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STRATEGIC EQUILIBRIA AND DECISIVE SET STRUCTURES FOR SOCIAL CHOICE MECHANISMS

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I. INTRODUCTION AND TABULATION OF RESULTS

The impossibility theorem of Arrow [1] has been generalized and extended in a variety of directions over the past quarter century, giving rise to a vital and elegant though not always tidy bundle of often negative results that we now describe as social choice theory. We focus in this paper on two separate ways in which Arrow's results have been extended. By utilizing established or new social choice and equilibrium conditions, we show that these two extensions are consistent and closely related.

One avenue of extension of Arrow's work admits the possibility of strategic behavior on the part of voters, locating social choice as a branch of cooperative game theory. The fundamental results in this direction were obtained by Gibbard and Satterthwaite [6,9], showing that the presence of a natural equilibrium strategy for each voter forces the social choice rule to be a dictatorship.

Another path of extension from Arrow's result shows that dictatorship can give way to less concentrated structures of power (namely oligarchy, collegial polity, and acyclic majority) as rationality conditions on the social preference are relaxed [7,4, and 3]. Such power structures have been most conveniently described by looking at the set-theoretic nature of the collection of "decisive sets" of voters [3]. In this formulation a dictatorship corresponds to an ultrafilter, an oligarchy to a filter, and a collegial polity to a prefilter. This terminology will be developed in the next section and used throughout.

In this paper we extend and unite the results cited in the previous two paragraphs, showing that various equilibrium and social choice conditions allow decisive set structures more "democratic" than a dictatorship. Reflecting the extreme cases treated by Satterthwaite and Gibbard, we consider both "true preferences" and abstract strategies in forming the domain for the social choice mechanism g whose range is the set X of alternatives. In the latter case the link with preferences is provided by an equilibrium selecting function h mapping preference profiles into strategy n-tuples. The results also depend on the notion of decisive set that is used. We consider three different definitions of decisive set, which we call <u>weakly preventing</u> (W). <u>strongly preventing</u> (P), and <u>controlling</u> ($\mathcal{C}$). The results we obtain are summarized in Table I.

[Table I inserted here]

New results obtained are those linking binary independence with filters and Pareto optimality with prefilters and acyclic majorities. A result of Ferejohn and Grether [5] linking strong Nash equilibria with prefilters is given a direct and simple proof. We now present the relevant notation and definitions to be followed by the stated results and their proofs.

TABLE I

Summary of Results

Domain of g	Decisiveness Notion	Equilibrium Concept	Structure of Decisive Sets
True Preferences (h = identity function) or Abstract Strategies	W = P	Paretian	Acyclic Majority ($\lvert X\rvert < n$)
			Prefilter ($\lvert X\rvert \geq n$)
	W	Binary Indep. & Paretian	Filter
		Nash Equilibrium	Ultrafilter (Satterthwaite)
Abstract Strategies	P	Paretian	Acyclic Majority ($\lvert X\rvert < n$)
			Prefilter ($\lvert X\rvert \geq n$)
		Free Binary Indep. and Free Strong Equil.	Filter
		Dominant Strategy	Ultrafilter
Abstract Strategies	$\mathcal{C}$	Strong Nash	Acyclic Majority ($\lvert X\rvert < n$)
			Prefilter ($\lvert X\rvert \geq n$) (Ferejohn-Grether)
		?	Filter
		Dominant Strategy	Ultrafilter (Gibbard)

II. NOTATION AND DEFINITIONS

We consider a finite set N of voters (and often game players) with $|N| = n$. A set X of alternatives with $|X| \geq 3$ is under consideration by the voters, who must choose a single alternative from X.

Let $\mathcal{R}$ denote the set of reflexive, transitive, total orderings (weak orders) on X. If $R_i \subset \mathcal{R}$ denotes the preference ordering over X for voter i, let

$$\pi = (R_1, R_2, \ldots, R_n) \in \mathcal{R}^n$$

denote the corresponding profile of preferences. Conversely, if $\pi \in \mathcal{R}^n$ is a preference profile, R_i will always denote the i^{th} component of π and P_i will denote the asymmetric part $(xP_iy \Leftrightarrow \sim yR_ix)$ of R_i. Likewise, $\bar{\pi} \in \mathcal{R}^n$ has i^{th} component $\bar{R}_i$.

