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THEORETICAL IMPLICATIONS OF THE PERMANENT INCOME HYPOTHESIS*

by

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Introduction

This paper explores several implications of what I term the permanent income hypothesis. By the permanent income hypothesis, I mean the assumption that the marginal utility of money of each consumer is constant. This assumption is an exaggeration of Milton Friedman's hypothesis [22]. I visualize the consumer as having a nearly fixed idea of the value of money to him. When he buys a good, he spends money on it up to the point at which the utility gained from the quantity bought with an additional dollar equals the fixed utility of a dollar. He fixes his utility of money so that his long-run average expenditure per period equals his long-run average income. He uses money to iron out short-run fluctuations in his net expenditures.

It may be seen that the permanent income hypothesis is appropriate only in short-run contexts and when the consumer's expectations about the future are fairly stable. Also, he must have had time to accumulate adequate money balances. The hypothesis has been justified rigorously in a previous paper [10].

The permanent income hypothesis leads naturally to a version of general equilibrium theory which can serve as an alternative to the Arrow-Debreu model. This version is, of course, applicable only in the limited setting in which the permanent income hypothesis is appropriate. In this theory, equilibrium is a stationary stochastic process of temporary equilibria and is also Pareto optimal. It is Pareto optimal even though there are no forward markets for contingent claims.

In part I, I formulate this theory. Consumers' utility functions and endowments and firms' technologies all fluctuate according to a stationary probability law. Prices also form a stationary stochastic process. All trading is for current delivery. Firms maximize long-run average profit per period. Consumers maximize their long-run average flow of utility, subject to the constraint that long-run average expenditure per period not exceed long-run average income. All firms and consumers have rational expectations. They also observe the underlying stochastic process which governs all exogenous fluctuations in the economy. The equilibrium defined is termed stationary equilibrium. In equilibrium, each consumer's marginal utility of money is constant.

The assumption that each agent observes the exogenous stochastic process is quite strong. In fact, agents do not necessarily need so much information. This issue is discussed in section I.4.

Stationary equilibrium turns out to be equivalent to general equilibrium in a two period economy in which goods in the second period are artificially transferred back to the first period. (This fact is proved in section I.6.) I prove that stationary equilibria exist by proving that the two period economy has an equilibrium. The two period economy necessarily has infinitely many commodities (unless there is no randomness at all). I prove that the two period economy has an equilibrium by applying results of a previous paper [9] on economies with infinitely many commodities.

A routine argument proves that stationary equilibrium is Pareto optimal. It is intuitively fairly clear why markets for contingent claims are not needed for Pareto optimality. In the first place, since the marginal utility of money is constant, a consumer has no need for insurance. In effect, he uses money in order to insure himself. In the second place, forward markets are not needed to coordinate intertemporal supply and demand, for agents have rational expectations and full information.

Stationary equilibrium may be interpreted as providing a theory of socially optimal inventory policy. In the stationary equilibrium model, a distinction is made between producible and primary goods. Since all fluctuations are thought of as short-lived, one must think of capital goods as primary goods in fixed supply. The producible goods are consumables and intermediate goods. (One can introduce an exogenous demand for capital goods, as I explain in section I.4.) Stocks of producible goods may fluctuate. The quantities held are Pareto optimal since the stationary equilibrium itself is Pareto optimal.

Some caution must be used in interpreting this theory of inventory holdings. Since markets are perfect, many of the usual motives for holding inventories are absent. Agents believe they can always buy whatever they need when they need it.

It is remarkable that Pareto optimal inventory and production decisions are made in stationary equilibrium, even though no individual knows the aggregate quantities held of the various goods. This fact shows how strong are the assumptions of rational expectations and full information.

Stationary equilibrium offers a limited answer to the question of why we do not in reality observe complete markets for contingent claims.

Arrow posed this question in [5]. The answer is that self-insurance and rational expectations can take care of every day fluctuations.

If there is one consumer and one firm, then a stationary equilibrium is a stationary optimal consumption and production plan, or a stochastic golden rule. Thus, I give a proof using equilibrium theory of the existence of such a golden rule and associated competitive prices. This problem has been studied by Radner [38], Evstigneev [21], and others. Their work is discussed in section I.5. Of course, the interpretation of stationary equilibrium is very different from that of the golden rule. The golden rule has to do with the theory of optimal capital investment. In stationary equilibrium, capital goods are in fixed supply.

In summary, stationary equilibrium provides a synthesis of general equilibrium theory, the permanent income hypothesis, rational expectations, temporary equilibrium theory and the theory of the golden rule.

In part II of this paper, I show that the permanent income hypothesis provides a solution to the stability problem of general equilibrium theory. By the stability problem, I mean the fact that the differential equation system

$$\frac{dp(t)}{dt} = Z(p(t)) \tag{1}$$

may be unstable. In this equation, $p(t)$ is the price vector at time t and $Z(\cdot)$ is the market excess demand function. This equation system may be thought of as representing the motion generated by a <u>tâtonnement</u> price adjustment process. In fact, this equation cannot really be justified rigorously. It simply expresses the idea that prices rise or fall as demand exceeds or falls short of supply. The fact that solutions of equation (1) may be unstable seems to have discouraged the development of a rigorous theory of price adjustment and to have led to the conclusion that there is no a priori reason that market prices should be stable. Of course there are many obstacles to the development of a convincing theory of price dynamics. For instance, it is hard to reconcile any realistic price adjustment process with perfect competition, as Arrow pointed out [3].

That the differential system (1) may be unstable was first demonstrated by examples of Scarf [43] and Gale [24]. Results of Sonnenschein [46, 47], Mantel [33, 34], and Debreu [18] lead to the conclusion that almost any market excess demand function is possible, so that the motion determined by equation (1) may be as unstable as one likes.

If demand functions are defined using the permanent income hypothesis, then differential system (1) is globally stable and there is a unique equilibrium. This price stability must be thought of only as short-run stability, for it is appropriate to think of the marginal utility of money as fixed only in the short-run.

One can imagine that each consumer adjusts his marginal utility of money slowly in response to the difference between expenditure and income. If one makes this assumption, one obtains a long-run adjustment process. This process may be unstable. In fact, an analogue of the Sonnenschein, Mantel, Debreu theorem applies to it. I prove this fact in another paper [11].

In part III, I point out that the permanent income hypothesis justifies the use of consumers' surplus. That is, if consumers' marginal utilities of money are constant, then changes in the sum of expenditure and consumers' surplus measure changes in a weighted sum of consumers' utilities. This fact is, of course, nearly obvious. The point is that one can defend vigorously the use of consumers' surplus provided the permanent income hypothesis is appropriate. It is appropriate only when the price changes involved are viewed by consumers as temporary. Also, the changes must not sharply reduce consumers' holdings of liquid assets.

Part I: Stationary Equilibrium

Stationary equilibrium inevitably involves the use of an infinite dimensional commodity space. In stationary equilibrium, prices and allocations depend on the infinite history of past values of an exogenous stationary random process. If this process is not periodic, it has a continuum of possible histories.

The use of an infinite dimensional commodity space makes stationary equilibrium a somewhat technical subject. Notation and definitions are introduced in the next section. Assumptions are listed in section 2, and the theorems are listed in section 3. In section 4, I discuss certain assumptions. Related work is discussed in section 5. The proofs follow. These are in large part routine, given knowledge of Debreu [17] and my own paper [9].

I.1 Definitions, Notation and the Model

The Underlying Stochastic Process

The exogenous stochastic process is denoted by $\{s_n\}_{n=-\infty}^{\infty}$. It influences utility functions, endowments, and production functions. (Doob [19] is the reference used for stochastic processes.) The random variables, s_n , take values in some measurable space $(M,\mathcal{M})$, where M is a set and $\mathcal{M}$ is a σ-field of subsets of M . The sample space of the process is $S = \{(\ldots,s_{-1},s_0,s_1,\ldots) \mid s_n \in M$, for all $n\}$. s denotes an element of S and s_n denotes the n^{th} component of s . The set of all measureable subsets of S is denoted by $\mathcal{A}$, and P denotes the probability on $\mathcal{A}$. $\mathcal{A}$ is the smallest complete σ-field such that all the random variables s_n are measureable with respect to $\mathcal{A}$. Sets $A \in \mathcal{A}$ such that $P(A) = 0$ are called sets of probability zero. An event occurs <u>almost surely</u> or <u>for almost every s</u> if it occurs for every s except for points s belonging to a set of probability zero.

E denotes the expectation operator corresponding to P . That is, if $X : S \to (-\infty,\infty)$ is integrable with respect to P , then $E X = \int X(s) P(ds)$.

$\mathcal{A}_n$ denotes the smallest complete σ-field with respect to which the random variables s_k , $k \leq n$, are measurable. Clearly, $\mathcal{A}_n \subset \mathcal{A}$. $\mathcal{A}_n$ represents the information available in period n to an observer of the process.

Let $\sigma : S \to S$ be the shift operator, defined by the formula $(\sigma s)_n = s_{n+1}$. $\{s_n\}$ is <u>stationary</u> if and only if σ is probability preserving. That is, σ must satisfy $P(\sigma B) = P(B)$, for every $B \in \mathscr{S}$. A measurable set B is said to be invariant if $P(B) = P(B \cap \sigma B)$. A stationary process is said to be <u>metrically transitive</u> if every invariant set is of probability 0 or 1. Metric transitivity is often referred to in probability theory as ergodicity.

I will assume that the process $\{s_n\}$ is <u>stationary</u> and <u>metrically transitive</u>.

Commodities

There are L types of commodities. $L_c \subset \{1,\ldots,L\}$ denotes the set of consumption goods. $L_o \subset \{1,\ldots,L\}$ denotes the set of primary commodities, such as labor and raw materials. Capital equipment should also be thought of as a primary good. $L_p = \{k = 1,\ldots,L \mid k \notin L_o\}$ denotes the set of producible goods. Goods not in either L_c or L_o should be thought of as intermediate goods or goods in process.

Vector Space Notation

R^L denotes L-dimensional Euclidean space. R^{L_c} denotes the subspace of R^L corresponding to L_c. That is, $R^{L_c} = \{x \in R^L \mid x_k = 0$ if $k \notin L_c\}$. R^{L_o} and R^{L_p} are defined similarly.

$\mathscr{L}_{\infty,L}(S,\mathscr{S}_n,P)$ denotes the space of equivalence classes of functions from S to R^L which are measurable with respect to $\mathscr{S}_n$ and are essentially bounded. $f : S \to R^L$ is said to be essentially bounded if for some $r > 0$, $P\{s \mid \|f(s)\| > r\} = 0$. The equivalence relation is that of almost sure equality. That is, if $f = S \to R^L$ and $g: S \to R^L$ are $\mathscr{A}_n$-measurable and essentially bounded, then f and g represent the same element of $\mathscr{L}_{\infty,L}(S,\mathscr{A}_n,P)$ if $f(s) = g(s)$ for almost every s .

$\mathscr{L}_{1,L}(S,\mathscr{A}_n,P)$ denotes the space of equivalence classes of functions from S to R^L which are $\mathscr{A}_n$-measurable and integrable with respect to P .

$\mathscr{L}_{\infty,L_c}(S,\mathscr{A}_n,P)$, $\mathscr{L}_{\infty,L_p}(S,\mathscr{A}_n,P)$ and $\mathscr{L}_{\infty,L_o}(S,\mathscr{A}_n,P)$ are the subspaces of $\mathscr{L}_{\infty,L}(S,\mathscr{A}_n,P)$ corresponding to L_c, L_p and L_o , respectively.

If $x \in R^L$, then "$x \geq 0$" means "$x_k \geq 0$, for all k ." "$x > 0$" means "$x \geq 0$ and $x \neq 0$." "$x \gg 0$" means "$x_k > 0$, for all k." R^L_+ denotes $\{x \in R^L | x \geq 0\}$, and R^L_- denotes $-R^L_+$. If $x \in R^L$, then $x^+ \in R^L_+$ is defined by $x^+_k = \max(0, x_k)$, for all k .

If $x \in \mathscr{L}_{q,L}(S,\mathscr{A}_n,P)$, where $q = 1$ or ∞ , I write "$x \geq 0$" if $x(s) \geq 0$, for almost every s . I write "$x > 0$" if $x \geq 0$ and $x \neq 0$. I write "$x \gg 0$" if $x(s) \gg 0$, for almost every s . Finally, "$x(s) \ggg 0$" means "there exists a positive real number r such that $x_k(s) \geq r$ almost surely, for all k ."

