). Because the constraints are convex and the objective function is concave, any solution of the first order conditions is an optimum.

Problem 13.1 is a variant of the maximization problem traditional in growth theory. I now simply adapt the well-known proofs of the turnpike theorem to the situation here.

First, I define an appropriate Liapunov function. For $K \in R_+^{L_p}$, let $V_\delta(K) = \sum_{t=0}^{\infty} \delta^t \sum_i \Lambda_i^{-1} [u_i(x_i^t) - u_i(\bar{x}_i)]$, where $((\underset{\sim}{x}_i), (\underset{\sim}{y}_j))$ is a solution to problem 13.1 with initial stocks $K$. (Of course, $V_\delta(K)$ exists only if problem 13.1 has a solution with initial stocks $K$.) Recall that $\underset{\sim}{\bar{p}}$ is of the form $\underset{\sim}{\bar{p}} = (\bar{p}, \delta \bar{p}, \delta^2 \bar{p}, \ldots.)$

Let $F_\delta(K) = \bar{p} \cdot (K - \bar{K}) - V_\delta(K)$. $F_\delta$ is the Liapunov function I will use.

The diagram below may help one visualize $F_\delta$.

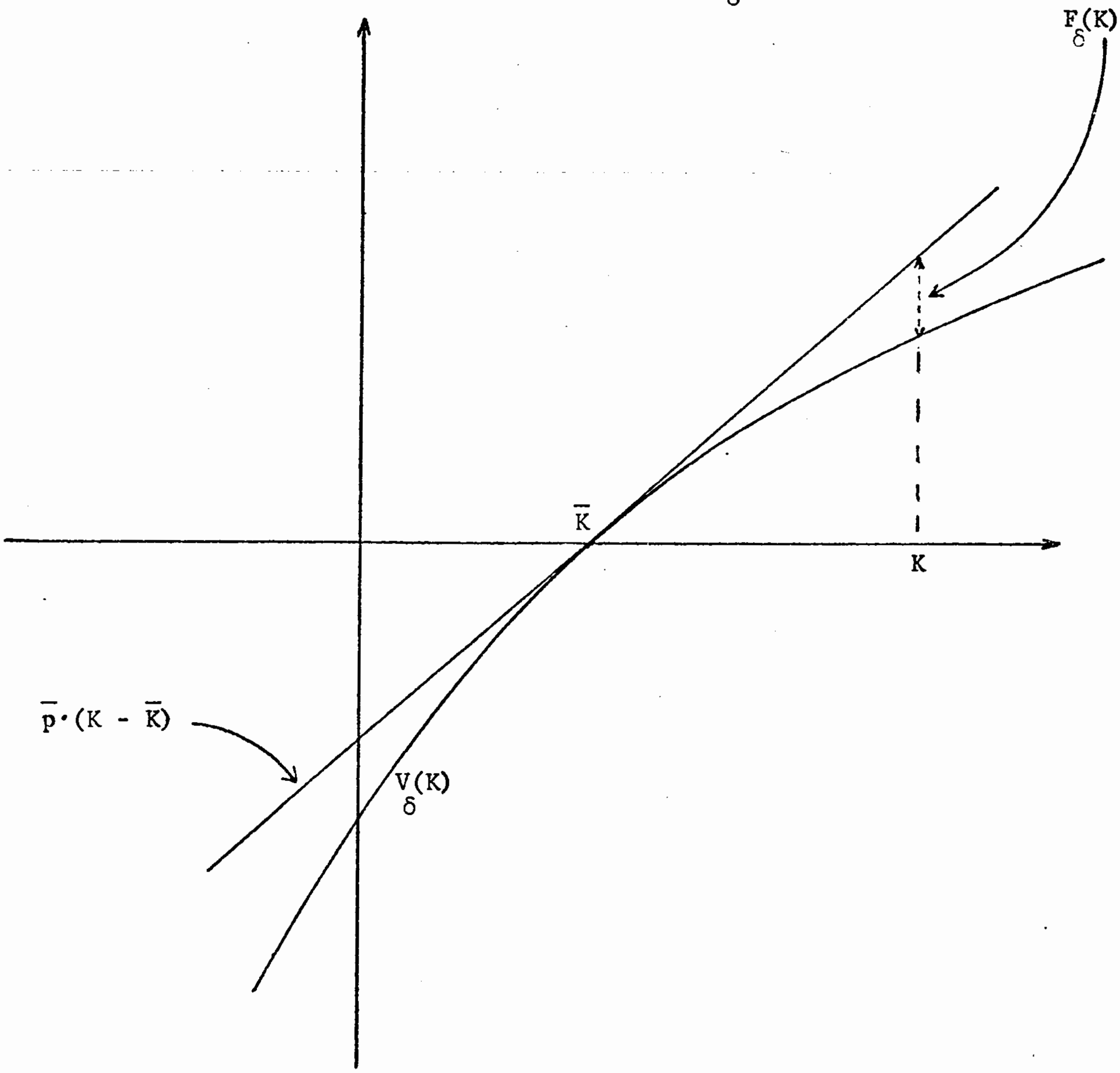


Figure 1

I now turn to a number of technical matters, which, unfortunately, take up a good deal of space. A series of lemmas then follow which establish properties of $F_\delta$ . The actual proof of convergence is contained in the last few paragraphs.

13.2) <u>Lemma</u> There exist $\lambda > 0$ such that $\Lambda_i > \lambda$ , for all $i$ , no matter what the value of $\delta$ .

<u>Proof</u> As in the proof of theorem 4.4, there exist numbers $a$ and $b$ such that $a \leqq \frac{\partial u_i(x_i^t)}{\partial x_k} \leqq b$ , for all $i$ , $t$ and $k$ .

By the definition of $\Lambda_i$, $\frac{\delta^t \partial u_i(x_i^t)}{\partial x_k} \leqq \Lambda_i p_k^t$ , with equality if $x_{ik}^t > 0$ , for all $i$ , $k$ and $t$ . Since $\sum_i \Lambda_i = 1$ , there is $i$ such that $\Lambda_i \geqq I^{-1}$ . At the end of the proof of theorem 4.1, I noted that the income of every consumer is positive in equilibrium. Therefore, $x_{ik}^t > 0$ , for some $i$ , $t$ and $k$ , so that $b \geqq \frac{\partial u_i(x_i^t)}{\partial x_k} = \Lambda_i \delta^{-t} p_k^t \geqq I^{-1} \delta^{-t} p_k^t$ . That is, $\delta^{-t} p_k^t \leqq bI$ . It now follows that for the same value of $t$ and for any $n = 1, \ldots, I$ , $a \leqq \frac{\partial u_n(x_n^t)}{\partial x_k} \leqq \Lambda_n \delta^{-t} p_k^t \leqq \Lambda_n bI$ . In conclusion, $\Lambda_n \geqq ab^{-1} I^{-1}$ .

Q.E.D.

I next show that I may assume that

13.3) if $((\underset{\sim}{x}_i), (\underset{\sim}{y}_j), \underset{\sim}{p})$ is any equilibrium for $\mathcal{E}$ , then $p^t >> 0$ and $\sum_i x_i^t = \sum_i \omega_i + \sum_j (y_{j0}^t + y_{j1}^t)$, for all $t$ .

The argument supporting this assumption requires some new terminology. I say that firm $j$ can use good $k_0$ to produce good $k$, if $M_{j0}$ contains the $k_0{}^{th}$ coordinate axis and $M_{j1}$ contains the $k_1{}^{th}$ coordinate axis. I call $ik_0$ or $ik_N j_N \ldots\ldots j_2 k_1 j_1 k_0$ a <u>productive sequence</u> if the following are true: 1) $k_n \in \{1,\ldots,L\}$, for all $n$; 2) $k_N \in L_c$; 3) $j_n \in \{1,\ldots,J\}$, for all $n$; 4) $i \in \{1,\ldots,I\}$; 5) for all $n$, firm $j_n$ can use good $k_{n-1}$ to produce good $k_n$; and 6) there are no repetitions in the sequences $k_0,\ldots,k_N$ and $j_1,\ldots,j_N$. I call good $k$ <u>productive</u> if $k = k_0$ in some productive sequence $ik_0$ or $ik_N j_N \ldots j_1 k_0$.

