Discussion Paper 402

BAYESIAN ANALYSIS OF COMPETITIVE DECISION-MAKING SITUATIONS

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1. Introduction

In analyzing competitive decision-making situations, two approaches may be taken: game-theoretic and a Bayesian decision-theoretic. The game-theoretic approach is static in its nature, assumes that the competitors do not assign probabilities to each other's choice of an action, allows for mixed strategies as an optimal solution, and emphasizes the existence and stability of competitive equilibrium. The Bayesian decision-theoretic approach is dynamic in its nature by allowing the decision maker to assign probabilities to the opponent's choices of actions and to revise them in light of new information. It prescribes the selection of pure strategies as an optimal behavior, and considers the optimality of the behavior of one competitor.

In this paper we study the dynamic behavior of the competitors (players), generated from Bayesian decision-theoretic considerations. In particular, we are interested here in the following questions:

(a) can any special patterns of behavior be observed across competitive situations? (b) over time? (c) do the behaviors under these schemes converge in the long-run to the equilibrium strategies prescribed by game-theory? (d) what is the direction and the rate of the convergence? and (e) what is the effect of the players' attitudes toward risk upon their dynamic behaviors?

Our aim is to provide answers to all of these questions, for two-by-two, two-person, non-zero-sum, repeated noncooperative games with incomplete information. Games with incomplete information are games in which the players do not know the complete description of the game; for instance, they do not know the other

2. The Competitive Situation and the Bayesian Model

The competitive situation studied here is the case in which there are two competitors (players) I and II, with two actions available to each competitor, and where the decisions are made simultaneously by the two players. Both players know this, and in addition, each player knows only his own possible payoffs. In the terminology of game theory, we are dealing with a two-person, non-zero-sum and non-cooperative game with incomplete information, represented in normal form by a 2 X 2 matrix. This same competitive situation is repeated many times and allows the competitors to learn about each other's past decisions which are observable. Of course, future decisions of the opponent are not known to the player and can be just inferred from his past behavior.

Although competitive situations depicted as 2 X 2 games are the simplest two-person games, they have attracted attention of researchers from many disciplines. Rapoport, Guyer and Gordon [1976] summarize and interpret what has been learned in the last fifteen years, through experimentation, about social interaction and behavior using this paradigm. Classifications for all 2 X 2 games have been suggested by Rapoport and Guyer [1966] and Harris [1969, 1972] to aid in combining together games with similar game-theoretic and behavioral aspects. Iterated Prisoners' Dilemma games have been also studied extensively [Grofman and Pool, 1975; Rapoport and Chammah, 1965]. Sequential games arise also in contexts such as economics [Shubik, 1960], gaming [Shubik, 1975], and stochastic processes [Sanghvi, 1978; Sanghvi and Sobel, 1977].

The Bayesian decision-theoretic model developed here assumes that the competitors regard each other's behavior as a stochastic decision process. This assumption is implicit in the "fictitious play" literature [Brown, 1951;

$$E(p|r,n) = \frac{r}{n} \tag{2.1}$$

and

$$V(p|r,n) = \frac{r(n-r)}{n^2(n+1)}. \tag{2.2}$$

The shape of $f_\beta(p|r,n)$ depends on r and n, and can accommodate large number of probability assessors. If the prior parameters at time t are r_t and n_t, and the sample results are r "successes" in n trials, the posterior parameters at time t+1, r_{t+1} and n_{t+1}, can be easily computed from:

$$n_{t+1} = n_t + n \tag{2.3}$$

and

$$r_{t+1} = r_t + r. \tag{2.4}$$

In our context, of course, n=1 whereas r=1 or 0, depending on whether or not the opponent selected his first action. We note therefore that n_t and r_t can be viewed as counters such that $n_t - n_0$ counts the number of simultaneous decisions that have been made and $r_t - r_0$ counts the number of times the opponent has used his first action. We also note that within the Bayesian decision-theoretic framework, the simultaneous decisions amounts to the selection of an action which does not influence the subjective probability of the random events (states) associated with this action. This case which is assumed throughout this paper is called the act - unconditional states case. An alternative Bayesian decision-making model, act - conditional states [Schoner and Mann, 1978] allows for the possibility that the selection of an alternative may influence the subjective probability. More formally, if $f^I(q|a_1)$ and $f^I(q|a_2)$ denote player I's subjective p·d·f over the event that player II is choosing his first action with probability q, given that he