$\mathscr{L}^+_{\infty,L}(S,\mathscr{A}_n,P)$ denotes $\{x \in \mathscr{L}_{\infty,L}(S,\mathscr{A}_n,P) \mid x \geq 0\}$. $\mathscr{L}^+_{\infty,L_c}(S,\mathscr{A}_n,P)$, $\mathscr{L}^+_{\infty,L_o}(S,\mathscr{A}_n,P)$ and $\mathscr{L}^+_{1,L}(S,\mathscr{A}_n,P)$ are defined similarly.

If $x \in \mathcal{L}_{\infty,L}(S,\mathcal{A}_n,P)$ and $p \in \mathcal{L}_{1,L}(S,\mathcal{A}_n,P)$, then $p \cdot x$ denotes $\sum_{k=1}^{L} \int p_k(s) x_k(s) P(ds)$.

If $x \in \mathcal{L}_{q,L}(S,\mathcal{A}_n,P)$, where $q = 1$ or ∞, then $\sigma^m x$ is defined by $\sigma^m x = x(\sigma^m s)$, where σ is the shift operator on S. Clearly, $\sigma^m x \in \mathcal{L}_{q,L}(S,\mathcal{A}_{n+m},P)$.

<u>Consumers</u>

There are I consumers, where I is a positive integer. The endowment of consumer i is determined by $\omega_i \in \mathcal{L}^{+}_{\infty,L_0}(S,\mathcal{A}_0,P)$. $\omega_i(\sigma^n s) \in R_+^{L_0}$ is his endowment vector at time n. Note that consumers are endowed only with primary goods.

The utility function of consumer i for consumption in period zero is $u_i : R_+^{L_c} \times S \to (-\infty,\infty)$. I assume that u_i is measurable with respect to $\mathcal{B} \otimes \mathcal{A}_0$, where $\mathcal{B}$ is the Borel σ-field of $R_+^{L_c}$ and $\mathcal{B} \otimes \mathcal{A}_0$ is the product σ-field generated by $\mathcal{B}$ and $\mathcal{A}_0$. The utility function for period n is $u_i(x,\sigma^n s)$.

Utility is additively separable with respect to time and satisfies the expected utility hypothesis. That is, if $x_n \in \mathcal{L}^{+}_{\infty,L_c}(S,\mathcal{A}_n,P)$, for $n = 1,\ldots,N$, then the total utility to consumer i of the bundle

$(x_1,\ldots x_n)$ is $$\sum_{n=1}^{N} \int u_i(x_n(s),\sigma^n s) P(ds) .$$

Firms

There are J firms, where J is a positive integer. The production function of firm j is $g_j: R_-^L \times S \to R_+^{Lp}$. I assume that g_j is measurable with respect to $\mathcal{B} \otimes \mathcal{A}_1$, where $\mathcal{B}$ is the Borel σ-field on R_-^L. g_j defines the technology at time zero for transforming inputs at time zero into outputs at time one. The technology at time n is defined by $g_j(\cdot, \sigma^n s)$. Inputs y_0 at time n may be transformed into outputs y_1 at time $n+1$ when the state of the world is s, if and only if $y_1 \leqq g_j(y_0, \sigma^n s)$. Inputs carry a negative sign and outputs a positive sign.

Firms are owned by consumers. Consumer i owns a proportion θ_{ij} of firm j, for $i = 1,\ldots,I$ and $j = 1,\ldots,J$. $0 \leqq \theta_{ij} \leqq 1$, for all i and j, and $\sum_{i=1}^{I} \theta_{ij} = 1$, for all j.

The Economy

The economy is denoted by $\mathcal{E}$. It is described by the list $\{ (u_i, \omega_i), g_j, \theta_{ij} : i = 1,\ldots,I \text{ and } j = 1,\ldots,J \}$.

Allocations

A <u>consumption program</u> for a particular consumer is of the form $(x^0, x^1, \ldots)$, where $x^n \in \mathcal{L}^+_{\infty, L_c}(S, \mathcal{A}_n, P)$, for all n. The consumption vector at time n determined by this program is $x^n(s) \in R_+^{L_c}$.

A stationary consumption program is a program such that $x^n = \sigma^n x$, for all n, where $x \in \mathcal{L}^+_{\infty, L_c}(S, \mathscr{S}_n, P)$. Such a program is denoted simply by x.

A production program is of the form $(y_1^{-1}, (y_0^0, y_1^0), (y_0^1, y_1^1), \ldots)$, where $y_1^{-1} \in \mathcal{L}_{\infty, L_p}(S, \mathscr{S}_0, P)$ and $(y_0^n, y_1^n) \in \mathcal{L}_{\infty, L}(S, \mathscr{S}_n, P) \times \mathcal{L}_{\infty, L_p}(S, \mathscr{S}_n, P)$, for $n \geq 0$. The program is feasible for firm j if $y_0^n \leqq 0$ and $y_1^n(s) \leqq g_j(y_0^n(s), s)$ almost surely, for all $n \geqq 0$. The input vector at time n is $y_0^n(s)$, and the output vector at time $n+1$ is $y_1^n(s)$. Notice that this definition of feasibility implies free disposibility.

A stationary production program is a program such that $y_1^{-1} = \sigma^{-1} y_1$ and $(y_0^n, y_1^n) = (\sigma^n y_0, \sigma^n y_1)$, for $n \geqq 0$, where $(y_{01}, y_1) \in \mathcal{L}_{\infty, L}(S, \mathscr{S}_0, P) \times \mathcal{L}_{\infty, L_p}(S, \mathscr{S}_1, P)$. Such a program is denoted simply by (y_0, y_1).

An allocation for the economy $\mathcal{E}$ is of the form $((x_i)_{i=1}^I, (y_j)_{j=1}^J)$, where each $x_i = (x_i^0, x_i^1, \ldots.)$ is a consumption program and each $y_j = (y_{j1}^{-1}, (y_{j0}^0, y_{j1}^0), (y_{j0}^1, y_{j1}^1), \ldots..)$ is a production program feasible for firm j. $((x_i), (y_j))$ is feasible if

$$\sum_{i=1}^I x_i^n = \sum_{i=1}^I \sigma^n \omega_i + \sum_{j=1}^J (y_{j0}^n + y_{j1}^{n-1}), \text{ for all } n \geqq 0 .$$

A stationary allocation is denoted by $((x_i)_{i=1}^{I}, (y_{j0}, y_{j1})_{j=1}^{J})$,

where each x_i is a stationary consumption program and each (y_{j0}, y_{j1}) is a stationary production program. The allocation is feasible if $\sum_{i=1}^{I} x_i = \sum_{i=1}^{I} \omega_i + \sum_{j=1}^{J} (y_{j0} + \sigma^{-1} y_{j1})$.

The Assumption of Full Information

There is an assumption of full information implied by the definition of allocation. Presumably, agents have enough information to choose the bundles determined by an allocation $((x_i), (y_j))$. At time n, consumer i chooses $x_i^n(s)$, where $x_i = (x_n^0, x_n^1, \ldots)$, and firm j chooses y_{j0}^n, where $y_j = (y_{j1}^{-1}, (y_{j0}^0, y_{j1}^0), \ldots)$. x_i^n, y_{j0}^n, and y_{j1}^{n-1} are all measurable with respect to $\mathcal{S}_n$. That is, they depend on possibly all the values of s_k, for $k \leq n$. In order to make such choices, the agents should know s_k, for $k \leq n$. That is, they should have full information.

In stationary equilibrium, the information requirements are less exaggerated since prices reveal information. I discuss this matter in section I.4.

Pareto Optimality

A feasible allocation $((x_i), (y_j))$ is said to be Pareto optimal if there exists no feasible allocation $((\overline{x}_i), (\overline{y}_j))$ such that

$$\liminf_{N \to \infty} N^{-1} \sum_{n=0}^{N} \int [u_i(\overline{x}_i^n(s), s) - u_i(x_i^n(s), s)] \; P(ds) \geq 0, \text{ for all } i,$$

with strict inequality for some i. Observe that Pareto optimality is defined in terms of long-run average flows of expected utility.

Prices

I define only stationary price systems. A price system is simply a non-zero element, p, of $L^{+}_{1,L}(S,\mathcal{S}_0,P)$. The vector of prices at time n defined by p is $p(\sigma^n s) \in R^L_+$.

If $x \in \mathcal{L}_{\infty,L}(S,\mathcal{S}_n,P)$, then $p\cdot x$ is the long-term average financial flow determined by p and x. That is,

$p\cdot x = \lim_{N\to\infty} N^{-1}\sum_{n=1}^{N} p(\sigma^n s)\cdot x(\sigma^n s)$, almost surely. This fact follows from the strong law of large numbers. The strong law of large numbers spplies since $\{s_n\}$ is stationary and metrically transitive. (See Doob [19], p. 465.)

Profit Maximization

Given a stationary price system p, each firm chooses a stationary program so as to maximize its long-run average flow of profit. If the price system is p, the average flow of profit from the stationary program (y_0,y_1) is $p\cdot(y_0 + \sigma^{-1}y_1)$.

The set of stationary programs feasible for firm j is

$Y_j = \{(y_0,y_1) \in \mathcal{L}_{\infty,L}(S,\mathscr{S}_0,P) \times \mathcal{L}_{\infty,L_p}(S,\mathscr{S}_1,P) \mid y_0 \leqq 0$ and $y_1(s) \leqq g_j(y_0(s),s)$ almost surely$\}$. Therefore, the problem of firm j is

$$\max \{p\cdot(y_0 + \sigma^{-1}y_1) \mid (y_0,y_1) \in Y_j\} .$$

$\eta_j(p)$ denotes the set of solutions to this problem. $\pi_j(p)$ denotes the maximum profit flow. That is, $\pi_j(p) = p\cdot(y_0 + \sigma^{-1}y_1)$, where $(y_0, y_1) \in \eta_j(p)$.

It is easy to see that if $(y_0,y_1) \in \eta_j(p)$, then the infinite horizon program $(\sigma^{-1}y_., (y_0,y_1), (\sigma y_0,\sigma y_1),\ldots..)$ solves the problem

$$\max \{ \limsup_{N\to\infty} N^{-1} \sum_{n=1}^{N} \int p(\sigma^n s) \cdot (y_0^n(s) + y_1^{n-1}(s)) P(ds) \mid$$
$$(y_1^{-1}, (y_0^0, y_1^0),\ldots.) \text{ is a program feasible for firm } j \} .$$

Thus, a program in $\eta_j(p)$ maximizes firm j's long-run average flow of expected profit.

Utility Maximization

Given the stationary price system p , consumer i's stationary budget set is $\beta_i(p) = \{x \in \mathcal{L}_{\infty,L_c}(S,\mathscr{A}_0,P) \mid p\cdot x \leq p\cdot\omega_i + \sum_{j=1}^{J}\theta_{ij}\pi_j(p)\}$.

His maximization problem is

$$\max \{ \int u_i(x(s),s) \; P(ds) \mid x \in \beta_i(p)\} .$$

$\xi_i(p)$ denotes the set of solutions to this problem.

If $x \in \xi_i(p)$ (and u_i satisfies assumptions I.2.5 and I.2.6 below), then the program $(x, \sigma x, \sigma^2 x, \ldots.)$ solves the infinite horizon maximization problem.

$$\max \{\limsup_{N\to\infty} N^{-1} \sum_{n=1}^{N} \int u_{in}(x_n(s),s) \; P(ds) \mid x_n \in \mathcal{L}^{+}_{\infty,L_c}(S,\mathscr{A}_n,P), \text{for all } n,$$

$$\text{and } \limsup_{N\to\infty} N^{-1} \sum_{n=1}^{N} p(\sigma^n(s)\cdot(x_n(s) - \omega_i(\sigma^n s)) \leq 0, \text{ almost surely}\} .$$

That is, every $x \in \xi_i(p)$ maximizes the long-run average flow of utility, subject to a long-run average budget constraint. This fact is easy to prove. The proof uses Fatou's lemma (Halmos [27], p. 113) and the following fact. If $x \in \xi_i(p)$, then there exists a positive number λ_i , such that for

almost every s , $u_i(z,s) \leq u_i(x(s),s) + \lambda_i \; p(s)\cdot(z - x(s))$, for every $z \in R_+^{L_c}$. That is, consumer i's behavior satisfies the permanent income hypothesis.

Stationary Equilibrium

A stationary equilibrium for $\mathcal{E}$ consists of $((x_i), (y_{j0}, y_{j1}), p)$, where

I.1.1) $((x_i), (y_{j0}, y_{j1}))$ is a feasible stationary allocation,

I.1.2) p is a stationary price system,

I.1.3) $(y_{j0}, y_{j1}) \in \eta_j(p)$, for all j, and

I.1.4) $x_i \in \xi_j(p)$, for all i.

A stationary equilibrium with transfer payments consists of $((x_i), (y_{j0}, y_{j1}), p)$, where these satisfy conditions I.1.1 - I.1.3 above and the following condition.