Our general forumlation, introduced in [6], allows a set S of abstract strategies for each $i \in N$ (there is no loss of generality in assuming the same S for each voter). A game form is then a function

$$g : S^n \rightarrow X$$

which maps each n-tuple $s \in S^n$ of strategies onto an alternative $g(s) \in X$. Given $s \in S^n$ and $C \subset N$, we let s_C denote the projection of s onto C. This leads to self-explanatory though not perfectly grammatical notation such as $(s_C, \bar{s}_{D-C}, \bar{\bar{s}}_{N-(D \cup C)})$ to denote the strategy n-tuple for which voters from C

choose strategies projected from $s \in S^n$, voters from $D-C$ choose strategies projected from $\bar{s} \in S^n$, and the remaining voters use strategies projected from $\bar{\bar{s}} \in S^n$. We use this exceedingly handy notation freely in what follows.

The link between preferences and strategies is provided by a function $h: \mathcal{R}^n \to S^n$. If we interpret h as selecting an equilibrium strategy n-tuple for each profile, then $g \circ h$ describes the social outcome for that profile. For the special case where strategies are announcements of preferences, we set $S^n = \mathcal{R}^n$. If "true" preference revelation is desired (cf.[9]), h is taken to be the identity function. If strategic behavior in announcing preferences is allowed (cf.[8]), more general functions $h: \mathcal{R}^n \to \mathcal{R}^n$ can be used.

Taking $g: S^n \to X$ as fixed, we now list various properties of strategy selecting functions $h: \mathcal{R}^n \to S^n$. For the most part these may be thought of as equilibrium conditions on the strategies selected by h.

h is <u>Paretian</u> if $\forall x,y \in X$, $\forall \pi \in \mathcal{R}^n$,

$$xP_iy \ \ \forall \ i \in N \ \Rightarrow \ g(h(\pi)) \neq y .$$

h is <u>binary independent</u> if $\forall\ x \neq y$, $\forall\ \pi,\pi' \in \mathcal{R}^n$,

$$xP_iy \Leftrightarrow xP_i'y \ \text{ and } \ yP_ix \Leftrightarrow yP_i'x \ \forall i \in N \ \text{ and } \ g(h(\pi)) = x \Rightarrow g(h(\pi')) \neq y .$$

Binary independence is a natural version of the well-known independence of irrelevant alternatives condition applied to our social choice setting. It says that two profiles which are identical in their pairwise rankings of x and y cannot result in a choice of x in one case and a choice of y in the other. We now strengthen this by allowing some of the voters to deviate from the equilibrium strategies $h(\pi)$ and $h(\pi')$.

h is <u>freely binary independent</u> if $\forall\, x \neq y$, $\forall\, \pi, \pi' \in \mathcal{R}^n$, $\forall\, s \in S^n$, $\forall\, C \subset N$,

$xP_i y \Leftrightarrow xP_i' y$ and $yP_i x \Leftrightarrow yP_i' x$ $\forall\, i \in N$ and $g(h(\pi)_C, s_{N-C}) = x$

$\Rightarrow g(h(\pi')_C, s_{N-C}) \neq y$.

Free binary independence says that *if* a coalition C obtains alternative x while $N-C$ remains fixed at an arbitrary strategy, then preference changes within C that do not alter the relationship between x and y cannot result in a change to y.

h is a <u>Nash equilibrium</u> if $\forall\, \pi \in \mathcal{R}^n$, $\forall\, s \in S^n$, $\forall\, i \in N$,

$g(h(\pi)) R_i\, g(s_i, h(\pi)_{N-\{i\}})$

h is a <u>strong Nash equilibrium</u> if $\forall$ nonempty $C \subset N$, $\forall\, \pi \in \mathcal{R}^n$, $\forall\, s \in S^n$,

$\exists\, i \in C \ni g(h(\pi)) R_i\, g(s_C, h(\pi)_{N-C})$.