It should be clear that all productive good will have positive price in any equilibrium for $\mathcal{E}$. Similarly, goods which are not productive will never be consumed and need never be used in production. These facts follow from monotonicity in consumption (assumption 3.4) and in production (assumptions 3.8 and 3.10). In conclusion, goods which are not productive may be eliminated from the economy.

The interiority assumption implies that all goods are productive. This fact, in turn, implies (13.3).

I now prove that

13.4) there exist numbers $\underline{q}$ and $\overline{q}$ such that $0 < \underline{q} \leq \overline{p} \leq \overline{q}$, for all $k$, if $\delta \geqq \underline{\delta}$,

where $\underline{\delta}$ is as in the interiority assumption.

With any productive sequence $ik_N j_N \ldots k_1 j_1 k_0$, I associate the number $q(ik_N j_N \ldots k_1 j_1 k_0) = \delta^N \Lambda_i^{-1} \dfrac{\partial u_i(\bar{x}_i)}{\partial x_{k_N}} \left( \dfrac{\partial q_{j_N}(\bar{y}_{j_N})}{\partial y_{1k_N}} \right)^{-1}$
$\left( \dfrac{\partial q_{j_N}(\bar{y}_{j_N})}{\partial y_{0k_{N-1}}} \right) \cdots\cdots \left( \dfrac{\partial q_{j1}(\bar{y}_{j_1})}{\partial y_{0k_0}} \right) > 0$. Similarly, with the productive sequence $ik_0$, I associate the number $q(ik_0) = \Lambda_i^{-1} \dfrac{\partial u_i(\bar{x}_i)}{\partial x_{k_0}} > 0$.

Let $q_k = \max \{ q(ik_N j_N \ldots j_1 k_0) \mid ik_N j_N \ldots j_1 k_0$ is a productive sequence and $k_0 = k \}$. There are only finitely many possible productive sequences, since there are no repetitions in the sequence $k_N, \ldots, k_0$. Therefore, the maximum in the above definition makes sense.

Since all goods are productive, $q_k$ 0, for all k.

If $\delta \geqq \underline{\delta}$, then by the interiority assumption every good is either produced or available as a primary good in the stationary equilibrium $((\bar{x}_i), (\bar{y}_j), \bar{p})$. It follows that $\bar{p} = q$.

It is now easy to see that (13.4) is true. Let B be as in lemma 7.2. By assumptions 3.3 and 3.4, $\frac{\partial u_i(x)}{\partial x_k}$ is uniformly bounded away from zero and infinity, for $|x| \leqq B$. Similarly by assumptions 3.9 and 3.10, the partial derivatives of $g_j$ at y are bounded away from zero and infinity, for $|y| \leqq B$. By lemma 13.2, $\lambda \leqq \Lambda_i \leqq 1$, for all i. Assertion (13.4) follows at once.

I now state an analogue of lemma 13.2, which applies to firms. For each j, there is $\rho_j > 0$ such that $\bar{p}_k \geqq \rho_j \frac{\partial q_j(\bar{y}_j)}{\partial y_{0k}}$, for all k, with equality if $\bar{y}_{j0k} < 0$. Also $\delta \bar{p}_k \leqq \rho_j \frac{\partial g_j(\bar{y}_j)}{\partial y_{1k}}$, with equality if $y_{j1k} > 0$. If $\bar{y}_j = 0$, I normalize $\rho_j$ by assuming that $\bar{p}_k = \rho_j \frac{\partial q_j(\bar{y}_j)}{\partial y_{0k}}$, for some k.

13.5) There are numbers $\underline{\rho}$ and $\bar{\rho}$ such that $0 < \underline{\rho} \leqq \rho_j \leqq \bar{\rho}$, for all j, if $\delta \geqq \underline{\delta}$.

This fact follows from (13.4) and because $|\bar{y}_j| \leqq B$ and $Dg_j(y)$ is a continuous function of $y$ .

I next define a process, which I call distributing surpluses. The idea is to modify a feasible allocation so as to absorb additional supplies. I do so in such a way as to control the size of the modification.

Let $((\underset{\sim}{x}_i), (\underset{\sim}{y}_j))$ be a feasible allocation and suppose that at time $T$ , $a > 0$ units of good $k$ are made available to the economy. I define a new allocation, $((\tilde{\underset{\sim}{x}}_i), (\tilde{\underset{\sim}{y}}_j))$ which makes use of these supplies. For simplicity, I assume that $T = 0$ .

In order to construct $((\tilde{\underset{\sim}{x}}_i), (\tilde{\underset{\sim}{y}}_j))$ , I need some new terminology. $e_k = (0,\ldots,0,1,0,\ldots,0)$ denotes the $k^{th}$ standard basis vector of $R^L$ . I say that a productive sequence $ik_Nj_N\cdot\ldots j_1k_0$ realizes $\bar{p}_k$ if $k_0 = k$ and $\bar{p}_k = q(ik_Nj_N\cdot\ldots j_1k_0)$ .

Let $ik_Nj_N\cdot\ldots j_1k_0$ be a productive sequence which realizes $\bar{p}_k$ . I assume that $N > 0$ . The construction for the case $N = 0$ will be obvious.

I next define a sequence of positive numbers, $a_0,\ldots,a_N$ . Let $a_0 = a$ . Given $a_{t-1}$ , let $a_t = \max\{b \geq 0 \mid (y^t_{j_t0} - a_{t-1}e_{k_{t-1}}, y^t_{j_t1} + be_{k_t}) \in Y_{j_t}\}$ . By assumptions 3.8 - 3.10, $a_t$ is well-defined and positive.

Now let $\tilde{y}_{j_t} = (y^t_{j_t0} - a_{t-1}e_{k_{t-1}}, y^t_{j_t1} + a_te_{k_t})$ , for $t = 0,\ldots,N$. Let $\tilde{x}^N_i = x^N_i + a_Ne_{k_N}$ . If $j \neq j_t$ , let $\tilde{y}_j = y_j$ . If $t \neq N$ or if $i' \neq i$ , let $x^t_{i'} = x^t_{i'}$ . This defines $((\tilde{\underset{\sim}{x}}_i), (\tilde{\underset{\sim}{y}}_j))$ .

I now show how to control the size of the numbers $a_n$ defined above. Again, I need new notation. The number

$$\left(\frac{\partial g_j(y)}{\partial y_{1k_1}}\right)^{-1}\left(\frac{\partial g_j(y)}{\partial y_{0k_0}}\right)$$ may be thought of as a

marginal rate of transformation. It is denoted by $MRT(j,k_0,k_1;y)$. It is well-defined only if firm $j$ can use good $k_0$ to produce good $k_1$.

By assumption 3.9 and 3.10,

13.6) there exists $\epsilon > 0$ such that

$$|MRT(j,k_0,k_1;y) - MRT(j,k_0,k_1,y')| \leq \tfrac{1}{2}MRT(j,k_0,k_1;y),$$

provided that these marginal rates of transformation are well-defined and that $|y - y'| < \epsilon$ and $|y| \leq B$.

where $B$ is as in lemma 7.2.

I now exploit the fact that if $ik_Nj_N\cdots j_1k_0$ realizes $\bar{p}_{k_0}$, then $MRT(j_n,k_{n-1},k_n;\bar{y}_{j_n}) = \delta^{-1}\bar{p}_{k_n}^{-1}\bar{p}_{k_{n-1}}$, for $n = 1,\ldots,N$. This fact and (13.6) imply the following.