I.1.5) For each i, x_i solves the problem

$$\max \left\{ \int u_i(x(s),s) \; P(ds) \;\middle|\; x \in \mathcal{L}^+_{\infty,L_c}(S,\mathcal{S}_n,P) \text{ and } p\cdot x \leqq p\cdot x_i \right\} .$$

The flow of transfer payments made by consumer i is

$$p\cdot\omega_i + \sum_{j=1}^{J} \theta_{ij}\,\pi_j(p) - p\cdot x_i$$

Associated with any stationary equilibrium with transfer payments are marginal utilities of money, $\lambda_1,\ldots,\lambda_I$. λ_i, the marginal utility of money of consumer i, is the multiplier associated with the budget constraint in I.1.5.

I.2) Assumptions

I list below all the assumptions I use. Some have already been mentioned.

The Stochastic Process

I.2.1) $\{s_n\}_{n=-\infty}^{\infty}$ is stationary and metrically transitive.

Non-Triviality

I.2.2) $L_c \neq 0$, $L_p \neq 0$ and $L_o \neq \emptyset$. $I \geqq 1$ and $J \geqq 1$.

Consumers

I.2.3) $\omega_i \in \mathscr{L}^{+}_{\infty, L_o}(S, \mathscr{A}_0, P)$, for all i.

I.2.4) For all i, $u_i : R_+^{L_c} \times S \to (-\infty, \infty)$ is measurable with respect to $\mathscr{B} \otimes \mathscr{A}_0$, where $\mathscr{B}$ is the Borel σ-field on $R_+^{L_c}$.

I.2.5) For each i, $u_i(x, \cdot) : S \to (-\infty, \infty)$ is integrable, for each $x \in R_+^{L_c}$.

I.2.6) For each i and every s, $u_i(\cdot, s) : R_+^{L_c} \to (-\infty, \infty)$ is continuous, concave and weakly monotone.

By weakly monotone, I mean that $u_i(x,s) > u_i(y,s)$ whenever $x \gg y$.

Firms

I.2.7) (Measurability) For each j, $g_j = R_-^L \times S \to R_+^{L_p}$ is measurable with respect to $\mathcal{B} \otimes \mathcal{S}_1$, where $\mathcal{B}$ is the Borel σ-field on R_-^L.

Assumptions I.2.8 - I.2.11 below hold for all j and s.

I.2.8) (Convexity) Each component of $g_j(y_0,s)$ is a continuous concave function of y_0.

I.2.9) (Free disposability) Each component of $g_j(y_0,s)$ is monotonely non-increasing with respect to y_0.

That is, if $y_0 \leqq \bar{y}_0$, then $g_j(y_0,s) \geqq g_j(\bar{y}_0,s)$.

I.2.10) (Constant Returns to Scale) $g_j(t\,y_0,s) = t\,g(y_0,s)$, for all $t \geqq 0$.

I.2.11) (Necessity of Primary Inputs) $g_j(y_0,s) = 0$, if $y_{0k} = 0$, for all $k \in L_o$.

Lipschitz Condition

I.2.12) There exists $K > 0$ such that for all s and j, $g_j(y_0,s)$ is a Lipschitz function of y_0 with constant K.

That is, $\| g_j(y_0,s) - g_j(\bar{y}_0,s) \| \leqq K \| y_0 - \bar{y}_0 \|$, for all j and s.

This is a technical assumption which is used to prove that production sets are closed. (It is used in the proof of lemma I.7.2.) It is intuitively reasonable, but excludes the convenient Cobb-Douglas production function.

Boundedness

The next assumption guarantees that the set of feasible stationary allocations for the economy is uniformly bounded.

I.2.13) There exists $g: R_{-}^{L} \to R_{+}^{Lp}$ such that $g_j(y_0, s) \leqq g(y_0)$, for all y_0 and s, and g satisfies assumptions I.2.8 - I.2.11.

Adequacy

The last two assumptions guarantee that no consumer would have a zero income in an equilibrium. I call these assumptions the adequacy assumptions. The need for adequacy is a well-known problem in equilibrium theory. The adequacy assumptions seem always to be the most awkward part of any equilibrium existence theorem. The theorem of this paper is no exception. Roughly speaking, my assumptions guarantee that any consumer's endowment makes possible a stationary production program for the whole economy which produces a positive amount of every good.

I.2.14) $\omega_i \ggg 0$, for all i .

This means that each consumer is endowed with every primary good.

Recall that $Y_j = \{(y_0, y_1) \in \mathcal{L}_{\infty,L}(S, \mathscr{A}_0, P) \times \mathcal{L}_{\infty,L_p}(S, \mathscr{A}_1, P) \mid$

$y_0 \leqq 0$ and $y_1(s) \leqq g_j(y_0(s), s)$ almost surely $\}$.

I.2.15) There exist $\omega \in \mathcal{L}^+_{\infty,L_o}(S, \mathscr{A}_0, P)$ and

$(y^n_{j0}, y^n_{j1}) \in Y_j$, for $j = 1,\ldots,J$ and $n = 1,\ldots,N$ such that

i) $\sum_{j=1}^{J} (y^n_{j0} + \sigma^{-1} y^{n-1}_{j1}) + \omega \geqq 0$, for $n = 1,\ldots,N$,

$\sum_{j=1}^{J} (y^1_{j0} + \sigma^{-1} y^N_{j1}) + \omega \geqq 0$ and

ii) there exists a positive number r such that

$\sum_{j=1}^{J} (y^1_{j0k}(s) + \sigma^{-1} y^N_{j1k}(s)) \geqq r$ almost surely, for all $k \in L_p$.

Strong Monotonicity

In one theorem, I make use of the following strengthening of the monotonicity of utility assumed in I.2.6.

I.2.16) For each i and every s , $u_i(\cdot, s) : R_+^{L_c} \to (-\infty, \infty)$ is strongly monotone.

By strongly monotone, I mean that $u_i(x, s) > u_i(y, s)$ whenever $x > y$.

I.3) Theorems

I assume that assumptions I.2.1 - I.2.15 apply.

I.3.1) Theorem There exists a stationary equilibrium.

I.3.2) Theorem Let $\lambda_1, \ldots, \lambda_I > 0$. There exists a stationary equilibrium with transfer payments such that for each i λ_i is the marginal utility of consumer i in the equilibrium.

I.3.3) Theorem The allocation of any stationary equilibrium with transfer payments is Pareto optimal.

I.3.4) Theorem Assume that the strong monotonicity assumption applies (I.2.16) . Then, any Pareto optimal stationary allocation is the allocation of a stationary equilibrium with transfer payments.

Theorem I.3.2 expresses better the intuitive notion of stationary equilibrium than does theorem I.3.1 . Stationary equilibrium is a way of visualizing what happens in a reasonably stable economy over a period of, say, one year. It is essential to this image that the marginal utility of money of each consumer be constant. It does not really matter whether consumers exactly balance their budgets.

The constancy of the marginal utility of money is, of course, an idealization. I think of each consumer as continually, but slowly, adjusting his marginal utility of money in his attempt to control his finances. It is the slowness that is important. The effect of theorem I.3.1 is simply to reassure us that indeed all consumers could simultaneously have just the right marginal utility of money.

I.4) Discussion of Assumptions

Information

I have already pointed out in section I.1 that I implicitly assume full information. However, a careful analysis of stationary equilibrium shows that agents do not necessarily need an exorbitant amount of information in stationary equilibrium.

Consider a consumer. If his utility function is strictly concave, than one can think of him as needing to know only current prices and the current state of his utility function. For let p be a stationary equilibrium price system and let λ_i be consumer i's marginal utility of money. If $u_i(\cdot,s)$ is a strictly concave function, then there is only one point $x(s) \in R_+^{L_c}$ satisfying $u_i(z,s) \leqq u_i(x(s),s) + \lambda_i p(s)\cdot(z - x(s))$, for all $z \in R_+^{L_c}$.

$x(s)$ is the demand of consumer i when state s occurs. Clearly, $x(s)$ is completely determined by $u_i(\cdot,s)$, $p(s)$ and λ_i . Of course, λ_i must be such that the consumer satisfies his long-term average budget constraint. If the consumer were to calculate this constraint, he would have to know the entire joint distribution of $u_i(\cdot,s)$ and $p(s)$. However, one can think of the consumer as having found the appropriate level of λ_i by trial and error.

A firm's problem is more difficult, for it must predict the future behavior of prices in making decisions. Let p be a stationary equilibrium price system and let $(y_0, y_1) \in \eta_j(p)$. Then for almost every s , $y_0(s)$ solves the problem.

$$\max_{y_0} [p(s)\cdot y_0 + E(\sigma^{-1}p(\cdot)\cdot g_j(y_0,\cdot) \mid \mathscr{I}_0)(s)]$$

In this expression, $E(\cdot \mid \mathscr{I}_0)$ denotes expectation conditioned on $\mathscr{I}_0$. (This concept is defined in Doob [19], pp. 15-20.) Thus, firm j needs to predict its own production function and the prices of its outputs. Presumably, the firm knows the probability distribution of its own production function. The problem is whether it would be able to predict the prices of its outputs. (Its prediction must be as good as that of anyone else in the economy.) This would be nearly the case in the following situation. Imagine that random variation is of two sorts, generalized (e.g., weather) and specific to individual agents (e.g., machine breakdowns). Suppose that all agents have the same information about generalized events and that the variation specific to an individual is statistically independent of all other forms of variation. Then, if there were a great many individual firms and consumers, the variation specific to an individual would have a negligible effect on prices. Only information about generalized events would be relevant for price prediction, and this information would be available to everybody.

Malinvaud [32] has made the distinction between variation specific to individuals and generalized variation. He pointed out its relevance to the theory of insurance and to the Pareto optimality of profit maximization.

Separability of Utility

A fundamental assumption is that utility is additively separable with respect to time. This assumption perhaps makes sense in the context of models of growth theory, where time periods are years or generations. It is hard to defend in the context I have in mind. If time periods are short, consumption in one period should affect appetites in immediately following periods. (This point was made to me forcefully by Christian von Weizsächer.) It makes more sense to suppose that the current flow of utility depends on current and past consumption. If this is so, a consumer might prefer a non-stationary program to any stationary one. I have not studied this problem in detail.

Ownership of Capital Equipment

It may seem strange that firms have no endowment, for I assume that capital goods are not producible and so are part of the economy's endowment. These goods are presumably owned by firms and cannot be exchanged, at least not in the short-run. However, it is possible to interpret my model in such a way that particular capital goods are assigned to particular firms.

Suppose that we wish to assign the bundle ω_j of capital goods to firm j , for each j . First of all, it is necessary to label these goods not only by their physical type but by the label j of the firm to which they are assigned. Thus, ω_j would belong to a subspace of the commodity space which pertains only to firm j . The production function of each firm must be so defined that it can use only capital goods with its own label. Finally, the bundle $\theta_{ij}\omega_j$ must be made part of the endowment of consumer i , for each i . That is, the endowment of each firm is assigned to the firm's owners. In stationary equilibrium, each firm would rent its capital equipment from its own stockholders. These rents may be thought of as the profits of the firm in the every day sense of the word. Since I assume constant returns to scale, the stockholders would receive no income from firms other than these rents.

Constant Returns to Scale

The assumption of constant returns to scale has the disturbing consequence that the scale of output of a firm is not determined by prices. However, if capital goods are fixed in the way I just described, then the output of each firm could indeed be completely determined by prices.

Intermediate Goods

Technologies have been defined as if all production processes required only one period. However, it is possible to represent multi-period production processes simply by appropriate labeling of commodities. For instance, if a wine must age, then a one period old wine would be a different commodity than a two period old wine. One period old wine would be transformed into two period old wine. This observation has often been made in the literature on intertemporal models.

Intermediate goods may be just as specific to a certain firm as are capital goods. Just as in the case of capital goods, this aspect of reality may be modeled by appropriate labeling of commodities and definition of production functions.

Production of Capital Goods

I assume that capital goods are an endowment, even though in reality they are continually being produced and replaced. It is perhaps appropriate to assume that the total stock of capital goods is fixed over a short period of time. It is not realistic to assume that none are produced. One could introduce an exogenous demand for capital goods. These goods would simply disappear, once produced. Their production would be financed by transfer payments from consumers. The stationary equilibrium prices would elicit the production of these goods. All this is easy to do. The only subtlety is that the demand for capital goods must be included in the adequacy assumption.

Another awkward part of my model is the sharp distinction made between capital goods, which are in fixed supply, and other goods used in production. In reality, of course, goods used in production exist in a spectrum of durability. There is no way of avoiding this difficulty, if one is to retain the simplification offered by the permanent income hypothesis.