3. Analysis of Competitive Situations

Let Matrix (i) represent a 2 X 2 game with the following payoffs to the two players:

		II	
		b_1	b_2
I	a_1	$R_I(\sigma_1)$, $R_{II}(\sigma_1)$	$R_I(\sigma_2)$, $R_{II}(\sigma_2)$
	a_2	$R_I(\sigma_3)$, $R_{II}(\sigma_3)$	$R_I(\sigma_4)$, $R_{II}(\sigma_4)$

Matrix (i)

where $\sigma_1 = (a_1b_1)$, $\sigma_2 = (a_1b_2)$, $\sigma_3 = (a_2b_1)$, and $\sigma_4 = (a_2b_2)$ are the four possible outcomes (states) of the game and R_I $(\cdot)$, R_{II} $(\cdot)$ are the payoffs to player I and II, respectively. Assuming that the payoffs are defined on an ordinal scale, let:

$$S = R_I(\sigma_4) - R_I(\sigma_2)$$

$$T = R_I(\sigma_3) - R_I(\sigma_1)$$

$$U = R_{II}(\sigma_4) - R_{II}(\sigma_3)$$

$$V = R_{II}(\sigma_2) - R_{II}(\sigma_1)$$

denote the differences in the payoffs. As noted before, in this section we assume that the players' utility functions are linear and we shall also assume strict preference ordering of the outcomes. It has been noted by Rapoport and Guyer [1966] that cases of indifference between two outcomes can be considered as limiting cases of strict preference.

We classify all 2 X 2 games according to the following relationships:

sequential game as a discrete time Semi-Markov process with a discrete state space and which possesses transition probabilities equal to zeros and ones. The state of the process is determined by the play's outcome $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ and we shall characterize the dynamic behavior of the process (game) using terminology from the theory of stochastic processes.

Proposition 1

Given the state of the process at time t and the ordinal property of the payoffs of the game, the next different state visited by the process is given by the entries of Table 1. More formally, if $P^s_{\sigma_i,\sigma_j}$ denotes the probability that given that the state of the process at time t is σ_i (i=1,2,3,4), the first transition into σ_j $(j\neq 1)$ will occur at time t + s, then for some games, there exist integers s and w such that $P^s_{\sigma_i,\sigma_j}=1$ and $P^w_{\sigma_i,\sigma_i}=1$ for any $1 \leq w < s$, for other games there does not exist such an s, and for additional classes of games, this information is insufficient to determine the existence of s.

Table 1

Current and Next-Visited State for each Class of Games

		Class of Games				
		(i)	(ii)	(iii)	(iv)	(v)
Current State	σ_1	n/a*	none **	not unique	σ_2	σ_3
	σ_2	n/a*	not unique	none**	σ_4	σ_1
	σ_3	n/a* or σ_4	not unique	none**	σ_1	σ_4
	σ_4	none**	none **	not unique	σ_3	σ_2

*not applicable

**whenever the game is in this state, it remains there on all future plays.

State σ_4 and Class (iii)

Consider now the following game:

	b_1	b_2
a_1	a,e	c,h
a_2	d,g	b,f

Again, given state σ_4 implies: $EV_t^I(a_1) \leq EV_t^I(a_2)$ and $EV_t^{II}(b_1) \leq EV_t^{II}(b_2)$. After one revision of the prior beta distributions we obtain:

$$EV_{t+1}^I(a_1) = \frac{n_t^I EV_t^I(a_1)+c}{n_t^I+1} \quad \text{and} \quad EV_{t+1}^I(a_2) = \frac{n_t^I EV_t^I(a_2)+b}{n_t^I+1}.$$

$$EV_{t+1}^{II}(b_1) = \frac{n_t^{II} EV_t^{II}(b_1)+g}{n_t^{II}+1} \quad \text{and} \quad EV_{t+1}^{II}(b_2) = \frac{n_t^{II} EV_t^{II}(b_2)+f}{n_t^{II}+1}.$$

It can be readily seen that the ordinal property of the payoffs and the current state of the game are not sufficient to determine uniquely the state of the process even at t+1, since we need also to know the difference in the expected payoffs relatively to the difference in the respective payoffs. Q.E.D.