13.7) Suppose that the allocation $((\underset{\sim}{x}_i),(\underset{\sim}{y}_j))$ satisfies $g_j(y_j^t) = 0$ and $|y_j^t - \bar{y}_j| < \epsilon/2$, for all $t$ and $j$, where $\epsilon$ is as in (13.6). Suppose also that $a > 0$ units of good $k$ are distributed in the manner described above, giving rise to the new allocation $((\tilde{\underset{\sim}{x}}_i),(\tilde{\underset{\sim}{y}}_j))$. Suppose that $a$ is so small that $3^L\underline{q}^{-1}\bar{q}\,a \leq \epsilon/2$. If

$\delta \geqq \frac{1}{2}$ , then $a_n \leqq 3^L \underline{q}^{-1} \bar{q}\, a$ , for all $n$ .

In the above, $\underline{q}$ and $\bar{q}$ are as in (13.4). (13.6) and the definition of $a_n$ imply that

$a_n \leqq \frac{3}{2} \delta^{-1} \bar{p}_{k_n}^{-1} \bar{p}_{k_{n-1}} a_{n-1} \leqq 3\, \underline{q}^{-1} \bar{q}\, a_{n-1}$, for all $n$. Since $N \leqq L$, $a_n \leqq 3^L \underline{q}^{-1} \bar{q}\, a_0 = 3^L \underline{q}^{-1} \bar{q}\, a$ , for all n. The condition on $a$ in (13.7) implies that (13.6) may be applied at each passage from $a_{n-1}$ to $a_n$, for $n = 0,\ldots,N$.

Now suppose that a vector, $z \in R^L_+$, of goods is made available at time $T$ and suppose that a feasible allocation $((x_i),(y_j))$ is given. Proceeding as above, distribute the first component $z_1$, obtaining an allocation $((^1x_i),(^1y_j))$. Then, distribute the second component, $z_2$, obtaining $((^2x_i),(^2y_j))$. Continuing inductively, one distributes all components of $z$, obtaining an allocation $((\tilde{x}_i),(\tilde{y}_j)) = ((^Lx_i),(^Ly_j))$. I now compute an upper bound on the distance between $((\tilde{x}_i),(\tilde{y}_j))$ and $((x_i),(y_j))$.

Suppose that $| y^t_j - \underline{y}_j | < \varepsilon/2$, for all $j$ and $t$, where $\varepsilon$ is as in (13.6). Suppose also that $g_j(y^t_j) = 0$ and that $| z | \leqq (L\, 3^L \underline{q}^{-1} \bar{q})^{-1} \varepsilon/2$.

The various goods in $z$ might be used to produce one good later. Taking this into account and using (13.7), I obtain the following.

13.8) If $\delta \geqq \frac{1}{2}$ , then $| ((\tilde{x}^{T+t}_i),(\tilde{y}^{T+t}_j)) - ((x^{T+t}_i),(y^{T+t}_j)) | \leqq L\, 3^L \underline{q}^{-1} \bar{q}\, | z |$, if $0 \leq t \leq L$. Otherwise, $((\tilde{x}^t_i),(\tilde{y}^t_j)) = ((x^t_i),(y^t_j))$.

I now turn to the properties of the Liapunov function $F_\delta$ defined earlier.

The next lemma says that figure 1 is correct. It says that $\bar{p}$ is a subgradient of $V$ at $\bar{K}$.

13.9 ) Lemma If $F_\delta(K)$ is well-defined, then $F_\delta(K) \geqq 0 = F_\delta(\bar{K})$.

Proof It is obvious that $F_\delta(\bar{K}) = 0$.

Let $((\underset{\sim}{x}_i),(\underset{\sim}{y}_j)) \in \mathcal{J}(K)$. I must show that

$$13.10) \quad \sum_{t=0}^{\infty} \delta^t \sum_i \Lambda_i^{-1}(u_i(x_i^t) - u_i(\bar{x}_i)) \leqq \bar{p} \cdot (K - \bar{K}) \quad .$$

Choose $(y_{j1}^{-1})$ arbitrarily so that $\sum_j y_{j1}^{-1} = K$. The following equation is simply an identity.

$$13.11) \quad \bar{p} \cdot (K - \bar{K}) - \sum_{t=0}^{\infty} \delta^t \sum_i \Lambda_i^{-1} (u_i(x_i^t) - u_i(\bar{x}_i))$$

$$= \sum_{t=0}^{\infty} \delta^t \sum_i [\bar{p} \cdot (x_i^t - \bar{x}_i) - \Lambda_i^{-1}(u_i(x_i^t) - u_i(\bar{x}_i))]$$

$$+ \sum_{t=0}^{\infty} \delta^t \bar{p} \cdot [ \sum_j (y_{j0}^t + y_{j1}^{t-1}) - (\bar{y}_{j0} + \bar{y}_{j1})$$

$$- \sum_i (x_i^t - \bar{x}_i)]$$

$$+ \sum_{t=0}^{\infty} \delta^t \sum_j [(\bar{p} \cdot \bar{y}_{j0} + \delta \bar{p} \cdot y_{j1}) - (\bar{p} \cdot y_{j0}^t + \delta \bar{p} \cdot y_{j1}^t)] \quad .$$

It is convenient to write the right hand side of the above as $S_1 + S_2 + S_3$, where $S_i$ is the $i^{th}$ infinite sum. $S_1$ can be rewritten as follows.

$$13.12) \quad S_1 = \sum_{t=0}^{\infty} \delta^t \sum_i [(\Lambda_i^{-1} u_i(\bar{x}_i) - \bar{p} \cdot \bar{x}_i)$$

$$-(\Lambda_i^{-1} u_i(x_i^t) - \bar{p} \cdot x_i^t)] .$$

In rewriting $S_2$, I use the fact that $\sum_i (\bar{x}_i - \omega_i) = \sum_j (\bar{y}_{j0} + \bar{y}_{j1})$.

13.13) $$S_2 = \sum_{t=0}^{\infty} \delta^t \bar{p} \cdot [\sum_j (y_{j0}^t + y_{j1}^{t-1}) + \sum_i (\omega_i - x_i^t)].$$

For future reference, I record the formula for $S_3$.

13.14) $$S_3 = \sum_{t=0}^{\infty} \delta^t \sum_j [(\bar{p} \cdot \bar{y}_{j0} + \delta\, \bar{p} \cdot \bar{y}_{j1}) - (\bar{p} \cdot y_{j0}^t + \delta\, \bar{p} \cdot y_{j1}^t)].$$

Clearly, $\bar{x}_i$ maximizes the function $\Lambda_i^{-1} u_i(x) - \bar{p} \cdot x$, so that $S_1 \geqq 0$. Since $((\underset{\sim}{x}_i),(\underset{\sim}{y}_j))$ is feasible, $\sum_j (y_{j0}^t + y_{j1}^{t-1}) + \sum_i (\omega_i - x_i^t) \geqq 0$,

so that $S_2 \geqq 0$. By profit maximization in the equilibrium $((\bar{\underset{\sim}{x}}_i),(\bar{\underset{\sim}{y}}_j),\bar{\underset{\sim}{p}})$, $S_3 \geqq 0$.

Q.E.D.

Recall that $\hat{K}^0$ is the vector of initial resources associated with the equilibrium $((\hat{\underset{\sim}{x}}_i),(\hat{\underset{\sim}{y}}_j),\hat{\underset{\sim}{p}})$. For $t > 0$, let $\hat{K}^t = \sum_j \hat{y}_{j0}^{t-1}$.

Since all goods are assumed to be productive, $\hat{K}^t = \sum_i (\hat{x}_i - \omega_i) - \sum_j \hat{y}_{j0}^t$.