Intertemporal Production

My representation of production is extremely simple in that output is entirely determined by inputs and the state of the world. I give firms no possibility of choosing among techniques. They cannot modify their production methods so as to produce more or less of various types of goods, given a fixed input. Moreover at time n, a firm is not able to trade off more output in some states s_{n+1} against less in other states, unless this trade off is affected by the choice of inputs. (In [38], Radner gives firms both types of choice.)

My production functions in fact represent techniques. It is possible to use my model in order to represent an economy in which firms have many techniques, as long as they have only finitely many. It is sufficient to label the techniques as distinct firms. These artificial firms would share the same stockholders and capital equipment.

Free Disposability

I assume free disposability and make a good deal of use of it.

(See, for instance, the proof of lemma I.7.15.) The assumption is contained in I.2.9 and in the definition of a feasible production program (section I.1). This assumption is particularly disturbing in a model representing short-run activity. It says that unused goods never get in the way. However, as always in equilibrium theory, free disposability could be dropped if sufficient monotonicity were postulated. Suppose that every good is always useful in consumption or can always be used to produce some good useful in consumption. Then, every good would have a positive price in stationary equilibrium and so would never be disposed of. Thus, free disposability need be assumed only hypothetically in the middle of proofs. For instance in proving existence of stationary equilibrium, one could use free disposability in order to obtain an equilibrium and then at the end show that the assumption was not necessary.

I.5) Review of the Literature

From a technical point of view, a stationary equilibrium is simply a general equilibrium in the usual sense, except that there are infinitely many commodities. This fact is made clear in the next section. Thus, stationary equilibrium is simply an interpretation of general equilibrium, just as the Arrow-Debreu model of equilibrium with contingent claims is an interpretation of general equilibrium. (The Arrow-Debreu model is described in [4] and in [17], chapter 7.)

General equilibrium with infinitely many commodities has been studied by Debreu [16], myself [9] and Stigum [48,49]. Debreu proved that equilibrium is Pareto optimal and that a Pareto optimum is a quasi-equilibrium. In [9], I proved that if the commodity space is $\mathcal{L}_\infty$, then an equilibrium exists with prices in $\mathcal{L}_1$. Stationary equilibrium may be viewed as a interpretation of the equilibrium in [9]. Stigum proved that equilibria exist and are Pareto optimal, and that a Pareto optimum is a quasi-equilibrium. However, he assumed that there are countably many commodities. In stationary equilibrium there are either finitely many or a continuum of commodities. Therefore, one cannot use Stigum's results to prove the existence of stationary equilibrium.

My work is related to much work on the interpretation and extension of equilibrium theory. In section I.4, I pointed out the relevance of Malinvaud's work [32]. My work is related to work on rational expectations and temporary equilibrium. Stationary equilibrium is, in fact, a form of temporary equilibrium with rational expectations. The first work on equilibrium with rational expectations is by Radner [37]. Grandmont [25] has surveyed the large literature on temporary equilibrium. In temporary equilibrium theory, it is assumed that consumers face a different budget constraint in every period. There is no permanent income hypothesis, so that equilibria are not necessarily Pareto optimal.

In Radner's rational expectations equilibrium in [37], different agents could have different information. One can imagine that those in ignorance could use prices to surmise the information learned by others. (In [37], Radner assumed that agents did not use prices in this way.) If agents do use prices to obtain information, then rational expectations equilibria may not exist, as examples by Green [26] and Kreps [31] show. Radner [40] and Allen [2] prove that equilibria exist generically. The literature on information revealing prices is large. A very incomplete list of references includes Futia [23], Jordan [30] and Radner [39].

The idea of rational expectations has appeared in many branches of economics. Shiller [45] surveys applications in macroeconomics. Muth [36] invented the concept of rational expectations.

There seems to be a prejudice within the economics profession that rational expectations and symmetric information imply optimality. This idea finds expression in growth theory, when optimal programs are characterized by separating prices (as in Dana [15] and Evstigneev [21]). The optimal programs can be interpreted as competitive equilibria for an economy with many identical consumers, who have rational expectations. The present paper seems to be the first in which the connection between Pareto optimality and rational expectations is made clear in a model with diverse consumers.

Similarly, the idea that consumers are to some extent self-insuring seems to be very old, though I have never seen it expressed formally.

I have already pointed out that a stationary equilibrium becomes a golden rule program if there are only one consumer and one firm. It is, of course, not at all appropriate to use equilibrium theory in order to study study the golden rule, for equilibrium theory is inherently much more difficult than the theory of the golden rule. However, the models used in the theory of stochastic growth models ressemble my own.

The first work on multi-sector stochastic growth models seems to be that of Radner [38]. Dana [15], Evstigneev [21] and Jeanjean [29] have also made contributions. Dana, Evstigneev and Jeanjean all addressed larger and more difficult issues that the existence of golden rule programs and their characterization using prices. They provedthat there exists an optimal infinite horizon program, starting from given initial conditions, and that this program converges to an optimal stationary program. Brock and Mirman [13] solved these same problems for a one sector model.

I.6) The Two Period Economy

In this section, I show that a stationary equilibrium for the economy $\mathcal{E}$ is equivalent to a general equilibrium in a two-period economy $\mathcal{E}_0$.

$\mathcal{E}_0$ is defined as follows. The commodity space is $\mathcal{L}_{\infty,L}(S,\mathcal{A}_0,P) \times \mathcal{L}_{\infty,L_p}(S,\mathcal{A}_1,P)$. The consumption set of each consumer is $\mathcal{L}^{+}_{\infty,L_c}(S,\mathcal{A}_0,P) \times \{0\} \subset \mathcal{L}_{\infty,L}(S,\mathcal{A}_0,P) \times \mathcal{L}_{\infty,L_p}(S,\mathcal{A}_1,P)$. The utility function of consumer i, $U_i^0 : \mathcal{L}^{+}_{\infty,L_c}(S,\mathcal{A}_0,P) \times \{0\} \to (-\infty,\infty)$, is defined by $U_i^0(x,0) = \int u_i(x(s),s)\ P(ds)$. The endowment of consumer i is $(\omega_i,0) \in \mathcal{L}_{\infty,L}(S,\mathcal{A}_0,P) \times \mathcal{L}_{\infty,L_p}(S,\mathcal{A}_1,P)$. The production possibility set of firm j is $Y_j = \{(y_0,y_1) \in \mathcal{L}_{\infty,L}(S,\mathcal{A}_0,P) \times \mathcal{L}_{\infty,L_p}(S,\mathcal{A}_1,P) \mid y_0 \leq 0$ and $y_1(s) \leq g_j(y_0(s),s)$ almost surely $\}$. An artifical production set Y_0 is introduced. $Y_0 = \{(y_0,y_1) \in \mathcal{L}_{\infty,L}(S,\mathcal{A}_0,P) \times \mathcal{L}_{\infty,L_p}(S,\mathcal{A}_1,P) \mid y_0 \leq -\sigma^{-1} y_1\}$. Y_0 represents the imaginary process of tranferring goods backwards or forwards through time to the same state of the world. I let $\theta_{i0} = I^{-1}$, for all i. In summary, $\mathcal{E}_0 = \{U_i^0, (\omega_i,0), Y_j, \theta_{ij} : i = 1,\dots I,\ j = 0,1,\dots,J\}$.

An allocation for $\mathcal{E}_0$ consists of $((x_i,0)_{i=1}^{I}, (y_{j0},y_{j1})_{j=0}^{J})$, where $(x_i,0) \in \mathcal{L}^{+}_{\infty,L_c}(S,\mathcal{A}_0,P) \times \{0\}$ and $(y_{j0},y_{j1}) \in Y_j$, for all

i and j . This allocation is feasible if $\sum_{i=1}^{I}(x_i, 0) = \sum_{i=1}^{I}(\omega_i, 0) + \sum_{j=0}^{J}(y_{j0}, y_{j1})$.

A price system for $\mathcal{E}_0$ consists of $(p_0, p_1) \in \mathcal{L}^+_{1,L}(S, \mathcal{S}_0, P) \times \mathcal{L}^+_{1,L_p}(S, \mathcal{S}_1, P)$, such that $(p_0, p_1) > 0$.

If (p_0, p_1) is a price system for $\mathcal{E}_0$, then the supply correspondence of firm j is $\eta^0_j(p_0, p_1) = \{(y_0, y_1) \in Y_j \mid p_0 \cdot y_0 + p_1 \cdot y_1 \geqq p_0 \cdot \bar{y}_0 + p_1 \cdot \bar{y}_1$, for all $(\bar{y}_0, \bar{y}_1) \in Y_j\}$. $\pi^0_j(p_0, p_1)$ denotes the maximum profit of firm j . That is, $\pi^0_j(p_0, p_1) = p_0 \cdot y_0 + p_1 \cdot y_1$, for $(y_0, y_1) \in \eta^0_j(p_0, p_1)$.

The budget set of consumer i is $\beta^0_i(p_0, p_1) = \{(x, 0) \in \mathcal{L}^+_{\infty, L_c}(S, \mathcal{S}_0, P) \times \{0\} \mid p_0 \cdot x \leqq p_0 \cdot \omega_i + \sum_{j=0}^{J} \theta_{ij} \pi^0_j(p_0, p_1)\}$. The demand correspondence of consumer i is $\xi^0_i(p_0, p_1) = \{(x, 0) \in \beta^0_i(p_0, p_1) \mid U^0_i(x, 0) \geq U^0_i(\bar{x}, 0)$, for all $(\bar{x}, 0) \in \beta^0_i(p_0, p_1)\}$.

An equilibrium for $\mathcal{E}_0$ consists of $((x_i, 0)_{i=1}^{I}\ (y_{j0}, y_{j1})_{j=0}^{J},\ (p_0, p_1))$, where

(i) $((x_i, 0), (y_{j0}, y_{j1}))$ is a feasible allocation for $\mathcal{E}_0$,

(ii) (p_0, p_1) is a price system for $\mathcal{E}_0$,

(iii) $(y_{j0}, y_{j1}) \in \eta^0_j(p_0, p_1)$, for $j = 0, \ldots, J$, and

(iv) $(x_i, 0) \in \xi^0_i(p_0, p_1)$, for $i = 1, \ldots, I$.

I now show that to every equilibrium for $\mathcal{E}_0$, there corresponds a stationary equilibrium for $\mathcal{E}$. It is easy to show that the reverse relation applies.

Let $((x_i,0)_{i=1}^{I}, (y_{j0},y_{j1})_{j=0}^{J}, (p_0,p_1))$ be an equilibrium for $\mathcal{E}_0$. I show that there exists $(\bar{y}_{j0},\bar{y}_{j1})_{j=1}^{J}$ such that $((x_i)_{i=1}^{I}, (\bar{y}_{j0},\bar{y}_{j1})_{j=1}^{J}, p_0)$ is a stationary equilibrium for $\mathcal{E}$.

Since $(y_{00},y_{01}) \in Y_0$, it follows that

$$-\sigma^{-1} y_{01} \geqq y_{00} \text{ .} \tag{I.6.1}$$

Since $(-\sigma^{-1}y_{01},y_{01}) \in Y_0$ and $0 = p_0 \cdot y_{00} + p_1 \cdot y_{01}$ = max $\{p_0 \cdot \hat{y}_0 + p \cdot \hat{y}_1 \mid (\hat{y}_0,\hat{y}_1) \in Y_0\}$, it follows from I.6.1 that

$$0 = p_0 \cdot (-\sigma^{-1} y_{01} - y_{00}) \text{ .} \tag{I.6.2}$$

Let $(\bar{y}_{00},\bar{y}_{01}) = (-\sigma^{-1}y_{01},y_{01})$. Clearly, $(\bar{y}_{00},\bar{y}_{01}) \in \eta_0^0 (p_0,p_1)$.

Let $(\bar{y}_{10},\bar{y}_{11}) = (y_{10} + \sigma^{-1}y_{01} + y_{00}, y_{11})$. I.6.1 implies that $(\bar{y}_{10},\bar{y}_{11}) \in Y_1$, since Y_1 satisfies free disposability (assumption I.2.9) . I.6.2 implies that $(\bar{y}_{10},\bar{y}_{11}) \in \eta_1^0 (p_0,p_1)$.

If $j > 1$, let $(\bar{y}_{j0},\bar{y}_{j1}) = (y_{j0},y_{j1})$.