State σ_4 and Class (iv)

Consider now the following game:

	b_1	b_2
a_1	a,g	b,h
a_2	c,e	d,f

where $EV_t^I(a_1) \leq EV_t^I(a_2)$ and $EV_t^{II}(b_1) \leq EV_t^{II}(b_2)$. It can be readily shown that after one revision the expected payoffs are:

Proof:

Straightforward from Table 1.

Corollary 1.1 helps us focus on the types of games to be studied further. Attention will be restricted to games in Class (iv) (similar results can be shown for Class (v)). Furthermore, we shall specifically examine Game 75 in the taxonomy proposed by Rapoport and Guyer which is a pure conflict game and can be represented by Matrix (ii):

	b_1	b_2
a_1	a,g	b,h
a_2	c,e	d,f

Matrix (ii)

where d>a>c>b and h>e>g>f. This game captures both non-zero-sum and zero-sum conditions.

For non-zero-sum games, the prescription provided by game-theory is given in terms of the following equilibrium mixed strategies:

$$p^* = P_r\{\text{I plays } a_1\} = (e-f)/(h+e-g-f) \quad (3.1)$$

and $$q^* = P_r\{\text{II plays } b_1\} = (d-b)/(d+a-c-b). \quad (3.2)$$

Zero-sum games can be obtained from Matrix (ii) when h=-b, e=-c, g=-a, f=-d, and then (3.1) and (3.2) become the minimax strategies for the two players.

Since the game proceeds by cycles (Corollary 1.1), it makes no difference when we start observing it when studying its long-run behavior. We shall therefore assume that at t=0 the process is in state σ_4 (the next states will be $\sigma_3 \to \sigma_1 \to \sigma_2$). We denote by $E_0^{I}(q)$ and $E_0^{II}(p)$ the expected values, at t=0, of the prior beta distributions of player I and II, respectively, over the probability that the opponent will

$$EV_0^{II}(b_1) = \frac{gr_0^{II} + e(n_0^{II} - r_0^{II})}{n_0^{II}} \quad \text{and}$$

$$EV_0^{II}(b_2) = \frac{n_0^{II} EV_0^{II}(b_1) + \delta}{n_0^{II}}$$

For any cycle t=1,2,3,... it is easy to verify (based on the order of the realized states) that:

$$EV^{II}_{i'_t+j'_{t-1}+k'_{t-1}+\ell'_{t-1}}(b_1) = \frac{n_0^{II} EV_0^{II}(b_1) + (i'_t + j'_{t-1})e + (k'_{t-1} + \ell'_{t-1})g}{n_0^{II} + i'_t + j'_{t-1} + k'_{t-1} + \ell'_{t-1}}$$

and $$EV^{II}_{i'_t+j'_{t-1}+k'_{t-1}+\ell'_{t-1}}(b_2) = \frac{n_0^{II} EV_0^{II}(b_1) + (i'_t + j'_{t-1})f + (k'_{t-1} + \ell'_{t-1})h + \delta}{n_0^{II} + i'_t + j'_{t-1} + k'_{t-1} + \ell'_{t-1}}$$

Hence, $EV^{II}_{i'_t+j'_{t-1}+k'_{t-1}+\ell'_{t-1}}(b_1) > EV^{II}_{i'_t+j'_{t-1}+k'_{t-1}+\ell'_{t-1}}(b_2)$ (i.e., the process enters state σ_3) if and only if (3.3) holds. Similar arguments prove (3.4), (3.5) and (3.6). Q.E.D.