13.15) <u>Lemma</u> $\delta F_\delta(\hat{K}^{t+1}) - F_\delta(\hat{K}^t)$

$$= \sum_i [(\Lambda_i^{-1} u_i(\hat{x}_i^t) - \bar{p} \cdot \hat{x}_i^t) - (\Lambda_i^{-1} u_i(\bar{x}_i) - \bar{p} \cdot \bar{x}_i)]$$

$$+ \sum_j [(\bar{p} \cdot \hat{y}_{j0}^t + \delta\, \bar{p} \cdot \hat{y}_{j1}^t) - (\bar{p} \cdot \bar{y}_{j0} + \delta\, \bar{p} \cdot \bar{y}_{j1})]$$

$$\leqq 0, \text{ for all } t \geqq 0.$$

<u>Proof</u> By the definition of $F_\delta$

13.16) $$\delta F_\delta(\hat{K}^{t+1}) - F_\delta(\hat{K}^{t+1}) = \delta p \cdot (\hat{K}^{t+1} - \bar{K}) - p \cdot (\hat{K}^{t} - \bar{K})$$

$$+ V_\delta(\hat{K}^t) - \delta V_\delta(\hat{K}^{t+1})$$

Clearly, $V_\delta(\hat{K}^t) = \sum_{n=t}^{\infty} \delta^{n-t} \sum_i \Lambda_i^{-1}(u_i(\hat{x}_i^n) - u_i(\bar{x}_i))$. Substituting this into (13.14) and rearranging, I obtain

$$\delta F_\delta(\hat{K}^{t+1}) - F_\delta(\hat{K}^t) = (\delta \bar{p} \cdot \hat{K}^{t+1} - \bar{p} \cdot \hat{K}^t)$$

$$- (\delta \bar{p} \cdot \bar{K} - \bar{p} \cdot \bar{K}) + \sum_i \Lambda_i^{-1}(u_i(\hat{x}_i^t) - u_i(\bar{x}_i)).$$

I substitutute $\sum_i (\hat{x}_i^t - \omega_i) - \sum_j \hat{y}_{j0}^t$ for $\hat{K}^t$, $\sum_j \hat{y}_{j1}^t$ for $\hat{K}^{t+1}$, $\sum_j \bar{y}_{j1}$ for the first $\bar{K}$ and $\sum_i(\bar{x}_i - \omega_i) - \sum_j \bar{y}_{j0}$ for the second $\bar{K}$. The result is the following.

$$\delta F_\delta(\hat{K}^{t+1}) - F_\delta(\hat{K}^t) = \delta \sum_j \bar{p} \cdot \hat{y}_{j1}^t$$

$$- \bar{p} \cdot [\sum_i(\hat{x}_i^t - \omega_i) - \sum_j \hat{y}_{j0}^t] - \delta \sum_j \bar{p} \cdot \bar{y}_{j1}$$

$$+ \bar{p} \cdot [\sum_i (\bar{x}_i - \omega_i) - \sum_j \bar{y}_{j0}] + \sum_i \Lambda_i^{-1}(u_i(\hat{x}_i^t) - u_i(\bar{x}_i)).$$

Upon rearrangement, the right hand side of this inequality becomes the expression in lemma 13.15. That this expression is non-positive follows immediately from the fact that $\Lambda_1,\ldots,\Lambda_I$ are the marginal utilities of expenditure in the equilibrium $((\bar{x}_i),(\bar{y}_j),\bar{p})$.

Q. E. D.

13.17) Lemma There exist $\alpha > 0$ and $\varepsilon > 0$ such that

$$\delta F_\delta(\hat{K}^{t+1}) - F_\delta(\hat{K}^t) \leqq - \alpha \min [\varepsilon^2, |((\hat{x}_i^t),(\hat{y}_j^t)) - ((\bar{x}_i),(\bar{y}_j))|^2],$$

provides that $\delta \geq \underline{\delta}$

Proof By assumption 3.5, the function of $y$, $u_i(y) - Du_i(x)\cdot y$, achieves a unique maximum at $y = x$. Let $B$ be as in lemma 7.1. Since $D^2u_i(x)$ is negative definite and continuous in $x$, there are $\varepsilon_c > 0$ and $\alpha_c > 0$ such that $(u_i(y) - Du_i(x)\cdot y) - (u_i(x) - Du_i(x)\cdot x)$ $\leqq -\alpha_c\,|y-x|^2$, for all $i$, if $|x| \leqq B$ and $|y-x| < \varepsilon_c$. Recall that $\Lambda_i \leqq 1$, for all $i$. It now follows that, for all $i$, $(\Lambda_i^{-1}u_i(\hat{x}_i^t) - \bar{p}\cdot\hat{x}_i^t) - (\Lambda_i^{-1}u_i(\bar{x}_i) - \bar{p}\cdot\bar{x}_i) \leqq (\Lambda_i^{-1}u_i(\hat{x}_i^t) - \Lambda_i^{-1}Du_i(\bar{x}_i)\cdot\hat{x}_i^t)$ $-(\Lambda_i^{-1}u_i(\bar{x}_i) - \Lambda_i^{-1}Du_i(\bar{x}_i)\cdot\bar{x}_i) \leqq -\alpha_c\,|\hat{x}_i^t - \bar{x}_i|^2$, provided that $|\hat{x}_i^t - \bar{x}_i| \leqq \varepsilon_c$. By concavity, $(\Lambda_i^{-1}u_i(\hat{x}_i^t) - \bar{p}\cdot\hat{x}_i^t) - (\Lambda_i^{-1}u_i(\bar{x}_i) - \bar{p}\cdot\bar{x}_i)$ $\leqq -\alpha_c\,\varepsilon_c^2$, when $|\hat{x}_i^t - \bar{x}_i| \geqq \varepsilon_c$.

I now apply the preceeding argument to the profits of firms. Let $B$ be as above. Suppose that $y' \in Y_j$ and $g_j(y') = 0$. Then by assumption 3.11, there exist $\alpha_p > 0$ and $\varepsilon_p > 0$ such that for all $j$, $Dg_j(y')\cdot(y-y')$ $\leqq -\alpha_p\,|y-y'|^2$, provided that $y \in Y_j$, $|y| \leqq B$ and $|y-y'| \leqq \varepsilon_p$. Let $\rho_j$ and $\underline{\rho}$ be as in (13.5). Then, $(\bar{p}\cdot\hat{y}_{j0}^t + \delta\bar{p}\cdot\hat{y}_{j1}^t)$ - $(\bar{p}\cdot\bar{y}_{j0} + \delta\bar{p}\cdot\bar{y}_{j1}) \leqq \rho_j Dg_j(\bar{y}_j)\cdot(\hat{y}_j^t - \bar{y}_j) \leqq -\underline{\rho}\alpha_p \min(\varepsilon_p^2, |\hat{y}_j^t - \bar{y}_j|^2)$.

It should no be clear that the lemma is true, with $\varepsilon = \min(\varepsilon_c, \varepsilon_p)$ and $\alpha = \min(\alpha_c, \underline{\rho}\,\alpha_p)$.

Q.E.D.

Lemmas 13.9 and 13.17 imply the following.

13.18) $\quad F_\delta(\hat{K}^t) \geqq \alpha \min\left(\varepsilon^2, |((\hat{x}_i^t), (\hat{y}_j^t)) - ((\bar{x}_i), (\bar{y}_j))|^2\right)$.

Hence, in order to demonstrate that $((\hat{\underset{\sim}{x}}_i), (\hat{\underset{\sim}{y}}_j))$ converges to $((\bar{x}_i), (\bar{y}_j))$ exponentially, it is sufficient to prove that $F_\delta(\hat{K}^t)$ converges to zero exponentially.

The next lemma is simply a corollary of the previous one.

13.19) Lemma There exist $\alpha > 0$ and $\epsilon > 0$ such that $F_\delta(\hat{K}^{t+1}) - \delta^{-1}F_\delta(\hat{K}^t) \leqq -2\alpha \min(\epsilon^2, |\hat{K}^t - \bar{K}|^2)$, no matter what the value of $\delta$ may be.