I now show that $((x_i)_{i=1}^{I}, (\bar{y}_{j0},\bar{y}_{j1})_{j=1}^{J})$ is a feasible allocation for $\mathcal{E}_0$. Since $((x_i)_{i=1}^I, (y_{j0},y_{j1})_{j=0}^J)$ is a feasible allocation for $\mathcal{E}_0$, it follows that

$$\sum_{i=1}^I x_i = \sum_{i=1}^I \omega_i + \sum_{j=0}^J y_{j0} \quad \text{and} \quad 0 = \sum_{j=0}^J y_{j1}$$

From the definition of $(\overline{y}_{j0}, \overline{y}_{j1})$, it follows immediately that

$$\sum_{i=1}^I x_i = \sum_{i=1}^I \omega_i + \sum_{j=0}^J \overline{y}_{j0} \quad \text{and} \quad 0 = \sum_{j=0}^J \overline{y}_{j1} . \tag{I.6.3}$$

Also, by the definition of $(\overline{y}_{00},\overline{y}_{01})$,

$$\overline{y}_{00} = -\sigma^{-1} \overline{y}_{01} . \tag{I.6.4}$$

The second equality of (I.6.3) implies that $-\overline{y}_{01} = \sum_{j=1}^J \overline{y}_{j1}$. Using (I.6.4), I obtain $\overline{y}_{00} = \sum_{j=1}^J \sigma^{-1} \overline{y}_{j1}$. Hence by the first equation of (I.6.3), $\sum_{i=1}^I x_i = \sum_{i=1}^I \omega_i + \sum_{j=1}^J (\overline{y}_{j0} + \sigma^{-1} \overline{y}_{j1})$. That is, $((x_i)_{i=1}^I , (\overline{y}_{j0}, \overline{y}_{j1})_{j=1}^J)$ is a feasible allocation for $\mathcal{E}$.

I next show that $p_{1k} = \sigma p_{0k}$, for all $k \in L_p$. Suppose that for some $k \in L_p$, $P\{s \in S \mid p_{1k}(s) < p_{0k}(\sigma s) \} > 0$. Let $A = \{s \in S \mid p_{1k}(s) < p_{0k}(\sigma s) \}$ and let $(\hat{y}_0,\hat{y}_1) \in Y_0$ be defined by

$$\hat{y}_{1m}(s) = \begin{cases} -1 , \text{ if } s \in A \text{ and } m = k \\ 0, \text{ if } s \notin A \text{ or } m \neq k, \text{ and} \end{cases}$$

$\hat{y}_0 = -\sigma^{-1} \hat{y}_1$.

Then, $p_0 \cdot \hat{y}_0 + p_1 \cdot \hat{y}_1 = \int_A (p_{0k}(\sigma s) - p_{1k}(s))\, P(ds) > 0$.

Since Y_0 is a cone with apex zero and $(\bar{y}_0, \bar{y}_1) \in \eta_0^0(p_0, p_1)$,

$0 = p_0 \cdot \bar{y}_0 + p_1 \cdot \bar{y}_1 \geqq p_0 \cdot \hat{y}_0 + p_1 \cdot \hat{y}_1$. This contradiction shows that $p_{1k} \geqq \sigma p_{0k}$, for $k \in L_p$. A similar argument shows that $p_{1k} \leqq \sigma p_{0k}$, for $k \in L_p$. This proves that $p_{1k} = \sigma p_{0k}$, for $k \in L_p$.

It now follows that $(p_0, p_1) > 0$ implies $p_0 > 0$, so that p_0 is a price system for $\mathcal{E}$. Also, if $(y_0, y_1) \in Y_j$, then $p_0 \cdot y_0 + p_1 \cdot y_1 = p_0 \cdot (y_0 + \sigma^{-1} y_1)$. Hence, $(\bar{y}_{j0}, \bar{y}_{j1}) \in \eta_j^0(p_0, p_1)$ implies that $(\bar{y}_{j0}, \bar{y}_{j1}) \in \eta_j(p_0)$, for $j = 1, \ldots, J$. Similarly, $(x_i, 0) \in \xi_i^0(p_0, p_1)$ implies $x_i \in \xi_i(p_0)$, for all i . I have shown that $((x_i)_{i=1}^{I}, (y_{j0}, y_{j1})_{j=1}^{J}, p_0)$ satisfies conditions (I.1.1) - (I.1.4) and so is a stationary equilibrium for $\mathcal{E}$.

I.7) Lemmas

In this section, I prove several lemmas, which allow me to apply the results of [9] to the economy $\mathcal{E}_0$.

Closedness of Production Sets

Recall that $Y_j = \{(y_0, y_1) \in \mathcal{L}_{\infty,L}(S, \mathcal{A}_0, P) \times \mathcal{L}_{\infty,L_p}(S, \mathcal{A}_1, P) \mid y_0 \leqq 0$ and $y_1(s) \leqq g_j(y_0(s), s)$ almost surely $\}$, for $j = 1, \ldots, J$.

I.7.1) Lemma For $j = 1,\ldots,J$, Y_j is closed in the weak-star topology.

The weak-star topology is defined as follows. Observe that any element p of $\mathcal{L}_{1,L}(S,\mathcal{A}_n,P)$ is a linear functional on $\mathcal{L}_{\infty,L}(S,\mathcal{A}_n,P)$. The linear functional is defined by $x \longmapsto p \cdot x$, for $x \in \mathcal{L}_{\infty,L}(S,\mathcal{A}_n,P)$. The weak-star topology on $\mathcal{L}_{\infty,L}(S,\mathcal{A}_n,P)$ is the weakest topology such that each functional $p \in \mathcal{L}_{1,L}(S,\mathcal{A}_n,P)$ is continuous. The weak-star topology on $\mathcal{L}_{\infty,L}(S,\mathcal{A}_0,P) \times \mathcal{L}_{\infty,L_p}(S,\mathcal{A}_1,P)$ is the product of the weak-star topologies on each of the component spaces.

The proof of lemma I.7.1 depends on the following lemma.

I.7.2) Lemma Let $y_0 = S \to R^L_-$ be measurable with respect to $\mathcal{A}_0$. For each $k = 1,\ldots,L$, there exists an integrable function $h_k\colon S \to R^L$, which is measurable with respect to $\mathcal{A}_1$ and such that for almost every $s \in S$, $g_{jk}(z,s) \leqq g_{jk}(y_0(s),s) + h_k(s)\cdot(z - y_0(s))$, for all $z \in R^L_-$.

Proof. I apply the following measurable choice theorem of Aumann [7]. Let (T,μ) be a σ-finite measure space, let X be a standard measure space and let H be a measurable subset of $T \times X$ whose projection on T is all of T. Then, there is a measurable function $h\colon T \to X$ such that $(t,h(t)) \in H$, for almost all $t \in T$.

I apply this theorem with $T = S$, $X = R_-^L$ and $\mu = P$. The σ-field on S is $\mathscr{I}_1$. Any Euclidean space is a standard measure space, as is required by the theorem (see Aumann [7]). I let $H = \{ (s,x) \in S \times R_-^L \mid g_{jk}(z,s) \leqq g_{jk}(y_0(s),s) + x\cdot(z - y_0(s)),$ for all $z \in R_-^L \}$. The projection of H onto S is all of S , for by assumption I.2.8 , $g_{jk}(z,s)$ is a concave function of z , for all z . In order to apply Aumann's theorem, I must show that H is measurable with respect to $\mathscr{I}_1 \otimes \mathscr{B}$, where $\mathscr{B}$ is the Borel σ-field of R_-^L .

For each $z \in R_-^L$, let $H(z) = \{ (s,x) \in S \times R_-^L \mid g_{jk}(z,s) - g_{jk}(y_0(s),s) - x\cdot(z - y_0(s)) \leqq 0 \}$. $H(z)$ is clearly measurable with respect to $\mathscr{I}_1 \otimes \mathscr{B}$, being the inverse image of the interval $(-\infty,0]$ with respect to an $\mathscr{I}_1 \otimes \mathscr{B}$-measurable function. Also, $H = \bigcap\{ H(z) \mid z \in R_-^L$ has rational coordinates $\}$. Therefore, H is measurable.

Thus, I may apply Aumann's selection theorem to obtain $h : S \to R_-^L$, which is measurable with respect to $\mathscr{I}_1$ and such that $(s,h(s)) \in H$ almost surely. By the assumption of free disposability (I.2.9) , $h(s) \leqq 0$ almost surely. Let h_k be defined by $h_{kn}(s) = \min(h_n(s),K)$, for $n = 1,\dots,L$, where K is the Lipschitz constant of assumption I.2.12.

It follows from this assumption that $(s,h_k(s)) \in H$ almost surely. Clearly, h_k is $\mathscr{A}_1$-measurable. Since it is essentially bounded, it is integrable.

Q.E.D.

Proof of Lemma I.7.1 Let (y_0^α, y_1^α), $\alpha \in A$, be a net in Y_j or a generalized sequence, where A is a directed set. (See Dunford and Schwartz [20], p.26, for a definition of generalized sequence.) Suppose that (y_0^α, y_1^α) converges to $(y_0, y_1) \in \mathscr{L}_{\infty,L}(S,\mathscr{A}_0,P) \times \mathscr{L}_{\infty,L_p}(S,\mathscr{A}_1,P)$ in the weak-star topology. I must show that for each $k = 1,\dots,L$, $y_{1k}(s) \leqq g_{jk}(y_0(s),s)$ almost surely. It is sufficient to show that

$$\int_B y_{1k}(s)\ P(ds) \leqq \int_B g_{jk}(y_0(s),s)\ P(ds)\ , \quad \text{for every } B \in \mathscr{A}_1 . \qquad (I.7.3)$$

By the previous lemma, there exists $h_k : S \to R_-^L$, which is integrable and $\mathscr{A}_1$-measurable and such that for almost every s, $g_{jk}(z,s) \leqq g_{jk}(y_0(s),s) + h_k(s) \cdot (z - y_0(s))$. Substituting $y_0^\alpha(s)$ for z, I obtain $y_1^\alpha(s) \leqq g_{jk}(y_0^\alpha(s),s) \leqq g_{jk}(y_0(s),s) + h_k(s)\cdot(y_0^\alpha(s) - y_0(s))$. Integrating this inequality over B, I have $\int_B y_1{}^\alpha(s)\ P(ds)$ $\leqq \int_B g_{jk}(y_0(s),s)\ P(ds) + \int_B h_k(s) \cdot (y_0^\alpha(s) - y_0(s))\ P(ds)$.

Since $\lim_{\alpha} \int_B y_1^{\alpha}(s)\ P(ds) = \int_B y_1(s)\ P(ds)$, it is sufficient to prove that

$$\lim_{\alpha} \int_B h_k(s)\cdot(y_0^{\alpha}(s) - y_0(s))\ P(ds) = 0 \qquad \text{(I.7.4)}$$

Since y_0^{α} converges to y_0 in the weak-star topology of $\mathcal{L}_{\infty,L}(S,\mathscr{A}_0,P)$, (I.7.4) would be obvious if B belonged to $\mathscr{A}_0$ and h_k were $\mathscr{A}_0$-measurable. But $B \in \mathscr{A}_1$, and h_k is $\mathscr{A}_1$-measurable. Therefore, I must use conditional expectations.

Let $f\colon S \to R^L$ be defined by $f(s) = E(\chi_B(\cdot)h_k(\cdot) \mid \mathscr{A}_0)\ (s)$, where χ_B is the indicator function of B defined by

$$\chi_B(s) = \begin{cases} 1\ , \text{ if } s \in B \\ 0, \text{ otherwise .} \end{cases}$$

By the definition of weak-star convergence,

$\lim_{\alpha} \int f(s)\cdot(y_0^{\alpha}(s) - y_0(s)\ )\ P(ds) = 0$. But,

$$\int f(s)\cdot(y_0^{\alpha}(s) - y_0(s)\ P(ds) = E\ [\ E(\chi_B\ h_k \mid \mathscr{A}_0)\ \cdot\ (y_0^{\alpha} - y_0)]$$

$$= E\ [\ E(\chi_B\ h_k\ \cdot\ (y_0^{\alpha} - y_0) \mid \mathscr{A}_0)\ ] = \int h_k(s)\cdot\ (y_0^{\alpha}(s) - y_0(s))\ \ P(ds),$$

by the elementary properties of conditional expectations (see Doob [19], pp. 22 and 35) . This proves (I.7.4)

Q.E.D.

Recall that $Y_0 = \{(y_0,y_1) \in \mathscr{L}_{\infty,L}(S,\mathscr{I}_0,P) \times \mathscr{L}_{\infty,L_p}(S,\mathscr{I}_1,P) \mid$

$y_0 \leqq -\sigma^{-1}y_1\}$.

I.7.5) Lemma Y_0 is closed in the weak-star topology.