Series for i'_t, j'_t, k'_t, and ℓ'_t can now be formed recursively from (3.3) - (3.6) since $\ell'_0 = k'_0 = j'_0 = 0$. In general, for any t: i'_t, j'_t, k'_t, and ℓ'_t will depend on: t, $\frac{d-b}{a-c}$, $\frac{e-f}{h-g}$, $\frac{\delta}{e-f}$, $\frac{\varepsilon}{a-c}$, $\frac{\delta}{h-g}$, and $\frac{\varepsilon}{d-b}$.

Let $\frac{d-b}{a-c} = K$, $\frac{e-f}{h-g} = L$, $\frac{\delta}{e-f} = M$ and $\frac{\varepsilon}{a-c} = N$, and suppose that K,L,M and N are any real numbers, then:

$$i'_t = f_i(t,K,L,M,N) \tag{3.7}$$

$$j'_t = f_j(t,K,L,M,N,) \tag{3.8}$$

$$k'_t = f_k(t,K,L,M,N) \tag{3.9}$$

$$\ell'_t = f_\ell(t,K,L,M,N) \tag{3.10}$$

Proposition 3

For the game represented in Matrix (ii) for $L = 1$; $M,N \geq 0$ and integers; $K \geq N + 3$ and integer, and under the Bayesian model, if at $t=0$ $E_0^{I}(q) \leq q^*$ and $E_0^{II}(p) \geq p^*$, then the following equations hold:

$$i_t' = t^2 + Mt \qquad \text{for } t=1,2,3,\ldots \tag{3.13}$$

$$j_t' = \begin{cases} Kt^2 + \left(MK + \frac{K-1}{2}\right)t - \frac{K-3}{2} + N & \text{for } t=1,3,5,\ldots \\ Kt^2 + \left(MK + \frac{K-1}{2}\right)t & \text{for } t=2,4,6,\ldots \end{cases} \tag{3.14}$$

$$k_t' = \begin{cases} Kt^2 + \left(MK + \frac{K+1}{2}\right)t - \frac{K-3}{2} + N & \text{for } t=1,3,5,\ldots \\ Kt^2 + \left(MK + \frac{K+1}{2}\right)t & \text{for } t=2,4,6,\ldots \end{cases} \tag{3.15}$$

$$\ell_t' = t^2 + (M+1)t \qquad \text{for } t=1,2,3,\ldots \tag{3.16}$$

Remark: Although we require in the proposition that $K \geq N + 3$, proofs for $2 \leq K < N + 3$ are similar but have to be considered separately. They are omitted to conserve space.

Proof:

By mathematical induction. For $t=1$, given Proposition 2 we obtain:

$$i_1' > M \Rightarrow i_1' = M + 1,$$

$$j_1' > (M+1)K + N \Rightarrow j_1' = (M+1)K + 1 + N,$$

$$k_1' \geq (M+1)K + 1 + N + M + 1 - M \Rightarrow k_1' = (M+1)K + 2 + N,$$

and $$\ell_1' \geq \frac{(M+1)K + 2 + N + (M+1)K + 1 + N}{K} - (M+1) - \frac{N}{K},$$

or $$\ell_1' \geq M + 1 + \frac{N+3}{K} \Rightarrow \ell_1' = M + 2 \quad \text{for } K \geq N + 3,$$

which verifies the proposition for $t=1$.

We assume now that for some t even, equations (3.13) - (3.16) hold, and we show that they also hold for $t+1$. That is, given (3.13) - (3.16) and some

Finally, the proof of (3.20) is based on (3.6), (3.17), (3.18) and (3.19):

$$\ell'_{t+1} \geq \left[K(t+1)^2 + \left(MK + \frac{K+1}{2}\right)(t+1) - \left(\frac{K-3}{2}\right) + N + K(t+1)^2 + \left(MK + \frac{K-1}{2}\right)(t+1) - \left(\frac{K-3}{2}\right) + N\right]/K - (t+1)^2 - M(t+1) - \frac{N}{K}$$

or $\quad \ell'_{t+1} \geq (t+1)^2 + M(t+1) + t + 1 - 1 + \frac{N+3}{K}$.

Hence, $\ell'_{t+1} = (t+1)^2 + (M+1)(t+1)$ for $K \geq N + 3$, which establishes (3.20) and completes the proof of Proposition 3. Q.E.D.