If we merely require one or more (rather than all) voters outside of C to stay at the equilibrium strategy $h(\pi)$, keeping the remaining voters fixed at an arbitrary strategy, we obtain a stronger version of strong Nash equilibrium:

h is a <u>freely strong equilibrium</u> if $\forall$ nonempty $C \subseteq N$,

$\forall$ nonempty $D \subset N-C$, $\forall\, \pi \in \mathcal{R}^n$, $\forall\, s \in S^n$,

$\exists\, i \in C \ni g(h(\pi)_C, h(\pi)_D, s_{N-(C \cup D)})\; R_i\; g(s_C, h(\pi)_D, s_{N-(C \cup D)})$.

Thus, voters in D stay fixed at the equilibrium strategy, while voters outside of D and C (the coalition threatening to desert the equilibruim) play arbitrary fixed strategies. Finally, if we restrict C to singleton sets while requiring D to be empty in the above definition, we obtain a familiar game-theoretic equilibrium condition:

h is a <u>dominant strategy equilibrium</u> if $\forall\, \pi \in \mathcal{R}^n$, $\forall\, s \in S^n$, $\forall\, i \in N$,

$g(h(\pi)_i, s_{N-\{i\}}) R_i\, g(s_i, s_{N-\{i\}})$.

In studying the power structure in social choice situations, various notions of "decisive sets" of voters have been fruitfully employed. The results to follow use three different decisiveness notions which we now define. The importance of blocking or preventing sets has been emphasized in [10] and [2].

Given a fixed $g: S^n \to X$ and a particular $h: \mathcal{P}^n \to S^n$, the <u>weakly preventing sets</u> for h are defined by

$$W_h = \{C \subset N \mid \forall x, y \in X, \forall \pi \in \mathcal{P}^n, xP_i y \; \forall_i \in C \Rightarrow g(h(\pi)) \neq y\} .$$

The <u>strongly preventing</u> sets for h are defined by

$$P_h = \{C \subset N \mid \forall x, y \in X, \forall \pi \in \mathcal{P}^n, \forall s \in S^n, xP_i y \; \forall i \in C \Rightarrow g(h(\pi)_C, s_{N-C}) \neq y\}$$

A third notion of decisive set within our context has also been used in the literature ([6] and [9]). The <u>controlling sets</u>, which depend only on g, are defined by

$$\mathcal{C} = \{C \subset N \mid \forall x \in X \; \exists \; s^x \in S^n \ni g(s^x_C, s_{N-C}) = x \; \forall s \in S^n\} .$$

The following lemmas show relationships among the three types of decisive sets that we have defined.

<u>Lemma 1</u>

(a) For any $h: \mathcal{P}^n \to S^n$, $P_h \subset W_h$.

(b) If $S^n = \mathcal{P}^n$ and h is the identity function, then $P_h = W_h$

<u>Proof</u>

(a) This is immediate via the definitions of P_h and W_h .

(b) It suffices from (a) to show that $W_h \subset P_h$. Given $C \in W_h$ and $\pi \in \mathcal{P}^n$ with $sP_i y \; \forall i \in C$ and given $\pi' \in \mathcal{P}^n$, consider the profile (strategy) $\pi'' = (\pi_C, \pi'_{N-C})$. Since h is the identity function, the fact that $C \in W_n$ and $x\pi''y \; \forall \; i \in C$ gives $g(\pi_C, \pi'_{N-C}) \neq y$. Thus $C \in P_n$.

QED

Lemma 2

(a) For any $h\colon \mathcal{P}^n \to S^n$, $P_h \subset \mathcal{C}$.

(b) For any strong Nash equilibrium $h\colon \mathcal{P}^n \to S^n$, $\mathcal{C} \subset W_n$.

Proof

(a) Given $C \in P_h$ and $x \in X$, take $\pi^x \in \mathcal{P}^n$ to be any profile for which x is every voter's undisputed first choice. Then $\forall \; y \neq x$, $\forall \; s \in S^n$, $g(h(\pi^x)_C, s_{N-C}) \neq y$. We must then have $g(h(\pi^x)_C, s_{N-C}) = x$. Thus $C \in \mathcal{C}$.

(b) Given $C \in \mathcal{C}$, suppose $\exists \; \pi \in \mathcal{P}^n$ and $s \in S^n$ such that $xP_iy \; \forall \; i \in C$, but $g(h(\pi)) = y$. Choose $s^x \in S^n$ such that $g(s^x_C, h(\pi)_{N-C}) = x$ (possible since $C \notin \mathcal{C}$) . This leads to a contradiction of the strong Nash equilibrium property of h since voters in C may switch from $h(\pi)$ to s^x and improve their lot.