Proof Let (y_0^α,y_1^α) , $\alpha \in A$, be a convergent net in Y_0 with $\lim_\alpha (y_0^\alpha,y_1^\alpha) = (y_0,y_1)$. Because σ is measure preserving, it follows that $\sigma^{-1}y_1^\alpha$ converges to $\sigma^{-1}y_1$. Hence, $\lim_\alpha (-y_0^\alpha - \sigma^{-1}y_1^\alpha)$ $= -y_0 - \sigma^{-1}y_1$. Since $-y_0^\alpha - \sigma^{-1}y_0^\alpha \geqq 0$, for all α , it follows that $-y_0 - \sigma^{-1}y_1 \geqq 0$. That is, $(y_0,y_1) \in Y_0$.

Q.E.D.

Boundedness

The next lemma asserts that the feasible allocations of $\mathscr{E}_0$ are uniformly bounded. The proof of this lemma depends on assumption I.2.13. Before stating the lemma, I must define the appropriate norm. If $y \in R^L$, let $|y| = \max_k |y_k|$. If $y \in \mathscr{L}_{\infty,L}(S,\mathscr{I}_n,P)$, let $|y| = \text{ess sup } |y(s)| \equiv \inf \{ r>0 : |y(s)| \leqq r \text{ almost surely} \}$.

Let Y_j , for $j = 0,1,\ldots,J$ be as in the definition of $\mathscr{E}_0$.

I.7.6) Lemma Let $\omega \in \mathscr{L}_{\infty,L}(S,\mathscr{A}_0,P)$ and let $B = \{ (y_{j0},y_{j1})_{j=0}^{J} \mid (y_{j0},y_{j1}) \in Y_j$, for all j and $\sum_{j=0}^{J}(y_{j0},y_{j1}) + (\omega,0) \geqq 0\}$. Then, B is bounded with respect to the norm $|\cdot|$.

I prove lemma I.7.6 by means of a sequence of lemmas which follow. Let $Y = \{ (y_0,y_1) \in R^L \times R^{L_p} \mid y_0 \leqq 0$ and $y_1 \leqq g(y_0) \}$, where g is as in assumption I.2.13.

I.7.7) Lemma Let $t_0 > 0$ and $t_1 > 0$. Then, $A = \{(y_0,y_1) \in Y \mid -t_0 \leqq y_{0k}$, for all $k \in L_o$, $-t_0 \leqq y_{1k}$, for all k , and $|y_0| \leqq |y_1| + t_1 \}$ is bounded.

Proof If the lemma were false, there would exist a sequence $(y_0^n,y_1^n) \in A$ such that $\lim_{n\to\infty} |y_1^n| = \infty$. Let $(\bar{y}_0^n,\bar{y}_1^n) = |y_1^n|^{-1}(y_0^n,y_1^n)$. Then, $|(\bar{y}_0^n,\bar{y}_1^n)| \leqq 1 + |y_1^n|^{-1} t_1$. Hence, $(\bar{y}_0^n,\bar{y}_1^n)$ has a convergent subsequence, which I index by n again. Let $(\bar{y}_0,\bar{y}_1)$ be the limit. Since Y_0 is a closed cone with apex zero, $(\bar{y}_0,\bar{y}_1) \in A$. Also, $\bar{y}_{0k} \geqq 0$, for $k \in L_o$, and $\bar{y}_1 \geqq 0$ and $|\bar{y}_1| = 1$. Hence, $\bar{y}_1 > 0$ even though there are no primary inputs. This contradicts assumption I.2.11, as applied to g .

Q.E.D.

Let $\hat{Y} = \{ (y_0, y_1) \in \mathcal{L}_{\infty,L}(S,\mathscr{A}_0,P) \times \mathcal{L}_{\infty,L_p}(S,\mathscr{A}_1,P) \mid$ $y_0 \leqq 0$ and $y_1(s) \leqq g(y_0(s))$ almost surely$\}$. By assumption I.2.13, $Y_j \subset \hat{Y}$, for $j = 1,\ldots,J$.

I.7.8) Lemma Given $\omega \in \mathcal{L}_{\infty,L}(S,\mathscr{A}_0,P)$, the set $A = \{(y_0,y_1) \in \hat{Y} \mid$ $y_0 + \sigma^{-1}y_1 + \omega \geqq 0 \}$ is bounded with respect to the norm $|\cdot|$.

Proof By lemma I.7.7, there exists $T > 0$ such that $|y_1| \leqq T$ if $(y_0,y_1) \in Y$ satisfies the following conditions: $-|\omega| \leqq y_{0k}$, for all $k \in L_o$, $-|\omega| \leqq y_{1k}$, for all k , and $|y_0| \leqq |y_1| + |\omega| + 1$.

Let $(y_0,y_1) \in A$. Then, if $k \in L_o$,

$$-|\omega| \leqq -\omega_{ik}(s) = -\omega_{ik}(s) - \sigma^{-1}y_{1k}(s) \leqq y_{0k}(s) \text{ almost surely} \tag{I.7.9}$$

If $k \in L_p$, then

$$y_{1k}(s) \geqq -y_{0k}(\sigma s) - \omega_k(\sigma s) \geqq -\omega_k(\sigma s) \geqq -|\omega| \text{ almost surely.} \tag{I.7.10}$$

Let $\bar{s} \in S$ be such that $|y_1(\bar{s})| \geqq |y_1| - 1$ and the following are true. $0 \leqq y_0(\bar{s}) + y_1(\sigma^{-1}\bar{s}) + \omega(\bar{s})$; (I.7.9) and (I.7.10) apply to

$(y_0(\bar{s}), y_1(\bar{s}))$; $y_0(\bar{s}) \leqq 0$; and $y_1(\bar{s}) \leqq g(y_0(\bar{s}))$. The last two inequalities simply assert that

$$(y_0(\bar{s}),\ y_1(\bar{s})) \in Y . \tag{I.7.11}$$

Also, $0 \leqq -y_1(\bar{s}) \leqq y_1(\sigma^{-1}\bar{s}) + \omega(\bar{s})$, so that

$$\begin{aligned} |y_0(\bar{s})| &\leqq |y_1(\sigma^{-1}\bar{s})| + |\omega(\bar{s})| \leqq |y_1| + |\omega| \\ &\leqq |y_1(\bar{s})| + |\omega| + 1 . \end{aligned} \tag{I.7.12}$$

It follows from (I.7.9) - (I.7.12) that $|y_1(\bar{s})| \leqq T$, so that $|y_1| \leqq T+1$. Hence, $|y_0| \leqq |y_1| + |\omega| \leqq T + |\omega| + 1$.

Q.E.D.

Proof of lemma I.7.6

Let $(y_{j0}, y_{j1})_{j=0}^{J} \in B$. It follows easily that $\omega + \sum_{j=1}^{J}(y_{j0} + \sigma^{-1} y_{j1}) \geqq 0$ (The argument used here is similar to that just following equation I.6.4.) $(y_{j0}, y_{j1}) \in \hat{Y}$, for $j = 1, \ldots, J$, since $Y_j \subset \hat{Y}$, for such j . Since $\hat{Y}$ is a cone, $\sum_{j=1}^{J}(y_{j0}, y_{j1}) \in \hat{Y}$. Hence by lemma I.7.8,

$\{\sum_{j=1}^{J}(y_{j0}, y_{j1}) \mid (y_{j0}, y_{j1})_{j=0}^{J} \in B$, for some $(y_{00}, y_{01})\}$ is bounded.

Since $y_{j0} \leqq 0$, for $j = 1, \ldots, J$, it follows that $C \equiv \{(y_{j0})_{j=1}^{J} \mid (y_{j0}, y_{j1})_{j=0}^{J} \in B$, for some (y_{00}, y_{01}) and some $(y_{j1})_{j=1}^{J}\}$ is bounded.

Let $T_0 = \max\{\max_{j=1,\ldots,J} |y_{j0}| = (y_{j0})_{j=1}^{J} \in C\}$.

Observe that $(y_0, y_1) \in Y$ implies that $(y_0, y_1^+) \in Y$.

Hence by lemma I.7.7 with $t_0 = t_1 = T_0$, I obtain that $D \equiv \{(y_{j1}^+)_{j=1}^J \mid (y_{j0}, y_{j1})_{j=0}^J \in B$, for some (y_{00}, y_{01}) and some $(y_{j0})_{j=1}^J \}$ is bounded. Let $T_1 = \max\{\max_{j=1,\ldots,J} |y_j^+| : (y_{j1}^+)_{j=1}^J \in D\}$. Since $\sum_{j=1}^J (y_{j0} + \sigma^{-1} y_{j1}) + \omega \geqq 0$ and $y_{j0} \leqq 0$, for $j=1,\ldots,J$, it follows that $y_{j1k}(s) \geqq -(J-1)\, T_1$, for all k. This proves that $\{(y_{j0}, y_{j1})_{j=1}^J \mid (y_{j0}, y_{j1})_{j=0}^J \in B$, for some $(y_{00}, y_{01})\}$ is bounded.

If $(y_{j0}, y_{j1})_{j=0}^J \in B$, then $-\omega - \sum_{j=1}^J y_{j0} \leqq y_{00} \leqq -\sigma^{-1} y_{01} \leqq \sum_{j=1}^J \sigma^{-1} y_{j1}$. It follows at once that B is bounded.

Q.E.D.

Adequacy

The next lemma states that $\mathcal{E}_0$ satisfies what in [9] I called the "adequacy assumption."

I.7.13) Lemma For each i, there exist $(y_{j0}, y_{j1}) \in Y_j$, for $j = 0,1,\ldots,J$, such that $\sum_{j=0}^J (y_{j0}, y_{j1}) + (\omega_i, 0) \ggg 0$.

The proof of this lemma makes use of the next lemma.

I.7.14) Lemma There exist $\omega \in \mathcal{L}^+_{\infty,L_o}(S,\mathcal{J}_0,P)$ and $(y_{j0},y_{j1}) \in Y_j$,

for $j = 1,\ldots,J$, such that

i) $\sum_{j=1}^{J}(y_{j0} + \sigma^{-1}y_{j1}) + \omega \geqq 0$ and

ii) for some $r > 0$, $\sum_{j=1}^{J}(y_{j0k}(s) + \sigma^{-1}y_{j1k}(s)) \geqq r$ almost surely, for all $k \in L_p$.

Proof Let r , ω and (y^n_{j0},y^n_{j1}), for $j = 1,\ldots,J$ and $n = 1,\ldots,N$, be as in assumption I.2.15. Let $(y_{j0},y_{j1}) = N^{-1}\sum_{n=1}^{N}(y^n_{j0},y^n_{j1})$. Then, $\sum_{j=1}^{J}(y_{j0} + \sigma^{-1}y_{j1}) + \omega$

$$= N^{-1}\left[\sum_{n=2}^{N}\left(\sum_{j=1}^{J}(y^n_{j0} + \sigma^{-1}y^{n-1}_{j1}) + \omega\right) + (y^1_{j0} + \sigma^{-1}y^N_{j1}) + \omega\right] \geqq 0 ,$$

by assumption I.2.15 . Thus, (i) of the lemma is valid. For $k \in L_p$, $\sum_{j=1}^{J}(y_{j0k}(s) + \sigma^{-1}y_{j1k}(s)) \geqq N^{-1}\sum_{j=1}^{J}(y^1_{j0k}(s) + \sigma^{-1}y^N_{j1k}(s)) \geqq N^{-1}r$, which is condition (ii) of the lemma.

Q.E.D.

Proof of lemma I.7.13 Let $\bar{r}$, $\bar{\omega}$ and $(\bar{y}_{j0},\bar{y}_{j1}) \in Y_j$, for $j = 1,\ldots,J$, satisfy the conditions of lemma I.7.14 . By assumption I.2.14 , there exists $t < 0$ such that $t\bar{\omega}_k(s) + t \leqq \bar{\omega}_{ik}(s)$ almost surely, for all $k \in L_o$. Let $(y_{j0},y_{j1}) = t(\bar{y}_{j0},\bar{y}_{j1})$, for $j = 1,\ldots,J$.

By assumption I.2.10, $(y_{j0}, y_{j1}) \in Y_j$, for all j. Let (y_{00}, y_{01}) be defined by

$$(y_{00k}(s),\ y_{01k}(s)) = \left(\sum_{j=1}^{J} \sigma^{-1} y_{j1k}(s) - 2^{-1} t\, \bar{r},\ - \sum_{j=1}^{J} y_{j1k}(s) + 2^{-1} t\, \bar{r} \right),$$

for all s and for $k \in L_p$. Clearly, $(y_{00}, y_{01}) \in Y_0$.