Proof It is enough to observe that $|((\hat{x}_i^t), (\hat{y}_j^t)) - ((\bar{x}_i), (\bar{y}_j)|$ $\geqq (I + J)^{-1} |\hat{K}^t - \bar{K}|$ .

Q.E.D.

The next lemma puts an upper bound on $F_\delta(\hat{K}^0)$,

13.20) Lemma There exist $C > 0$ such that $F_\delta(\hat{K}^0) \leqq C$ , for all $\delta > 0$.

I prove this lemma by adapting an argument of Gale [1967] (the proof of theorem 6 on p.12). It is at this point that I use the hypothesis that $\sum_j y_{j1k}^{-1} = \hat{K}_k^0 > 0$ , for all $k \in L_p$ .

Proof By assumption 3.15 and 3.16, there exist $y_j \in Y_j$, for $j = 1, \ldots, J$, such that $\sum_i \omega_i + \sum_j (y_{j0} + y_{j1}) >> 0$ . I may assume that $\sum_j y_{j1} \leqq \sum_j y_{j1}^{-1}$ , for I may multiply the $y_j$ by an arbitrarily small positive constant. Hence, I may assume that $\sum_i \omega_i + \sum_j (y_{j0} + y_{j1}^{-1}) >> 0$ .

Choose $\alpha$ such that $0 < \alpha < 1$ and $\alpha$ is so close to one that

13.21) $$(1 - \alpha) \sum_i \bar{x}_i \leqq \sum_i \omega_i + (1 - \alpha) \sum_j \bar{y}_{j0} + \alpha \sum_j y_{j0} + \sum_j y_{j1}^{-1} .$$

Let $x_i^t = (1 - \alpha^{t+1}) \bar{x}_i$ and let $y_j^t = (1 - \alpha^{t+1})\bar{y}_j + \alpha^{t+1} y_j$. Clearly,

$((x_i),(y_j))$ is an allocation. I prove that it is feasible.

First, I remark that $((x_i^{t+1}), (y_j^{t+1})) = (1-\alpha)((\bar{x}_i), (\bar{y}_j)) + \alpha((x_i^t), (y_j^t))$, for all $t$.

I now prove by induction on $t$ that

13.22) $\sum_i x_i^t \leqq \sum_i \omega_i + \sum_j (y_{j0}^t + y_{j1}^{t-1})$, for all $t$.

By (13.21), $\sum_i x_i^0 = (1-\alpha) \sum_i \bar{x}_i \leqq \sum_i \omega_i + (1-\alpha) \sum_j \bar{y}_{j0} + \alpha \sum_j y_{j0} + \sum_j y_{j1}^{-1}$ $= \sum_i \omega_i + \sum_j y_{j0}^0 + \sum_j y_{j1}^{-1}$. This proves (13.22) for the case $t = 0$.

Suppose that (13.22) is true for $t$. Then, $\sum_i x_i^{t+1} = (1-\alpha) \sum_i \bar{x}_i + \alpha \sum_i x_i^t \leqq (1-\alpha)[\sum_i \omega_i + \sum_j (\bar{y}_{j0} + \bar{y}_{j1})] + \alpha[\sum_i \omega_i + \sum_j (y_{j0}^t + y_{j1}^{t-1})] =$ $\sum_i \omega_i + \sum_j (y_{j0}^{t+1} + y_{j1}^t)$. The inequality follows from the inductive hypothesis. This proves (13.22).

Observe that $\lim_{t \to 0} x_i^t = \bar{x}_i$ exponentially. Since $u_i$ is differentiable, it follows that there is $c > 0$ such that $\sum_{t=0}^{\infty} (u_i(x_i^t) - u_i(\bar{x}_i)) \geqq -c$, for all $i$. Let $\lambda$ be as in lemma 13.2. Then, $V_\delta(\hat{K}^0) = \sum_{t=0}^{\infty} \delta^t \sum_i \Lambda^{-1}$ $(u_i(x_i^t) - u_i(\bar{x}_i)) \geqq - I\lambda^{-1}c$. Hence, $F_\delta(\hat{K}^0) = \bar{p} \cdot (\hat{K}^0 - \bar{K}) - V_\delta(\hat{K}^0) \leqq$ $\bar{p} \cdot (\hat{K}^0 - \bar{K}) + I\lambda^{-1}c \leqq \bar{q} \cdot (\hat{K}^0 - \bar{K}) + I\lambda^{-1}c$. The last inequality follows from (13.4).

Q.E.D.

13.23) Lemma There exist $\varepsilon > 0$ and $A > 0$ such that if $|K - \bar{K}| < \varepsilon$ and $\delta \geqq \underline{\delta}$, then $F_\delta(K)$ is well-defined and $F_\delta(K) \leqq A|K - \bar{K}|^2$.

This lemma implies, of course, that the value function, $V_\delta$, is differentiable at $\bar{K}$ and that $\bar{p} = DV_\delta(\bar{K})$. $V_\delta$ is probably differentiable everywhere. See Benveniste and Scheinkman [1979] and Araujo and Scheinkman [1977].

Proof It is sufficient to prove the following.

13.24) There exist $\varepsilon > 0$ and $A > 0$ such that if $|K-\bar{K}| < \varepsilon$, then there exists $((\underset{\sim}{x}_i), (\underset{\sim}{y}_j)) \in \mathcal{J}(K)$ such that $\bar{p} \cdot (K-\bar{K}) - \sum_{t=0}^{\infty} \delta^t \sum_i \Lambda_i^{-1} (u_i(x_i^t) - u_i(\bar{x}_i)) \leqq A\,|K-\bar{K}|^2$ .

If (13.24) is true, then a Cantor diagonalization argument proves that $F_\delta(K)$ exists. (Such an argument is given in Brock [1970], proof of lemma 5, pp.277-8.)

I start by defining a feasible allocation $(({}^0x_i), ({}^0y_j))$ which converges to $((\bar{x}_i), (\bar{y}_j))$ exponentially and from below. I do so by using the construction of Gale, which I have already used in proving lemma 13.20. Clearly,

$\frac{1}{2} \sum_i \bar{x}_i \leqq \sum_i \omega_i + \frac{1}{2} \sum_j \bar{y}_{j0} + \frac{1}{4} \sum_j \bar{y}_{j0} + \sum_j \bar{y}_{j1}$ . This is simply (13.21), with $\alpha = \frac{1}{2}$ , with $y_{j0} = \frac{1}{2} \bar{y}_{j0}$ and with $y_{j1}^{-1} = \bar{y}_{j1}$ . Let ${}^0x_i^t = (1 - (\frac{1}{2})^{t+1})\bar{x}_i$ and ${}^0y_j^t = (1 - (\frac{1}{2})^{t+2})\, \bar{y}_j$ . Then by argument following (13.21), $(({}^0\underset{\sim}{x}_i), ({}^0\underset{\sim}{y}_j))$ is a feasible allocation.

I now modify $(({}^0\underset{\sim}{x}_i), ({}^0\underset{\sim}{y}_j))$ so as to obtain an allocation $(({}^1\underset{\sim}{x}_i), ({}^1\underset{\sim}{y}_j))$ such that $g_j({}^1y_j^t) = 0$ , for all $j$ and $t$ . For all $j$ and $t$ , let $({}^1y_{j0}^t, {}^1y_{j1}^t) = ({}^0y_{j0}^t, a_j^t\, {}^0y_{j1}^t)$, where $a_j^t = \max\{a \geqq 1 \mid ({}^0y_{j0}^t, a\, {}^0y_{j1}^t) \in Y_j\}$. This defines $(({}^1\underset{\sim}{x}_i),({}^1\underset{\sim}{y}_j))$. Since ${}^1y_{j1}^t \geqq {}^0y_{j1}^t$, for all $j$ and $t$, $(({}^1\underset{\sim}{x}_i),({}^1\underset{\sim}{y}_j))$ is feasible.