QED

We now present definitions of the various set-theoretic structures that describe collections of decisive sets. We proceed from most to least democratic.

An <u>acyclic majority</u> is a collection $\mathscr{D}$ of subsets of N satisfying the following properties:

$$\text{I:} \quad \emptyset \notin \mathscr{D};\ N \in \mathscr{D}$$

$$\text{II:} \quad C \in \mathscr{D} \text{ and } C \subset D \Rightarrow D \in \mathscr{D}$$

$$\text{III}_{\text{AM}}: \quad C_i \in \mathscr{D}\ \forall i \in I \text{ and } |I| \leq |X| \Rightarrow \bigcap_{i \in I} C_i \neq \emptyset$$

The notion of acyclic majority was introduced in [3]. The condition III_{AM} becomes increasingly restrictive as the set X of alternatives grows. The cardinality condition $|I| \leq |X|$ has relevance only when $|X| < |N|$ (this will be argued later). With $|X| = 3$ such democratic structures as three-fourths rule ($|N| = 4$) and five-sevenths rule ($|N| = 7$) qualify as acyclic majorities.

A <u>prefilter</u> is a collection $\mathscr{D} \subset 2^N$ satisfying I, II, and

$$\text{III}_{\text{PF}}: \quad C_i \in \mathscr{D}\ \forall\ i \in I \Rightarrow \bigcap_{i \in I} C_i \neq \emptyset$$

The nonempty set $\bigcap_{C \in \mathscr{D}} C$ in a prefilter is called the <u>collegium</u>.

If a collection of decisive sets satisfies the properties of a prefilter, its collegium has a form of veto power though it generally needs outside support (enough to make it into a decisive set) to effect its collective will.

A <u>filter</u> is a collection $\mathscr{D}$ of subsets of N satisfying I, II, and

$$\text{III:} \quad C, D \in \mathscr{D} \Rightarrow C \cap D \in \mathscr{D}$$

In a filter on a finite set N the collegium is the smallest decisive set. This set can be regarded as an <u>oligarchy</u> which controls the voting rule or game under consideration.

Finally, an <u>ultrafilter</u> is a filter that also satisfies

IV: $C \notin \mathcal{D} \Rightarrow N\text{-}C \in \mathcal{D}$

Since N is taken to be finite, it follows readily that an ultrafilter is generated by (consists of all supersets of) a singleton set $\{i\} \in \mathcal{D}$. Voter i can then be regarded as a <u>dictator</u> in just the way that a dictator emerges in the theorems of Arrow and Gibbard-Satterthwaite.

III. RESULTS AND PROOFS

We now establish connections between the various conditions on $h: \mathcal{R}^n \to S^n$ and the properties of the corresponding decisive sets. We begin with the most straightforward case in which $S^n = \mathcal{R}^n$ and h is the identity function. From Lemma 1 we then have $P_h = W_h$

<u>Theorem 1</u>. Given $S^n = \mathcal{R}^n$ and h the identity function.

(a) h Paretian and $|X| < |N| \Rightarrow P_h(W_h)$ is an acyclic majority.

(b) h Paretian and $|X| \geq |N| \Rightarrow P_h(W_h)$ is a prefilter.

<u>Proof</u>. Properties I and II for acyclic majorities (and prefilters) follow directly from the definition of Paretian and of P_h. Properties III_{AM} and III_{PF} will emerge from the following common argument.

Given $C_1, C_2, \ldots, C_k \in P_h$ and <u>suppose</u> $\bigcap_{j=1}^{k} C_j = \emptyset$.