I now show that $\sum_{j=1}^{J} (y_{j0}, y_{j1}) + (\omega_i, 0) \ggg 0$. Let $k \in L_p$. Then, $\sum_{j=0}^{J} y_{j1k}(s) = 2^{-1} t\, \bar{r} > 0$ and $\sum_{j=0}^{J} y_{j0k}(s) \geq \sum_{j=1}^{J} (y_{j0k}(s) + \sigma^{-1} y_{j1k}(s)) - 2^{-1} t\, \bar{r} \geq 2^{-1} t\, \bar{r} > 0$, almost surely. If $k \in L_o$, then $\sum_{j=0}^{J} y_{j0k}(s) + \omega_{ik}(s) = \sum_{j=1}^{J} y_{j0k}(s) + \omega_{ik}(s) \geq t(\sum_{j=1}^{J} y_{j0k}(s) + \bar{\omega}_k(s)) + t \geq t > 0$, almost surely. This proves that $\sum_{j=0}^{J} (y_{j0}, y_{j1}) + (\omega_i, 0) \ggg 0$.

Q.E.D.

The Exclusion Assumption

I now show that what I called the "exclusion assumption" in [9] applies to $\mathcal{E}_0$. In order to describe this assumption, I must introduce some new terminology.

$\mathcal{L}_\infty(S, \mathcal{A}_n, P)$ denotes $\mathcal{L}_{\infty,1}(S, \mathcal{A}_n, P)$. $ba(S, \mathcal{A}_n, P)$ denotes the set of bounded additive set functions defined on $\mathcal{A}_n$ which are absolutely continuous with respect to P. $ba(S, \mathcal{A}_n, P)$ is the set of linear functionals on $\mathcal{L}_\infty(S, \mathcal{A}_n, P)$ continuous with respect to the supremum norm $|\cdot|$.

If $\nu \in ba(S,\mathscr{A}_n,P)$ is non-negative, then ν is said to be purely finitely additive if $\varphi = 0$ whenever φ is a countably additive set function defined on $\mathscr{A}_n$ such that $0 \leqq \varphi \leqq \nu$. A theorem of Yosida and Hewitt asserts that if $\nu \in ba(S,\mathscr{A}_n,P)$ and $\nu \leqq 0$, then there exist $\nu_c \leqq 0$ and $\nu_p \geqq 0$ such that ν_c is countably additive, ν_p is purely finitely additive and $\nu = \nu_c + \nu_p$ ([51], p. 52, theorem 1.23). ν_c is termed the countably additive part of ν and ν_p is termed the purely finitely additive part of ν.

Another theorem of Yosida and Hewitt asserts that if $\nu_p \in ba(S,\mathscr{A}_n,P)$ is purely finitely additive and $\nu_c \in ba\ (S,\mathscr{A}_n,P)$ is countably additive, then there exists a sequence of sets $S_n \in \mathscr{A}_0$ such that $\nu_p(S \setminus S_n) = 0$, for all n, and $\lim_{n\to\infty} \nu_c(S_n) = 0$. ([51], p. 50, theorem 1.19).

$ba_L^+(S,\mathscr{A}_n,P)$ denotes $\{ (\nu_k)_{k=1}^L \mid \nu_k \in ba(S,\mathscr{A}_n,P)$ and $\nu_k \geqq 0$, for all $k \}$. If $\nu \in ba_L^+(S,\mathscr{A}_n,P)$, $\nu = \nu_c + \nu_p$ is the Yosida-Hewitt decomposition of ν. If $(\nu_0,\nu_1) \in ba_L^+(S,\mathscr{A}_0,P) \times ba_{L_p}^+(S,\mathscr{A}_1,P)$, then the decomposition $(\nu_0,\nu_1) = (\nu_{0c},\nu_{1c}) + (\nu_{0p},\nu_{1p})$ is defined similarly.

If $\nu \in ba_L(S,\mathcal{S}_n,P)$ and $A \in \mathcal{S}_n$, then $\nu(A) = (\nu_k(A))_{k=1}^{L}$.

Exclusion Assumption $Y \subset \mathcal{L}_{\infty,L}(S,\mathcal{S}_0,P) \times \mathcal{L}_{\infty,L_p}(S,\mathcal{S}_1,P)$ satisfies the exclusion assumption if for every $(\nu_0,\nu_1) \in ba_L^+(S,\mathcal{S}_0,P) \times ba_{L_p}^+(S,\mathcal{S}_1,P)$, there exist sequences $S_{0n} \in \mathcal{S}_0$ and $S_{1n} \in \mathcal{S}_1$, $n = 1,2,\ldots,$ such that

1) $\lim_{n\to\infty} \nu_{0c}(S_{0n}) = \lim_{n\to\infty} \nu_{1c}(S_{1n}) = 0$,

2) $\nu_{0p}(S \setminus S_{0n}) = \nu_{1p}(S \setminus S_{1n}) = 0$, for all n and

3) if $(y_0,y_1) \in Y$, then $(y_0 \chi_{S\setminus S_{0n}}, y_1 \chi_{S\setminus S_{1n}}) \in Y$, for all n.

In (3) above, $\chi_{S\setminus S_{in}}$ is the indicator function of the set $S\setminus S_{in}$, for $i = 0,1$. Also, $y_i \chi_{S\setminus S_{in}}$ is defined by $(y_i \chi_{S\setminus S_{in}})(s) = \chi_{S\setminus S_{in}}(s)\, y_i(s)$, for $i = 0,1$.

I.7.15) Lemma The production sets Y_j, $j = 0,1,\ldots,J$, of the economy $\mathcal{E}_0$ satisfy the exclusion assumption.

Proof Let $(\nu_0,\nu_1) \in ba_L^+(S,\mathcal{S}_0,P) \times ba_{L_p}^+(S,\mathcal{S}_1,P)$ and let $(\nu_0,\nu_1) = (\nu_{0c},\nu_{1c}) + (\nu_{0p},\nu_{1p})$ be the Yosida-Hewitt decomposition of (ν_0,ν_1).

For each $k = 1,\dots,L$, there exists a sequence $A_{0kn} \in \mathscr{A}_0, n = 1,2,\dots,$ such that $\nu_{0pk}(S \setminus A_{0kn}) = 0$ and $\lim_{n\to\infty} P(A_{0kn}) = 0$. Let $S_{0n} = \bigcup_{k=1}^{L} A_{0kn}$. Then, $\lim_{n\to\infty} P(A_{0kn}) = 0$ and so $\lim_{n\to\infty} \nu_{0c}(S_{0n}) = 0$. Clearly, $\nu_{0p}(S \setminus S_{0n}) = 0$.

For each $k \in L_p$, there exists a sequence $A_{1kn} \in \mathscr{A}_1, n = 1,2,\dots,$ such that $\nu_{1pk}(S \setminus A_{1kn}) = 0$ and $\lim_{n\to\infty} P(A_{1kn}) = 0$.

Now consider Y_j, for $j \geqq 1$. Let $S_{1n} = S_{0n} \cup \left(\bigcup_{k \in L_p} A_{1kn}\right)$. Then, $\lim_{n\to\infty} \nu_{1c}(S_{1n}) = 0$ and $\nu_{1p}(S \setminus S_{1n}) = 0$.

I must show that if $(y_0, y_1) \in Y_j$, then $(y_0 \chi_{S \setminus S_{0n}}, y_1 \chi_{S \setminus S_{1n}}) \in Y_j$, for all n. That is, I must show that $\chi_{S \setminus S_{1n}}(s)\ y_1(s) \leq g_j(\chi_{S \setminus S_{0n}}(s)\ y_0(s), s)$, almost surely. If $s \in S_{1n}$, $\chi_{S \setminus S_{1n}}(s)\ y_1(s) = 0 \leqq g_j(\chi_{S \setminus S_{0n}}(s)\ y_0(s), s)$, since g_j maps into $R_+^{L_p}$. If $s \notin S_{1n}$, then $\chi_{S \setminus S_{1n}}(s)\ y_1(s) = y_1(s) \leqq g_j(y_0(s), s) = g_j(\chi_{S \setminus S_{0n}}(s)\ y_0(s), s)$. This completes the proof that Y_j satisfies the exclusion assumption if $j \geqq 1$.

I now show that Y_0 satisfies the exclusion assumption. Let $S_{0n} = \left(\bigcup_{k=1}^{L} A_{0kn}\right) \cup \left(\bigcup_{k \in L_p} \sigma^{-1} A_{1kn}\right)$ and let $S_{1n} = \sigma S_{0n}$. Since

P is invariant with respect to σ, it follows that

$\lim_{n\to\infty} P(S_{0n}) = \lim_{n\to\infty} P(S_{1n}) = 0$. Hence, $\lim_{n\to\infty} \nu_{0c}(S_{0n}) = \lim_{n\to\infty} \nu_{1c}(S_{1n}) = 0$.

Also, $\nu_{0p}(S \setminus S_{0n}) = \nu_{1p}(S \setminus S_{1n}) = 0$, for all n. It should be clear from the definition of Y_0 that $(y_0 \chi_{S \setminus S_{0n}}, y_1 \chi_{S \setminus S_{1n}}) \in Y_0$, if $(y_0, y_1) \in Y_0$. Hence, Y_0 satisfies the exclusion assumption.

Q.E.D.

I.8) Proof of Theorem I.3.1

By what has been shown in section I.6., it is enough to prove that $\mathscr{E}_0$ has an equilibrium. In order to prove that this is so, I apply results from [9].

In [9], the commodity space is $\mathscr{L}_\infty(M,\mathscr{M},\mu)$, where $(M,\mathscr{M},\mu)$ is a σ-finite measure space. ($\mathscr{L}_\infty(M,\mathscr{M},\mu)$ is $\mathscr{L}_{\infty,1}(M,\mathscr{M},\mu)$, in the notation of this paper.) The first task is to interpret the commodity space of $\mathscr{E}_0$ as such a space. Recall that the commodity space of $\mathscr{E}_0$ is $\mathscr{L}_{\infty,L}(S,\mathscr{S}_0,P) \times \mathscr{L}_{\infty,L_p}(S,\mathscr{S}_1,P)$. Let $M = S \times (\{a_1,\ldots,a_L\} \cup L_p)$, where $a_1,\ldots,a_L$ are L distinct symbols, none of them included in L_p. The measurable subsets of $S \times \{a_k\}$, for $k = 1,\ldots,L$, are of the form $A \times \{k\}$, where $A \in \mathscr{S}_0$. Similarly, the measurable subsets of $S \times \{k\}$, for $k \notin L_p$, are of the form $A \times \{k\}$, where $A \in \mathcal{A}_1$.

μ is defined by $\mu(A \times \{a_k\}) = P(A)$ and $\mu(A \times \{k\}) = P(A)$.

It should be obvious how to identify $\mathcal{L}_\infty(M,\mathcal{M},\mu)$ with

$\mathcal{L}_{\infty,L}(S,\mathcal{A}_0,P) \times \mathcal{L}_{\infty,L_p}(S,\mathcal{A}_1,P)$.

A key assumption of [9] is that preference orderings are continuous with respect to the Mackey topology. I do not define this topology here. (It is defined in [9].) It is sufficient to point out that by appendix II of [9], the utility functions $U_i^0(x,0) = \int u_i(x(s),s)\, P(ds)$ are Mackey continuous.

Another key assumption of [9] is that production possibility sets are Mackey closed. The weak-star topology is weaker than the Mackey topology. Hence, lemmas I.7.1 and I.7.5 imply that the production possibility sets of $\mathcal{E}_0$ are Mackey closed.

In [9], I also assumed that production possibility sets are convex cones with apex zero. This is obviously true of Y_0 . That it is true of Y_j , for $j \geq 1$, follows from assumptions I.2.8 and I.2.10.

Another assumption of [9] is the "monotonicity assumption." In order that this assumption be satisfied, it is enough that

1) $Y_j - (\mathcal{L}^+_{\infty,L}(S,\mathcal{A}_0,P) \times \mathcal{L}^+_{\infty,L_p}(S,\mathcal{A}_1,P)) \subset Y_j$, for some j , and

2) $L_c \neq \emptyset$, and $U_i(x,0) > U_i(z,0)$, if $x >\!>\!> z$ and x and z both belong to $\mathcal{L}^+_{\infty,L_c}(S,\mathcal{A}_0,P)$.

(1) is true for all j , since I assume free disposability. That (2) is

true follows from assumptions I.2.2 and I.2.6 .

Lemma I.7.6 says that $\mathcal{E}_0$ satisfies the boundedness assumption of [9] . Lemmas I.7.13 and I.7.15 imply that the "adequacy" and "exclusion" assumptions of [9] are satisfied.

The assumptions I have mentioned include all the major assumptions of theorems 1 and 3 of [9]. It is easy to check that all the minor assumptions are satisfied. These theorems imply that $\mathcal{E}_0$ has an equilibrium.

Q.E.D.