Let $\varepsilon'$ be as small as the $\varepsilon$ in (13.6) and (13.7). Let $\varepsilon > 0$ be so small that

$$(13.25) \quad \varepsilon < \min(\zeta/4, (10\, L 3^L \underline{q}^{-1} \overline{q})^{-1} \varepsilon'),$$

where $\zeta$ is as in the interiority assumption. (I also assume that $\varepsilon \leq 1/5$.) I will now prove that (13.24) is true for a suitable choice of A and for the $\varepsilon$ just defined.

Suppose that $0 < |K-\overline{K}| < \varepsilon$ and let $T$ be the largest non-negative integer such that $(\frac{1}{2})^{T+2}\, \overline{K}_k \geqq |K-\overline{K}|$, for all k. Such a T exists since $\varepsilon > \zeta/4$ and $\overline{K}_k \geqq \zeta$, for all k.

I claim that the part of $(({}^1\underset{\sim}{x}_i), ({}^1\underset{\sim}{y}_j))$ from T on belongs to $\mathcal{T}(K)$. For all $k \in L_p$, $K_k \geqq \overline{K}_k - |K-\overline{K}| \geqq (1-(\frac{1}{2})^{T+2})\, \overline{K}_k$. Therefore,

$$\sum_i {}^1x_{ik}^T - \sum_j {}^1y_{j0k}^T = (1 - (\tfrac{1}{2})^{T+1}) \sum_i \overline{x}_{ik} - (1 - (\tfrac{1}{2})^{T+2}) \sum_j \overline{y}_{j0k} \leqq (1 - (\tfrac{1}{2})^{T+2})(\sum_i \overline{x}_{ik} - \sum_j \overline{y}_{j0k}) = (1 - (\tfrac{1}{2})^{T+2})\, \overline{K}_k \leqq K_k,$$

for all $k \in L_p$. This proves the claim.

I now define surplus vectors $z^t \in R_+^L$, for $t \geqq T$, as follows. Let $z^T = K + \sum_i \omega_i - (\sum_i {}^1x_i^T - \sum_j {}^1y_{j0}^T)$. If $t > T$, let $z^t = \sum_j {}^1y_{j1}^{t-1} + \sum_i \omega_i - (\sum_i {}^1x_i^t - \sum_j {}^1y_{j0}^t)$. I distribute these surpluses in the manner described near the beginning of this section. This distribution changes the allocation $(({}^1\underset{\sim}{x}_i),({}^1\underset{\sim}{y}_j))$ into $(({}^2\underset{\sim}{x}_i),({}^2\underset{\sim}{y}_j))$.

I next show that

(13.26) $|z^t| \leq (5\zeta^{-1}B)\ (\frac{1}{2})^{t-T}\ |K-\overline{K}|$, for all $t \geqq T$,

where $\zeta$ is as in the interiority assumption and $B$ is as in lemma 7.2.

It should be clear that ${}^1y^t_{j1} \leq \overline{y}_{j1}$, for all $t$ and $j$. Therefore, for all $k \in L_p$, $(1-(\frac{1}{2})^{t+1})\overline{K}_k = (1-(\frac{1}{2})^{t+1})[\sum_i \overline{x}_{ik} - \sum_j \overline{y}_{j0k}] \leq (1-(\frac{1}{2})^{t+1}) \sum_i \overline{x}_{ik} - (1-(\frac{1}{2})^{t+2}) \sum_j \overline{y}_{j0k} = \sum_i {}^1x^t_{ik} - \sum_j {}^1y^t_{j0k} \leq \sum_j {}^1y^{t-1}_{j1k} \leq \overline{K}_k$. Hence, $0 \leq z^t_k \leq (\frac{1}{2})^{t+1}\ |\ \overline{K}\ |\ \leq (\frac{1}{2})^{t+1}B$, for all $k \in L_p$.

If $k \in L_o$, then $z^t_k = \sum_i \omega_{ik} - (\sum_i {}^1x^t_{ik} - \sum_j {}^1y^t_{j0k}) = \sum_i \overline{x}_{ik} - \sum_j \overline{y}_{j0k} - [(1-(\frac{1}{2})^{t+1}) \sum_i \overline{x}_{ik} - (1-(\frac{1}{2})^{t+2}) \sum_j \overline{y}_{j0k}] = (\frac{1}{2})^{t+1} \sum_i \overline{x}_{ik} - (\frac{1}{2})^{t+2} \sum_j \overline{y}_{j0k} \leq (\frac{1}{2})^{t+1}(\sum_i \overline{x}_{ik} - \sum_j \overline{y}_{j0k}) = (\frac{1}{2})^{t+1} \sum_i \omega_{ik} = (\frac{1}{2})^{t+1}B$.

Recall that $T$ is the largest non-negative integer such that $|K-\overline{K}| \leq (\frac{1}{2})^{T+2}\ \overline{K}_k$. Hence, $|K-\overline{K}| \geq (\frac{1}{2})^{T+3}\ \overline{K}_k \geq (\frac{1}{2})^{T+3}\ \zeta$, for some $k$. It follows that

(13.27) $(\frac{1}{2})^{t+1}B \leq 4\zeta^{-1}\ B\ (\frac{1}{2})^{t-T}\ |K-\overline{K}|$.

This inequality together with the inequalities of the previous two paragraphs imply (13.26) for $t > T$.

By what has been proved, $|z^T| \leq |K-\overline{K}| + |\ \overline{K} + \sum_i \omega_i - (\sum_i {}^1x^T_i - \sum_j {}^1y^T_j)\ | \leq |K-\overline{K}| + 4\zeta^{-1}B\ |K-\overline{K}|$. This completes the proof of (13.26).

I may assume that $\underline{\delta} \geqq \frac{1}{2}$, where $\underline{\delta}$ is as in the interiority assumption. This means that I may use inequality (13.8), which requires that $\delta \geqq \frac{1}{2}$. (13.8) and (13.26) imply that $|((^2x_i^t),(^2y_j^t)) - ((^1x_i^t),(^1y_j^t))|$ $\leqq L\, 3^L \underline{q}^{-1} \bar{q} \sum_{n=0}^{L} |z^{t-n}| \leqq (5\zeta^{-1}B)\ (6^L)(L^2 + L)\underline{q}^{-1}\bar{q}\ |K-\bar{K}|(\frac{1}{2})^{t-T}$. In applying (13.8), I have used restriction (13.25) on $\varepsilon$.

It is easy to see that

$|((^1x_i^t),\ (^1y_j^t)) - ((\bar{x}_i),(\bar{y}_j))| \leqq (\frac{1}{2})^{t+1}B \leqq 4\zeta^{-1}B\ (\frac{1}{2})^{t-T}|K-\bar{K}|$. The second inequality is (13.27). By the triangle inequality,

$$(13.28)\qquad |(^2x_i^t),(^2y_j^t)) - ((\bar{x}_i),(\bar{y}_j))\ | \leqq (\zeta^{-1}B)\ 6^{L+1}(L^2+L)\ \underline{q}^{-1}\bar{q}\ |K-\bar{K}|(\frac{1}{2})^{t-T}.$$

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I now let $((x_i^t),(y_j^t)) = ((^2x_i^{T+t}),(^2y_j^{T+t}))$. I will show that $((\underset{\sim}{x}_i),(\underset{\sim}{y}_j))$ satisfies (13.24), for a suitable choice of A.

I now define A. Let B be as in lemma 7.2. By assumptions 3.3 and 3.5, there exists $b_c > 0$ such that for all i,

$u_i(x) + D\,u_i\,(x)\cdot(x' - x) - u_i(x') \leqq b_c\ |\ x - x'\ |^2$, provided

that $|\ x\ | \leqq B$ and $|\ x - x'\ | \leqq 1$. Let $\lambda$ be as in lemma 13.2. Then, for any i, $\Lambda_i^{-1}\ u_i(\bar{x}_i) - \bar{p}\cdot\bar{x}_i$

$- (\Lambda_i^{-1}u_i(x) - \bar{p}\cdot x) \leqq \lambda^{-1}\ b_c\ |\ x - \bar{x}_i|^2$, provided that $x_k = \bar{x}_{ik} = 0$

whenever k is such that $\bar{p}_k \neq \Lambda_i^{-1}\dfrac{\partial u_i(\bar{x}_i)}{\partial x_k}$ .