For (a) we may also assume $k \leq |X|$. For (b) there is no loss of generality in assuming $k \leq |N|$ (for each $i \in N$ take one C_j that excludes i and the resultant sets C_j will form a collection of size $\leq |N|$ with empty intersection). Since $|N| \leq |X|$, we again may assume $k \leq |X|$, just as for (a). It is then possible to choose distinct $x_1, x_2, \ldots, x_k \in X$ and a profile $\pi \in \mathcal{R}^n$ such that everyone has $x_1, \ldots, x_k$ among their top k choices and

$$x_1 P_i x_2 \ \forall\ i \in C_1$$
$$x_2 P_i x_3 \ \forall\ i \in C_2$$
$$\vdots$$
$$x_{j-1} P_i x_j \ \forall\ i \in C_{j-1}$$
$$\vdots$$
$$x_k P_i x_1 \ \forall\ i \in C_k .$$

Note that transitivity of preferences in π is possible by our supposition that $\bigcap_{j=1}^{k} C_j = \emptyset$. Since h (the identity function) is taken to be Paretian, we must have $g(\pi) = x_j$ for some $j = 1,2,\ldots,k$. However, C_{j-1} (or C_k if $j = 1$) being in P_h gives $g(\pi) \neq x_j$. This contradicts the supposition that $\bigcap_{j=1}^{k} C_j = \emptyset$ and establishes the theorem.

QED

<u>Theorem 2.</u> Given $S^n = \mathcal{R}^n$ and h the identity function. Then h Paretian and binary independent $\Rightarrow P_h(W_h)$ is a filter.

<u>Proof</u>. We work with the definition of W_h in this proof.

Given $C, D \in W_h$, suppose some $\pi \in \mathcal{R}^n$ has $xP_iy \ \forall\ i \in C \cap D$. Choose $\bar{\pi} \in \mathcal{R}^n$ with x,y, and z $(z \neq x$ or $y)$ ranked among the top three alternatives for all voters in such a way that:

$$x\bar{P}_i z \bar{P}_i y \ \forall\ i \in C \cap D$$

$x\bar{P}_i z$ and $\bar{P}_i$ "agrees with" P_i on $\{x,y\} \ \forall\ i \in C-D$

$z\bar{P}_i y$ and $\bar{P}_i$ "agrees with" P_i on $\{x,y\} \ \forall\ i \in D-C$.

Then $C \in W_h \Rightarrow g(\bar{\pi}) \neq z$ and $D \in W_h \Rightarrow g(\bar{\pi}) \neq y$, from which it follows from the Pareto property that $g(\bar{\pi}) = x$. Since π and $\bar{\pi}$ agree on $\{x,y\}$, binary independence then yields $g(\pi) \neq y$. Thus $C \cap D \in W_h$.

QED

<u>Theorem 3</u>. (Satterthwaite) Given $S^n = \mathcal{P}^n$ and h the identity function.

Then g onto and h a Nash equilibrium $\Rightarrow P_h(W_h)$ is an ultrafilter.

<u>Proof</u>. This follows directly from (and is in fact equivalent to) Satterthwaite's result [9]. Indeed, Satterthwaite's dictator is readily seen to be a singleton preventing set, thus generating an ultrafilter. A direct approach in terms of preventing sets for the special case of strict preferences can be found in [2].

QED

We now consider the general case where S^n is an n-fold Cartesian product of abstract individual strategy sets S. The function $g: S^n \to X$ together with a strategy selecting function $h: \mathcal{P}^n \to S^n$ is then closely akin to Gibbard's notion of a <u>game form</u> [6]. By taking $S^n = \mathcal{P}^n$ with h not necessarily the identity function, we capture situations where equilibrium strategies may involve revelation of "false" preferences (more open-mindedly, strategic behavior of a sophisticated nature). This ties into results of Peleg [8].

The case of the weakly preventing sets W_h yields essentially the same results as those of Theorems 1, 2, and 3. This is because the definition of W_h enables us to treat $g \circ h : \mathcal{P}^n \to X$ just as we previously treated $g: \mathcal{P}^n \to X$. The following corollary formalizes this.

Corollary. Given $g: S^n \to X$ and $h: \mathcal{R}^n \to S^n$, then

1. (a) h Paretian and $|X| < |N| \Rightarrow W_h$ is an acyclic majority.

   (b) h Paretian and $|X| \geqq |N| \Rightarrow W_h$ is a prefilter.

2. h Paretian and binary independent $\Rightarrow W_h$ is a filter.

3. g onto and h a Nash equilibrium $\Rightarrow W_h$ is an ultrafilter.

Proof. Apply Theorems 1,2, and 3 to $g' = g \circ h : \mathcal{R}^n \to X$ (with the identity as the equilibrium selecting function). Since the definition of W_h can be expressed directly in terms of g', the conclusions of Theorems 1, 2, and 3 yield the corresponding results for $g: S^n \to X$ in the case of weakly preventing sets.

