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Antonio Camacho
Northwestern University

Jon Sonstelie
Northwestern University

www.kellogg.nwu.edu/research/math

CMS-EMS
The Center for
Mathematical Studies
in Economics &
Management Sciences

Northwestern University
2001 Sheridan Road 580 Leverone Hall Evanston, IL 60208-2014 USA

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In 1955 Professor John Harsanyi[1] proposed three appealing postulates for social choice under uncertainty and tried to show (Theorem V, p.314) that these postulates lead to a social welfare function which is a weighted sum of the utility functions of the individuals.

Although the proof of Theorem V given by Harsanyi seems to be incorrect, his theorem still holds and the main purpose of this note is to present an alternative proof. Indeed, we will prove a theorem which is stronger than Harsanyi's.

Harsanyi's three postualtes are:

<u>Postulate a</u>. Social preferences R satisfy Marschak's[2] Postulates I, II, III′, and IV.

<u>Postualte b</u>. Individual preferences R_i $(i=1,\ldots,n)$ satisfy the same four postulates.

<u>Postulate c</u>. If two prospects P and O are indifferent from the standpoint of every individual, they are also indifferent from a social standpoint.

Let $X_1,\ldots,X_j,\ldots,X_m$ represent the m sure prospects available to society. From postulate <u>a</u> it follows that there exists a social welfare function W, representing the social preferences, that satisfies the following property [α]: If the value of W corresponding to the sure prospect X_j $(j=1,\ldots,m)$ is w_j, then the value of W for any prospect $P = (p_1,\ldots,p_j,\ldots,p_m)$, $p_j \geq 0$, $\sum_{j-1}^{m} p_j = 1$, is

$$W(p_1,\dots,p_j,\dots,p_m) = \sum_{j=1}^{m} p_j w_j \; .$$

This social welfare function is unique up to a positive linear transformation. Similarly, from postulate b it follows that for each individual i (i=1,...,n) there exists a utility function U_i, representing his preferences R_i, and satisfying property [α]:

$$U_i(p_1,\dots,p_j,\dots,p_m) = \sum_{j=1}^{m} p_j u_{ij}$$

where u_{ij} is the value of U_i that corresponds to the sure prospect X_j. This utility function is unique up to a positive linear transformation.

Harsanyi's Theorem V can now be stated as follows: If the individual preferences R_i (i=1,...,n) satisfy postulate b, and the social preferences R satisfy postulates a and c, and the corresponding utility functions U_i and the social welfare function W are chosen in such a way that for a given prospect, all take the value 0, then the social welfare function W can be represented as a weighted sum of the individual utilities of the form

$$W = \sum_{i=1}^{n} a_i U_i$$

where the a_i's are constant real numbers; i.e., they do not depend on the prospects P.

A close look at Harsanyi's paper suffices to show that his proof of Theorem V is based on the existence of prospects such that, after choosing a common origin for the individual utility functions, the utility of each of these prospects is 1 for a given individual and 0 for the rest of the individuals in the society.

For those situations where there are prospects S_k $(k=1,\ldots,n)$ such that

$$U_i(S_k) = \begin{cases} 1, & \text{if } i = k \\ 0, & \text{if } i \neq k \end{cases} .$$

Harsanyi's proof and his interpretation of the coefficients a_i's are both correct. But in those cases where, after choosing a common origin for all the individual utility functions, there is no prospect whose utility is 1 for a given individual and 0 for the others, neither his proof nor his interpretation of the coefficients a_i apply. That there are such cases can easily be shown by examples as the following:

Consider a society with three individuals and three sure prospects X_1, X_2, X_3. Take a common origin, say X_1, for the three utility functions U_1, U_2, U_3, and let the values of $U_i(X_j) = u_{ij}$ be: $u_{11} = 0$, $u_{12} = 1$, $u_{13} = 2$; $u_{21} = 0$, $u_{22} = 1$, $u_{23} = 3$; $u_{31} = 0$, $u_{32} = 2$, $u_{33} = 4$. It is easy to check that there is not a prospect $P = (p_1, p_2, p_3)$ such that the utility of P is 1 for one individual and 0 for the others.

The proof that we will present below is not based on the existence of such prospects and makes use only of a postulate $\underline{c}'$ that is weaker than Harsanyi's postulate $\underline{c}$. Let $\overline{P}$ be any given prospect with all its components strictly positive: $\overline{P} = (\overline{p}_1,\ldots,\overline{p}_j, \ldots,\overline{p}_m)$, $\overline{p}_j > 0$, $j=1,\ldots,m$, $\sum_{j=1}^{m} p_j = 1$.

<u>Postulate c′</u>. If a prospect P is indifferent to $\overline{P}$ from the standpoint of every individual, then it is also indifferent to $\overline{P}$ from a social standpoint.

We will now prove the following <u>Theorem V′</u>: If the individual preferences R_i $(i=1,\dots,n)$ satisfy postulate <u>b</u>, and the social preferences R satisfy postulates <u>a</u> and <u>c</u>′, and the corresponding utility functions U_i and the social welfare function W are chosen in such a way that for a given prospect all take the value 0, then the social welfare function W can be represented as a weighted sum of the individual utilities of the form

$$W = \sum_{i=1}^{n} a_i U_i$$

where the a_i's are constant real numbers, i.e., they do not depend on the prospects P.