By assumptions 3.9 and 3.11, there exists $b_p > 0$ such that for all $j$, $g_j(y') + Dg_j(y') \cdot (y_j' - y) - g_j(y) \leqq b_p |y' - y|^2$, provided that $|y'| \leqq B$ and $|y - y'| \leqq 1$ and $g_j(y) = g_j(y') = 0$. Let $\rho_j$ and $\bar{\rho}$ be as (13.5). Then, $(\bar{p} \cdot \bar{y}_{j0} + \delta\, \bar{p} \cdot \bar{y}_{j1}) - (\bar{p} \cdot y_0 + \delta\, \bar{p} \cdot y_1)$ $= \rho_j Dg_j(\bar{y}_j) \cdot (\bar{y}_j - y) \leqq \rho_j b_p\, |y - \bar{y}_j|^2 \leqq \bar{\rho}\, b_p\, |y - \bar{y}_j|^2$, provided that the following conditions hold: 1) $y_{0k} = \bar{y}_{j0k} = 0$ whenever $\bar{p}_k \neq \rho_j \dfrac{\partial g_j(\bar{y}_j)}{\partial y_{0k}}$ ; 2) $y_{1k} = \bar{y}_{j1k} = 0$, whenever $\delta \bar{p}_k \neq \rho_j \dfrac{\partial g_j(\bar{y}_j)}{\partial y_{1k}}$ ; and 3) $g_j(y) = 0$.

I have been careful to choose $((\underset{\sim}{x}_i), (\underset{\sim}{y}_j))$ so that the following are true: 1) $x_{ik}^t = \bar{x}_k = 0$, whenever $\bar{p}_k \neq \Lambda_i^{-1} \dfrac{\partial u_i(\bar{x}_i)}{\partial x_k}$, for all $i$ and $t$; 2) $y_{j0k}^t = y_{j0k} = 0$, whenever $\bar{p}_k \neq \rho_j \dfrac{\partial q_j(\bar{y}_j)}{\partial y_{ok}}$ ; 3) $y_{j1k}^t = \bar{y}_{j1k} = 0$, whenever $\delta \bar{p}_k \neq \rho_j \dfrac{\partial g_j(\bar{y}_j)}{\partial y_{1k}}$ ; and 4) $g_j(y_j^t) = 0$, for all $j$ and $t$.

It follows that

$$(13.29) \quad \sum_i [(\Lambda_i^{-1} u_i(\bar{x}_i) - \bar{p} \cdot \bar{x}_i) - (\Lambda_i^{-1} u_i(x_i^t) - p \cdot x_i^t)]$$

$$+ \sum_j [\bar{p} \cdot \bar{y}_{j0} + \delta\, \bar{p} \cdot \bar{y}_{j1}) - (\bar{p} \cdot y_{j0}^t + \delta\, \bar{p} \cdot y_{j1}^t)]$$

$$\leqq b(I + J)\, |((x_i^t), (y_j^t)) - ((\bar{x}_i), (\bar{y}_j))|^2, \text{ for all } t,$$

where $b = \max (b_c, b_p)$.

Let $A = \frac{4}{3} b (I+J)(\zeta^{-1}B)^2 6^{2L+2} (L^2+L)^2 (\underline{q}^{-1}\bar{q})^2$.

I now show that (13.24) is true for $\varepsilon$, $A$ and $((\underset{\sim}{x}_i),(\underset{\sim}{y}_j))$ as defined above. By (13.11), $\bar{p} \cdot (K - \bar{K}) - \sum_{t=0}^{\infty} \delta^t \sum_i \Lambda_i^{-1}(u_i(x_i^t) - u_i(\bar{x}_i))$ $\leq S_1 + S_2 + S_3$, where $S_1$, $S_2$ and $S_3$ are defined by (13.12), (13.13) and (13.14), respectively. By (13.29) and (13.28),

$$S_1 + S_3 \leq b(I+J) \sum_{t=0}^{\infty} \delta^t \left|((x_i^t),(y_j^t)) - ((\bar{x}_i),(\bar{y}_j))\right|^2 \leq$$

$$\leq b(I+J)(\zeta^{-1}B)^2 6^{2L+2}(L^2+L)^2(\underline{q}^{-1}\bar{q})^2 \left(\sum_{t=0}^{\infty} (\tfrac{1}{2})^{2t}\right) |K-\bar{K}|^2 \leq A\,|K-\bar{K}|^2.$$

Since $\sum_j (y_{j0}^t + y_{j1}^{t-1}) + \sum_i (\omega_i - x_i^t) = 0$, for all $t$, it follows that $S_2 = 0$.

This proves (13.24)

Q.E.D.

I now may prove that $\lim_{t\to\infty} F_\delta(\hat{K}^t) = 0$ exponentially. By (13.18), this proof will complete the proof of the theorem.

Choose a small positive number no larger than the $\varepsilon$ of lemma 13.19 and the $\varepsilon$ of lemma 13.23. Call this number $\varepsilon$ again. Let $\alpha$ be as in lemma 13.19 and let $A$ be as in lemma 13.23. Clearly, I may assume that $\alpha < A$. Finally, let $C$ be as in lemma 13.20. Then, I know that $F_\delta(\hat{K}^0) \leq C$.

If $1 > \delta \geq \underline{\delta}$ and $|\hat{K}^t - \bar{K}| \leq \varepsilon$, then $F_\delta(\hat{K}^t) \leq A\,|\hat{K}^t - \bar{K}|^2$.

Also, $F_\delta(\hat{K}^{t+1}) - \delta^{-1}F_\delta(\hat{K}^t) \leq -2\alpha \min(\varepsilon^2, |\hat{K}^t - \bar{K}|^2)$, for all $t$.

Let $\bar{\delta} = \max\left(\underline{\delta}, \frac{A}{A+\alpha}, \frac{C}{C+\alpha\epsilon^2}\right)$ . From now on, I assume that $\delta \geqq \bar{\delta}$ .

I claim that

13.30) $\quad F_\delta(\hat{K}^{t+1}) \leqq F_\delta(\hat{K}^t) - \alpha\min\left(\epsilon^2, |\hat{K}^t - \bar{K}|^2\right)$ .

The argument involves induction on $t$ . Recall that $F_\delta(\hat{K}^0) \leqq C$ . Assume by induction that $F_\delta(\hat{K}^t) \leqq C$ . If $|\hat{K}^t - \bar{K}| \leqq \epsilon$ , then $F_\delta(\hat{K}^{t+1}) \leqq F_\delta(\hat{K}^t) + (\delta^{-1} - 1) F_\delta(\hat{K}^t) - 2\alpha |\hat{K}^t - \bar{K}|^2 \leqq F_\delta(\hat{K}^t) + [(\delta^{-1} - 1)A - 2\alpha] |\hat{K}^t - \bar{K}|^2 \leqq F_\delta(\hat{K}^t) - \alpha |\hat{K}^t - \bar{K}|^2$ . The last inequality follows from the choice of $\bar{\delta}$ . If $|\hat{K}^t - \bar{K}| \geqq \epsilon$ , then $F_\delta(\hat{K}^{t+1}) \leqq F_\delta(\hat{K}^t) + (\delta^{-1} - 1) F_\delta(\hat{K}^t) - 2\alpha\epsilon^2 \leqq F_\delta(\hat{K}^t) + (\delta^{-1} - 1)C - 2\alpha\epsilon^2 \leqq F_\delta(\hat{K}^t) - \alpha\epsilon^2$ . The last inequality again follows from the choice of $\bar{\delta}$ . It now follows that $F_\delta(\hat{K}^{t+1}) \leqq C$ . Hence, I may continue the above argument inductively. This proves (13.30) .