The next three theorems develop the structure of the strongly preventing sets for general strategy spaces.

Theorem 4. Given $g: S^n \to X$ and $h: \mathcal{R}^n \to S^n$. Then

(a) h Paretian and $|X| < |N| \Rightarrow P_h$ is an acyclic majority.

(b) h Paretian and $|X| \geqq |N| \Rightarrow P_h$ is a prefilter.

Proof. From 1 (a) and (b) of the Corollary the appropriate intersections of weakly preventing sets are nonempty. By Lemma 1(a) the collection of strongly preventing sets is included in the collection of weakly preventing sets.

Therefore the appropriate intersections of strongly preventing sets must be nonempty.

QED

Theorem 5. Given $g: S^n \to X$ and $h: \rho^n \to S^n$. Then h freely binary independent and a freely strong equilibrium $\Rightarrow P_h$ is a filter.

Proof. Given $C, D \in P_h$ and any $\pi \in \mathcal{R}^n$ with $xP_iy \; \forall \; i \in C \cap D$.

Pick $z \in X$ not equal to x or y and pick a $\bar{\pi} \in \mathcal{R}^n$ such that each voter has x, y, and z as top choices and

$$x\bar{P}_i z\bar{P}_i y \quad \forall \quad i \in C \cap D$$

$$y\bar{P}_i x\bar{P}z \quad \forall \quad i \in C - D$$

$$z\bar{P}_i y\bar{P}_i x \quad \forall \quad i \in D - C$$

Then $C \in P_h \Rightarrow g(h(\bar{\pi})_{C \cap D}, h(\bar{\pi})_{C-D}, s_{D-C}, s_{-(C \cup D)}) = x \text{ or } y \quad \forall \; s \in S^n$.

Also, $D \in P_h \Rightarrow g(h(\bar{\pi})_{C \cap D}, s_{C-D}, h(\bar{\pi})_{D-C}, s_{-(C \cup D)}) = x \text{ or } z \quad \forall \; s \in S^n$.

Thus $g(h(\bar{\pi})_{C \cap D}, h(\bar{\pi})_{C-D}, h(\bar{\pi})_{D-C}, s_{-(C \cup D)}) = x \quad \forall \; s \in S^n$.

It then follows that

$$g(h(\bar{\pi})_{C \cap D}, h(\bar{\pi})_{C-D}, s_{D-C}, s_{-(C \cup D)}) = x \quad \forall \; s \in S^n$$

(otherwise voters in D-C could switch from $h(\bar{\pi})$ to s and all benefit, violating the free strong equilibrium property). Now take any $\bar{\bar{\pi}} \in \mathcal{R}^n$

such that $\bar{\bar{\pi}}_{C\cap D} = \pi_{C\cap D}$ and $\bar{\bar{\pi}}_{C-D} = \bar{\pi}_{C-D}$. Then $\forall\ s \in S^n$,

$$g(h(\bar{\bar{\pi}})_{C\cap D}, h(\bar{\bar{\pi}})_{C-D}, s_{D-C}, s_{-(C\cap D)}) \neq y \quad \text{(free binary independence)}.$$

Hence, $g(h(\bar{\bar{\pi}})_{C\cap D}, s_{C-D}, s_{D-C}, s_{-(C\cap D)} \neq y$ (otherwise C-D can switch from $h(\bar{\bar{\pi}})$ to s and violate the freely strong equilibrium property.) Since $\bar{\bar{\pi}}_{C\cap D} = \pi_{C\cap D}$, free binary independence requires

$$g(h(\pi)_{C\cap D}, s_{-(C\cap D)}) = g(h(\bar{\bar{\pi}})_{C\cap D}, s_{-(C\cap D)}) \ \forall\ s \in S^n.$$

Hence $g(h(\pi)_{C\cap D}, s_{-(C\cap D)}) \neq y\ \ \forall\ \ s \in S^n$ and $C \cap D \in P_h$.

QED

The "freeness" requirements (whereby a set of voters stays fixed at an arbitrary strategy) seem quite strong. The freely strong equilibrium does not, however, imply a dominant strategy equilibrium (and hence a dictatorship) since some nonempty set of voters must stay fixed at the equilibrium $h(\pi)$. The possibility remains open that the hypotheses of Theorem 5 may imply more than a filter structure or that these hypotheses can be weakened somewhat.