From equations (3.13) - (3.16) we can also derived the number of times each state is visited during the $t+1^{\text{th}}$ cycle (i.e., the cycle that begins at time t and ends at time t+1) which we denote by i_t, j_t, k_t, ℓ_t, for states σ_4, σ_3, σ_1, and σ_2, respectively:

$$i_t = i'_{t+1} - i'_t = 2t + M + 1 \qquad \text{for } t=1,2,3,\ldots \tag{3.21}$$

$$j_t = j'_{t+1} - j'_t = \begin{cases} 2Kt + K(M+2) - 2 - N & \text{for } t=1,3,5,\ldots \\ 2Kt + K(M+1) + 1 + N & \text{for } t=2,4,6,\ldots \end{cases} \tag{3.22}$$

$$k_t = k'_{t+1} - k'_t = \begin{cases} 2Kt + K(M+2) - 1 - N & \text{for } t=1,3,5,\ldots \\ 2Kt + K(M+1) + 2 + N & \text{for } t=2,4,6,\ldots \end{cases} \tag{3.23}$$

$$\ell_t = \ell'_{t+1} - \ell'_t = 2t + M + 2 \qquad \text{for } t=1,2,3,\ldots \tag{3.24}$$

The asympototic dynamic behavior of the Bayesian competitive decision-making model can now be compared with the game-theoretic equilibrium strategies ((3.1) and (3.2)) by computing the limit of the empirical relative frequencies. Corollary 3.1 shows that the empirical distributions converge to the game-theoretic equilibrium strategies.

The proportions of time that a_1 and b_1 have been selected during the $t+1^{\underline{th}}$ cycle, for some t odd, are given by:

$$pp_t(a_1) = \frac{k_t + \ell_t}{i_t + j_t + k_t + \ell_t} = \frac{2(K+1)t + M(K+1) + 2K - N + 1}{4(K+1)t + 2M(K+1) + 4K - 2N} \qquad (3.31)$$

and $$pp_t(b_1) = \frac{j_t + k_t}{i_t + j_t + k_t + \ell_t} = \frac{4Kt + 2K(M+2) - 2N - 3}{4(K+1)t + 2M(K+1) + 4K - 2N}, \qquad (3.32)$$

and clearly, $\lim_{t\to\infty} pp_t(a_1) = \frac{1}{2}$ and $\lim_{t\to\infty} pp_t(b_1) = \frac{K}{K+1}$, which completes the proof of the corollary. Q.E.D.

This interesting result, that the empirical cumulative distributions converge to the equilibrium strategies, has been conjectured by Brown [1951] and proved by Robinson [1951] for finite two-person zero-sum games. Corollary 3.1 however, generalizes this result to non-zero-sum games, and shows that the result also holds, in the long-run, during the $t+1^{\underline{th}}$ cycle of visited states, as defined here (a well known result for regenerative stochastic processes).

From (3.27), (3.28), (3.31) and (3.32) we can obtain an insight as to the directions and the rates of the convergence to the equilibrium strategies. Clearly, $pp'_t(a_1)$ and $pp_t(a_1)$ approach their limit monotonically from above. As for $pp'_t(b_1)$, the convergence is from above if $K < 2N + 3$, and from below if $K > 2N + 3$. In the special case where $K = 2N + 3$; $pp'_t(b_1) = \frac{K}{K+1} = q^*$ for any t. It should be noted, however, that when $K \neq 2N + 3$, $pp'_t(b_1) = \frac{K}{K+1}$ for any t <u>even</u>, hence, in this case unlike $pp'_t(a_1)$, the convergence is not monotonic. However, $pp'_t(b_1)$ converges more rapidly than $pp'_t(a_1)$. Finally, the convergence of $pp_t(b_1)$ is oscillating around the equilibrium strategy for successive cycles.

makers (characterized by their attitude toward risk) may start the process in different states, for a <u>given</u> pair of expectations: $E_0^I(q)$ and $E_0^{II}(p)$. This claim is based on the changes of p^* and q^* as θ changes. The investigation of these functions has been reported elsewhere by Eliashberg and Winkler [1978] and just one case will be considered graphically here, for illustrative purposes.