<u>Proof</u>. Observe first that in view of property [α], to prove that $W = \sum_{i=1}^{n} a_i U_i$, it suffices to show that $w_j = \sum_{i=1}^{n} a_i u_{ij}$, for $j=1,\dots,m$.

Observe also that if the theorem holds when a given prospect is chosen as the common origin of the utility functions and the social welfare function, then it also holds if any other prospect is chosen as the common origin. Thus, without loss of generality, let the prospect $\overline{P}$ be chosen as the common origin. Consider now the two following systems of equations:

$$[1] \quad \begin{cases} u_{11}\, p_1 + \dots + u_{1m}\, p_m = 0 \\ \vdots \\ u_{n1}\, p_1 + \dots + u_{nm}\, p_n = 0 \, ; \end{cases}$$

$$[2] \quad \begin{cases} w_1 p_1 + \dots + w_m p_m = 0 \\ u_{11} p_1 + \dots + u_{1m} p_m = 0 \\ \vdots \qquad \vdots \qquad\qquad \vdots \\ u_{n1} p_1 + \dots + u_{nm} p_m = 0 \end{cases}.$$

Clearly we have the following:

(i) Prospect $\overline{P} = (\overline{p}_1, \dots, \overline{p}_j, \dots, \overline{p}_m)$ is a solution of both system [1] and system [2], and $\overline{p}_j > 0$, for $j=1,\dots,m$. Thus both systems have a common solution with all its components strictly greater than 0. The set of all solutions of system [1] which are prospects is precisely the set of all prospects which are indifferent to prospect $\overline{P}$ from the standpoint of every individual. The set of all solutions of system [2] which are prospects is precisely the set of all prospects which are indifferent to $\overline{P}$ from the standpoint of every individual and from the standpoint of society.

(ii) Any solution of system [2] is also a solution of system [1] since system [2] is obtained from [1] by adding the equation $w_1 p_1 + \dots + w_m p_m = 0$.

(iii) If $(p_1, \dots, p_j, \dots, p_m)$, not necessarily a prospect, is a solution of [1] and $p_j > 0$ for $j=1,\dots,m$, then $\left(\frac{p_1}{\sum_{j=1}^{m} p_j}, \dots, \frac{p_m}{\sum_{j=1}^{m} p_j} \right)$, which is a prospect, is a solution of [1] and, by postulate c′, it must also be a solution of [2], which implies that $(p_1, \dots, p_m)$ is a solution of [2].

From (i), (ii) and (iii) it follows that: (a) $(\bar{p}_1,\ldots,\bar{p}_m)$, where $\bar{p}_j > 0$ for all j, is a solution of both system [1] and system [2]; and (b) any $(p_1,\ldots,p_m)$, with $p_j > 0$ for all j, which is a solution of [1] is also a solution of [2] and vice versa. But (a) and (b) imply that systems [1] and [2] are equivalent, i.e., every solution of [1] is a solution of [2] and vice versa.[3] Since system [2] is derived from system [1] by adding the equation $w_1p_1 + \ldots + w_mp_m = 0$, it follows that

$$w_j = \sum_{i=1}^{n} a_i u_{ij}$$

as desired.

Footnotes

1. Harsanyi, John, "Cardinal Welfare, Individualistic Ethics, and Interpersonal Comparisons of Utility," Journal of Political Economy, LXIII (August, 1955), pp. 309-21.

2. Marschak, Jacob, "Rational Behavior, Uncertain Prospects, and Measurable Utility," Econometrica, XVIII (1950), pp. 111-41.

3. Let $(q_1,\ldots,q_j,\ldots,q_m)$ be any solution of [1]. We can write $q_j = \bar{p}_j + \alpha\varepsilon_j$, where $\bar{p}_j + \varepsilon_j > 0$ for $j=1,\ldots,m$. Then, since $(\bar{p}_1,\ldots,\bar{p}_m)$ is a solution of [1], $(\alpha\varepsilon_1,\ldots,\alpha\varepsilon_m)$, $(\varepsilon_1,\ldots,\varepsilon_m)$ and $(p_1 + \varepsilon_1,\ldots,p_m + \varepsilon_m)$ are also solutions of [1]. Since $\bar{p}_j + \varepsilon_j > 0$ for all j, it follows that $(\bar{p}_1 + \varepsilon_1,\ldots,$

   $\bar{p}_m + \varepsilon_m)$ is a solution of [2]. And since $(\bar{p}_1,\ldots,\bar{p}_m)$ is a solution of [2], it follows that $(\varepsilon_1,\ldots,\varepsilon_m)$, $(\alpha\varepsilon_1,\ldots,\alpha\varepsilon_m)$ and $(\bar{p}_1 + \alpha\varepsilon_1,\ldots,\bar{p}_m + \alpha\varepsilon_m) = (q_1,\ldots,q_m)$ are also solutions of [2]. In the same way we can show that any solution of [2] is a solution of [1].