I now prove that

13.3 ) $\quad F_\delta(\hat{K}^{t+1}) \leqq \max\left[(1 - \alpha A^{-1}) F_\delta(\hat{K}^t), F_\delta(\hat{K}^t) - \alpha\epsilon^2\right]$

By (13.30), $F_\delta(\hat{K}^{t+1}) \leqq F_\delta(\hat{K}^t) - \alpha\epsilon^2$ , if $|\hat{K}^t - \bar{K}| \geqq \epsilon$ . If $|\hat{K}^t - \bar{K}| \leqq \epsilon$ , then $F_\delta(\hat{K}^{t+1}) \leqq F_\delta(\hat{K}^t) - \alpha|\hat{K}^t - \bar{K}|^2 \leqq F_\delta(\hat{K}^t) - \alpha A^{-1} F_\delta(\hat{K}^t)$ . This proves (13.3 ).

I complete the proof by proving the following.

13.32) There exists a positive integer T such that $F_\delta(\hat{K}^t) \leqq C - t\alpha\epsilon^2$, if $t \leqq T$, and $F_\delta(\hat{K}^t) \leqq (1 - \alpha A^{-1})^{t-T} A\epsilon^2$, if $t \geqq T$.

If $F_\delta(\hat{K}^t) \geqq A\epsilon^2$, then $\alpha A^{-1} F_\delta(\hat{K}^t) \geqq \alpha\epsilon^2$, so that by (13.31), $F_\delta(\hat{K}^{t+1}) \leqq F_\delta(\hat{K}^t) - \alpha\epsilon^2$. Similarly, if $F_\delta(\hat{K}^t) \leqq A\epsilon^2$, then $\alpha A^{-1} F_\delta(\hat{K}^t) \leqq \alpha\epsilon^2$, so that $F_\delta(\hat{K}^{t+1}) \leqq (1 - \alpha A^{-1}) F_\delta(\hat{K}^t)$. Let T be the smallest positive integer such that $A\epsilon^2 \geqq C - \alpha\epsilon^2 T$. Assertion (13.32) follows.

Q.E.D.

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REFERENCES

Araujo, Aloisio and José Scheinkman, "Smoothness, Comparative Dynamics and the Turnpike Property," Econometrica, 45 (1977), 601-620.

Arrow, Kenneth, "Le rôle des valeurs boursières pour la répartition la meilleure des risques," Econométrie, (Centre Nationale de la Recherche Scientifique, Paris : 1953), 41-48.

Arrow, Kenneth and Gerard Debreu, "Existence of Equilibrium in a Competitive Economy," Econometrica, 27 (1954), 265-290.

Atsumi, Hiroshi, "Neoclassical Growth and the Efficient Program of Capital Accumulation," Review of Economic Studies, 32 (1965), 127-136.

Becker, Robert,"The Existence of a Perfect Foresight Competitive Equilibrium in a Simply Dynamic Model of Capital Accumulation with Heterogeneous Households," Ch. IV., Ph.D. dissertation, University of Rochester (1978).

______________, "On the Long-Run Steady State in a Simple Dynamic Model of Equilibrium with Heterogeneous Households," forthcoming in Quarterly Journal of Economics.

Benhabib, Jess and Kazuo Nishimura, "Notes on the Saddle Point Stability and Regularity of a Steady State in a Multi-Sector Optimal Growth Model," Working Paper, (revised June 1979), University of Southern California.

Benveniste, Lawrence and José Scheinkman, "Duality Theory for Dynamic Optimization Models: the Continuous Time Case," forthcoming in Journal of Economic Theory.

Bewley, Truman, "Existence of Equilibria in Economies with Infinitely Many Commodities," Journal of Economic Theory, 4 (1972), 514-540.

______________, "Theoretical Implications of the Permanent Income Hypothesis," Center for Mathematical Studies in Economics and Management Science, Discussion Paper No. 398, Northwestern University, Sept. 1979.

Brock, William, "On Existence of Weakly Maximal Programmes in a Multi-Sector Economy," Review of Economic Studies, 37 (April 1970), 275-280.

Brock, William and Jose Scheinkman, Global Asymptotic Stability of Optimal Control Systems with Applications to the Theory of Economic Growth," Journal of Economic Theory, 12 (1976), 164-190.

Cass, David and Karl Shell, "The Structure and Stability of Competitive Dynamical Systems," Journal of Economic Theory, 12 (1976), 31-70.

Debreu, Gerard, "Valuation Equilibrium and Pareto Optimum," Proceedings of the National Academy of Science, 40 (1954), 588-592.

______________, Theory of Value (Wiley: New York: 1959).

Dunford, Nelson and Jacob Schwartz, Linear Operators. Part I: General Theory (Wiley, New York: 1957).

Gale, David, "On Optimal Development in a Multi-Sector Economy," Review of Economic Studies, 34 (1967), 1-18.

Kelley, John, Isaac Namioka, et al., Linear Topological Spaces (van Nostrand, Princeton, NJ : 1963).

Magill, Michael, "Some New Results on the Local Stability of the Process of Capital Accumulation," Journal of Economic Theory, 15 (1977), 174-210.

McKenzie, Lionel, "On the Existence of General Equilibrium for a Competitive Market," Econometrica, 27 (1959), 54-71.

______________, "Turnpike Theorems for a Generalized Leontief Model," Econometrica, 31 (1963), 165-180.

______________, "Accumulation Programs of Maximum Utility and the von Neumann Facet," in Value Capital and Growth, ed. by J.N. Wolfe (Edinburgh University Press: 1968).

______________, "Capital Accumulation Optimal in the Final State," in Contributions to the von Neumann Growth Model, ed. by G. Bruckmann and W. Weber, (Springer-Verlag, Vienna: 1974).

______________, "Turnpike Theory," Econometrica, 44 (1976), 841-865.

______________, "Optimal Growth and Turnpike Theorems," (1979), to appear in The Handbook of Mathematical Economics, edited by Arrow and Intriligator.

Peleg, Bezalel and Harl Ryder, Jr., "The Modified Golden Rule of a Multi-Sector Economy," Journal of Mathematical Economics, 1 (1974), 193-198.

Peleg, Bezalel and Menahem Yaari, "Markets with Countably Many Commodities," International Economic Review, 11 (1970), 369-377.

Rader, Trout, The Economics of Feudalism (Gordon and Breach, New York: 1971).

_____________, "Utility over Time: The Homothetic Case," mimeographed, Department of Economics, Washington University, St. Louis, Missouri (1979).

Radner, Roy, "Paths of Economic Growth that are Optimal with Regard only to Final States," Review of Economic Studies, 28 (1961), 98-104.

Ramsey, Frank, "A Mathematical Theory of Saving," Economic Journal, 38 (1928, 543-59.

Rockafellar, R. Tyrell, "Saddle Points of Hamilitonian Systems in Convex Lagrange Problems Having a Nonzero Discount Rate," Journal of Economic Theory, 12 (1976), 71-113.

Schaefer, Helmut, Topolgical Vector Spaces, (Springer, New York· 1971).

Scheinkman, José, "On Optimal Steady States of n-Sector Growth Models when Utility is Discounted," Journal of Economic Theory, 12 (1976), 11-20.

Stigum, Bernt, "Competitive Equilibria with Infinitely Many Commodiites," Metroeconomica 24 (1972), 221-244.

_____________, "Competitive Equilibria with Infinitely Many Commodities(II)," Journal of Economic Theory, 6 (1973), 415-445.

Sutherland, W., "On Optimal Development in a Multi-Sectoral Economy: the Discounted Case," Review of Economic Studies. 37 (1970), 585-589.

Yano, Makoto, "Competitive Equilibria on Turnpikes in a McKenzie Economy," (July 1980), mimeographed, Dept. of Economics, University of Rochester, Rochester, New York., 14627