Suppose that player I is a risk-taker and player II is a risk-avoider. and that their monetary payoffs are such that $a+c>b+d$ and $h+f>g+e$. In this case p^* is monotonic increasing in θ whereas q^* decreases initially until it reaches a minimum and then is monotonic increasing in θ and like p^*, approaching 1 as $\theta\to\infty$ (see Eliashberg and Winkler [1978] for the proof of this claim). Figure 1 demonstrates how, for a given pair of expectations, it is possible to start the process in different states.

Insert Figure 1 here

For $0 \le \theta \le \theta_1$, $E_0^I(q) \le q^*$ and $E_0^{II}(p) \ge p^*$, and hence the first visited state is σ_4. However, for $\theta_1 < \theta < \theta_2$, $E_0^I(q) > q^*$ and $E_0^{II}(p) < p^*$, resulting in σ_1 as the initial state. Finally, for $\theta_2 \le \theta$, $E_0^I(q) \le q^*$ and $E_0^{II}(p) < p^*$, implying that the first visited state is σ_3.

To study the effect of risk attitudes upon the length of the cycle and compare it with the results obtained in Section 3 for risk-neutrals, we shall assume now that player II's payoffs are such that: $L(\theta) = \dfrac{U^{II}(e) - U^{II}(f)}{U^{II}(h) - U^{II}(g)} = 1$ for some strictly positive θ, hence, $p^*(\theta) = \frac{1}{2}$ for this θ, and that $E_0^{II}(p) > \frac{1}{2}$. We shall slao assume that $q^*(\theta)$ is monotonic increasing in θ (this is true when player I is a risk-taker and his payoffs are such that $a+c\le b+d$) and that $q^*(\theta) > E_0^I(q)$ for all $0 \le \theta < \infty$. In this case the initial visited state is σ_4 and thus we can use the results from Proposition 3, to study the effect of risk attitude upon the length of the cycle at some given time t. Although the order of the states visited by the process will remain the same as before (since the utility functions are monotonic increasing in their payoffs), the number of cumulative times that each state is visited during the first t cycles may change as θ changes. Equations (3.13), (3.14), (3.15), (3.16) indicate that $i_t^{'}$, $j_t^{'}$, $k_t^{'}$ and $\ell_t^{'}$ may change now in θ through K, M and N. Denoting the players' initial expected utilities by $EU_0(\ )$, we note that:

$$K(\theta) = \frac{U^I(d) - U^I(b)}{U^I(a) - U^I(c)} = \frac{q^*(\theta)}{1 - q^*(\theta)} ,$$

$$M(\theta) = \frac{n_0^{II}\left[EU_0^{II}(b_2) - EU_0^{II}(b_1)\right]}{U^{II}(e) - U^{II}(f)} = n_0^{II}\left[2E_0^{II}(p) - 1\right] ,$$

and

$$N(\theta) = \frac{n_0^{I}\left[EU_0^{I}(a_2) - EU_0^{I}(a_1)\right]}{U^{I}(a) - U^{I}(c)} = n_0^{I}\left[[1 - E_0^{I}(q)]K(\theta) - E_0^{I}(q)\right] ,$$

it can seen the both $i_t^{'}$ and $\ell_t^{'}$ are not changed compared with the risk-neutrals case $(\theta \to 0)$ since $M(\theta)$ is not dependent on θ. However, since K and N are both monotonic increasing in θ (recall that we are assuming now that $q^*(\theta)$ is monotonic increasing in θ) both $j_t^{'}$ and $k_t^{'}$ are also monotonic increasing in θ, implying that the cycles become longer in this case.

Several possibilities exist for future work in the same spirit as the work reported here. Obvious generalizations involve changes in the details of the game. One possible direction is the relaxation of assumption that each player knows his own possible payoffs. The relaxation of such assumption leads into questions concerning the revision of probabilities and the value of various parcels of information. Finally, the results reported here can be used for generating hypotheses regarding actual dynamic behavior in competitive situations. These hypotheses can then be tested in experimental gaming setting and may provide stepping stones to developing behavioral theory in the direction of greater relevance to "real life."

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