Theorem 6. Given $g\colon S^n \to X$ and $h\colon \mathcal{P}^n \to S^n$. Then g onto and h a **dominant** strategy equilibrium $\Rightarrow P_h$ is an ultrafilter.

Proof. Gibbard's well-known result [6] says that the controlling sets $\mathcal{C}$ form an ultrafilter. Let $i \in N$ be the dictator corresponding to the ultrafilter. Then $\forall\ x \in X\ \exists\ s^x$ such that $g(s_i^x, s_{N-\{i\}}) = x\ \ \forall\ s \in S^n$.

Suppose now that for some $\pi \in \mathcal{P}^n$ with xP_iy, $g(h(\pi)_i, s_{N-\{i\}}) = y$.

Then i can switch from $h(\pi)$ to s^x and obtain a better alternative, contradicting the assumption that h is a dominant strategy equilibrium. Thus $\{i\} \in P_h$ and P_h is an ultrafilter.

QED

We note that since generally $P_h \subset \mathcal{C}$ (Lemma 2), Theorem 6 provides a slight strengthening of Gibbard's result.

Results for the controlling sets $\mathcal{C}$ are not complete. One difficulty lies in the fact that $\mathcal{C}$ (unlike W_h and P_h) does not link the equilibrium selecting function h directly with the notion of decisiveness. We do, however, obtain the following.

<u>Theorem 7</u>. Given $g\colon S^n \to X$ onto and $h\colon \mathcal{R}^n \to S^n$. Then

(a) h a strong Nash equilibrium and $|X| < |N| \Rightarrow \mathcal{C}$ is an acyclic majority.

(b) h a strong Nash equilibrium and $|X| \geq |N| \Rightarrow \mathcal{C}$ is a prefilter.

<u>Proof</u>. First note that g onto and h a strong Nash equilibrium implies that h is Paretian. Given $C_1, C_2, \ldots, C_k \in \mathcal{C}$, we proceed as in the proof of Theorem 1. As in that proof, suppose that $\bigcap_{j=1}^{k} C_j \neq \emptyset$ and assume without loss of generality for both (a) and(b) that $k \leq |X|$. Using the profile $\pi \in \mathcal{R}^n$ described in the proof of Theorem 1, we again obtain $g(\pi) = x_j$ (for convenience assume $j \neq 1$) using the Pareto property of h. But $C_{j-1} \in \mathcal{C} \Rightarrow \exists\ s^{x_{j-1}} \in S^n$ such that

$g(s_{C_{j-1}}^{x_{j-1}}, h(\pi)_{N-C_{j-1}}) = x_{j-1}$. But then voters in C_{j-1} could switch from $h(\pi)$ to $s^{x_{j-1}}$ and all be better off, contradicting the strong Nash equilibrium property. It follows that $\bigcap_{j=1}^{k} C_j \neq \emptyset$ and $\mathcal{C}$ is an acyclic majority (part (a)) or a prefilter (part (b)).

QED

Theorem 7(b) provides an alternative and more direct proof of a result first obtained by Ferejohn and Grether [5]. Theorem 7 also provides a partial answer to a conjecture offered by Peleg [8].

The following well-known result gives conditions for $\mathcal{C}$ to be an ultrafilter.

Theorem 8. (Gibbard) Given $g: S^n \to X$ and $h: \mathcal{R}^n \to S^n$. Then g **onto** and h a dominant strategy equilibrium $\Rightarrow \mathcal{C}$ is an ultrafilter.

Proof. See Gibbard [6].

Referring to the summary of results provided by Table I, we leave open the question of what (minimal) conditions on h make $\mathcal{C}$ into a filter. Likewise, it is unclear at this point what the analogs to Theorems 1 and 2 might be with $\mathcal{C}$ as the decisiveness concept. Since it is well-known for $S^n = R^n$ and h = the identity function that Nash equilibrium is equivalent to dominant strategy equilibrium, an equilibrium concept weaker than Nash is desired. Since $P_h \subset \mathcal{C}$ from Lemma 2, the ultrafilter result for $\mathcal{C}$ analogous to Theorem 3 carries over directly.

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