I.9) Proof of Theorem I.3.2

I modify the economy $\mathcal{E}$ by replacing the I consumers by a single consumer with utility function $u : R_+^{L_c} \times S \to (-\infty,\infty)$, defined by $u(x,s) = \max \{\sum_{i=1}^{I} \lambda_i^{-1} u_i(x_i,s) \mid x_i \in R_+^{L_c}$, for all i , and $\sum_{i=1}^{I} x_i = x \}$. The endowment of the consumer is $\sum_{i=1}^{I} \omega_i$. Firms remain the same as in $\mathcal{E}$. Call the new economy $\hat{\mathcal{E}}$. It is easy to verify that $\hat{\mathcal{E}}$ satisfies all of assumptions I.2.1 - I.2.15 . Therefore by theorem I.3.1, $\hat{\mathcal{E}}$ has a stationary equilibrium , $(x,(y_{j0},y_{j1})_{j=1}^{J},p)$. I assume that p is so normalized that the marginal utility of money of the single consumer is one. By applying Aumann's measurable choice theorem [7] to $H = \{(s,x_1,\ldots,x_I) \in S \times (R_+^{L_c} \times \ldots \times R_+^{L_c}) \mid u(x(s),s) = \sum_{i=1}^{I} \lambda_i^{-1} u_i(x_i,s)\}$, one obtains

$x_i \in \mathcal{L}^+_{\infty,L_c}(S,\mathcal{S}_0,P)$, for $i = 1,\ldots,I$, such that

$u(x(s),s) = \sum_{i=1}^{I} \lambda_i^{-1} u_i(x_i(s),s)$ almost surely. It follows easily that $((x_i)_{i=1}^{I} , (y_{j0},y_{j1})_{j=1}^{J}, p)$ is a stationary equilibrium for $\mathcal{E}$ with transfer payments, which satisfies the conditions of the theorem.

Q.E.D.

I.10) Proof of Theorem I.3.3

The proof is nearly standard, except that I use the permanent income hypothesis.

Let $((x_i), (y_{j0},y_{j1}), p)$ be a stationary equilibrium with transfer payments and let λ_i be the marginal utility of money of consumer i in this equilibrium, for $i = 1,\ldots,I$. Let $((\bar{x}_i), (\bar{y}_j))$ be a feasible allocation which is not necessarily stationary. (Recall that $\bar{x}_i = (\bar{x}_i^0, \bar{x}_i^1, \ldots)$ and $\bar{y}_j = (\bar{y}_{j1}^{-1}, (\bar{y}_{j1}^0, (\bar{y}_{j0}^0, \bar{y}_{j1}^0), (\bar{y}_{j0}^1, \bar{y}_{j1}^1), \ldots).)$ Suppose that

$$\liminf_{N\to\infty} N^{-1} \sum_{n=0}^{N} [u_i(\bar{x}_i^n(s),s) - u_i(x_i(\sigma^n s),s)] P(ds) \geq 0 \text{ , for all } i \text{ ,}$$

with strict inequality for some i , say for $i = 1$. I derive a contradiction.

Let $\varepsilon > 0$ be such that $\liminf_{N\to\infty} N^{-1} \sum_{n=0}^{N} \int [u_1(\bar{x}_1^n(s),s) - u_1(x_1(\sigma^n s),s)] P(ds) \geq 3 \varepsilon \lambda_1$. Choose $\bar{N}$ so large that if $N \geqq \bar{N}$, then

$$N^{-1}\sum_{n=0}^{N}\int [u_1(\bar{x}_1^{\,n}(s),s) - u_1(x_1(\sigma^n s),s)]\,P(ds) \geqq 2\epsilon\lambda_1 ,$$

and

$$N^{-1}\sum_{n=0}^{N}\int [u_i(\bar{x}_i^{\,n}(s),s) - u_i(x_i(\sigma^n s),s)]\,P(ds) \geqq -I^{-1}\epsilon\lambda_i \ , \text{ for } i>1 .$$

Next observe that $u_i(\bar{x}_i^{\,n}(s),s) \leqq u_i(x_i(\sigma^n(s),s) + \lambda_i p(\sigma^n s)\cdot(\bar{x}_i^{\,n}(s)$ $- x_i(\sigma^n s))$, for all i . By combining these inequalities with the previous ones and adding the resulting inequalities, I obtain

$$\sum_{n=0}^{N}\int p(\sigma^n s)\cdot[\sum_{i=1}^{I}(\bar{x}_i^{\,n}(s) - x_i(\sigma^n s)]\,P(ds) \geqq \epsilon N, \text{ if } N \geqq \bar{N} . \qquad (I.10.1)$$

Since $((\bar{x}_i), (\bar{y}_j))$ and $((x_i), (y_{j0}, y_{j1}))$ are feasible allocations, I obtain $\sum_{n=1}^{I} \bar{x}_i^{\,n} = \sum_{i=1}^{I} \sigma^n\omega_i + \sum_{j=1}^{J} (\bar{y}_{j0}^{\,n} + \bar{y}_{j1}^{\,n-1})$, for all n , and $\sum_{i=1}^{I} x_i = \sum_{i=1}^{I} \omega_i + \sum_{j=1}^{J} (y_{j0} + \sigma^{-1} y_{j1})$. By substituting these expressions into (I.10.1) and rearranging, I obtain

$$\sum_{j=1}^{J}\{ p\cdot(\bar{y}_{j1}^{\,-1} - \sigma^{-1}y_{j1}) + p\cdot(\sigma^{-N}\bar{y}_{j0}^{\,N} - y_{j0}) + \sum_{n=0}^{N-1} [p\cdot(\sigma^{-n}\bar{y}_{j0}^{\,n} \qquad (I.10.2)$$

$$+ \sigma^{-(n+1)}\bar{y}_{j1}^{\,n}) - p\cdot(y_{j0} + \sigma^{-1}y_{j1})]\} \geqq \epsilon N, \text{ if } N \geqq \bar{N} .$$

Since $(y_{j0}, y_{j1}) \in \eta_j(p)$, it follows that $p\cdot(\sigma^{-n}\bar{y}_{j0}^{\,n} +$ $\sigma^{-(n+1)}\bar{y}_{j1}^{\,n+1}) \leqq p\cdot(y_{j0} + \sigma^{-1}y_{j1})$, for all j and n . Therefore, (I.10.2) implies that

$$\sum_{j=1}^{J} [p\cdot(\bar{y}_{j1}^{-1} - \sigma^{-1} y_{j1}) + p\cdot(\sigma^{-N}\, \bar{y}_{j0}^{N} - y_{j0})] \geqq \varepsilon N , \qquad \text{(I.10.3)}$$

for all $N \geqq \bar{N}$.

Since $(\sigma^{-N} y_{j0}^{N} , \sigma^{-N} y_{j1}^{N}) \in Y_j$, it follows that $y_{j0}^{N} \leqq 0$, for all j . Since $p > 0$, $p\cdot\sigma^{-N} y_{j0}^{N} \leqq 0$. Hence, I.9.4 becomes

$\sum_{j=1}^{J} p\cdot(y_{j1}^{-1} - \sigma^{-1} y_{j1} - y_{j0}) \geqq \varepsilon N$, for all $N \geqq \bar{N}$. That is,

$\sum_{j=1}^{J} p\cdot(y_{j1}^{-1} - \sigma^{-1} y_{j1} - y_{j0}) = \infty$, which is impossible. This contradiction proves the theorem.

Q.E.D.

I.11) Proof of Theorem I.3.4

I call an equilibrium for $\mathcal{E}_0$ a modified equilibrium if the prices belong to $ba_L^{+}(S,\mathcal{A}_0,P) \times ba_{L_p}^{+}(S,\mathcal{A}_0,P)$ rather than to $\mathcal{L}_{1,L}^{+}(S,\mathcal{A}_0,P) \times \mathcal{L}_{1,L_p}^{+}(S,\mathcal{A}_0,P)$. ($ba_L(S,\mathcal{A}_0,P) \times ba_{L_p}(S,\mathcal{A}_0,P)$ is the set of linear functionals on $\mathcal{L}_{\infty,L}(S,\mathcal{A}_0,P) \times \mathcal{L}_{\infty,L_p}(S,\mathcal{A}_1,P)$ which are continuous with respect to the supremum norm. See Dunford and Schwartz [20], p. 296, theorem 16.)

I first apply a standard separation argument to obtain prices for the economy $\mathcal{E}_0$. Strong monotonicity implies that I obtain a modified equilibrium for $\mathcal{E}_0$ with transfer payments. By a theorem in [9], I may assume that the prices belong to $\mathcal{L}_1$, so that I can drop the qualifier "modified." The argument of section I.6 then implies that I have a stationary equilibrium for $\mathcal{E}$.

Let $((\bar{x}_i), (\bar{y}_{j0}, \bar{y}_{j1}))$ be a Pareto optimal stationary allocation for $\mathcal{E}$. Let $\bar{y}_{01} = -\sum_{j=1}^{J} \bar{y}_{j1}$ and let $\bar{y}_{00} = -\sigma^{-1}\bar{y}_{01}$. It is easy to see that $((\bar{x}_i, 0), (\bar{y}_{j0}, \bar{y}_{j1})_{j=1}^{J})$ is a Pareto optimal allocation for $\mathcal{E}_0$.

Let $A = \{\sum_{i=1}^{I}(x_i, 0) \mid U_i^0(x_i, 0) > U_i^0(\bar{x}_i, 0)$, for all $i\}$, where U_i^0 is the utility function of consumer i in the economy $\mathcal{E}_0$. Let $B = \sum_{i=1}^{I}(\omega_i, 0) + \sum_{j=1}^{J} Y_j$. The usual argument says that A and B are convex and do not intersect. B contains $\sum_{i=1}^{I}(\omega_i, 0) - (\mathcal{L}^+_{\infty, L_p}(S, \mathcal{S}_0, P) \times \mathcal{L}^+_{\infty, L_p}(S, \mathcal{S}_1, P))$, and so has non-empty interior with respect to the supremum norm, $|\cdot|$. By the separation theorem (Dunford and Schwarz [20], p. 417, theorem 8) there exists a non-zero $(p_0, p_1) \in ba^+_L(S, \mathcal{S}_0, P) \times ba^+_{L_p}(S, \mathcal{S}_1, P)$ such that $(p_0, p_1)\cdot(a_0, a_1) \geqq (p_0, p_1)\cdot(b_0, b_1)$, for all $(a_0, a_1) \in A$ and $(b_0, b_1) \in B$. Clearly, $(p_0, p_1) > 0$.

Since I have assumed that utility functions are monotone, standard arguments prove (I.11.1) and (I.11.2) below. (See Debreu [16] or [17].)

$$(\bar{y}_{j0},\bar{y}_{j1}) \in \eta_j^0 (p_0,p_1) \text{ , for } j = 0,1\ldots,J \text{ ,} \tag{I.11.1}$$

where η_j^0 is the supply correspondence for firm j in the economy $\mathscr{E}_0$.

$$\text{For all } i \text{ , } p_0 \cdot x_i \geqq p_0 \cdot \bar{x}_i \text{ , whenever } U_i^0(x_i,0) > U_i^0(\bar{x}_i,0) \text{ .} \tag{I.11.2}$$

I now prove that

$$\text{for all } i \text{ , } p_0 \cdot x_i > p_0 \cdot \bar{x}_i \text{ , whenever } U_i^0(x_i,0) > U_i^0(\bar{x}_i,0) \text{ .} \tag{I.11.3}$$

A standard argument proves the following.

$$\text{For all } i \text{ , if } p_0 \cdot \bar{x}_i > 0 \text{ , then } p_0 \cdot x_i > p_0 \cdot \bar{x}_i \text{ whenever } U_i^0(x_i,0) > U_i^0(\bar{x}_i,0) \text{ .} \tag{I.11.4}$$

(See Debreu [17], "Remark" on p. 591.) I apply I.11.4 to prove I.II.3.

By lemma I.7.13, there exist $(y_{j0},y_{j1})_{j=0}^{J}$ such that $(y_{j0},y_{j1}) \in Y_j$, for all j , and $\sum_{i=1}^{I}(\omega_i,0) + \sum_{j=0}^{J}(y_{j0},y_{j1}) \ggg 0$. Since $(p_0,p_1) > 0$, it follows that

$(p_0,p_1) \cdot (\sum_{i=1}^{I}(\omega_i,0) + \sum_{j=0}^{J}(y_{j0},y_{j1})) > 0$. Since the allocation $((\bar{x}_i,0),\ (\bar{y}_{j0},\bar{y}_{j1})_{j=0}^{J})$ is feasible and $(\bar{y}_{j0},\bar{y}_{j1}) \in \eta_j^0(p_0,p_1)$, for all j , I obtain $(p_0,p_1) \cdot (\sum_i(\bar{x}_i,0)) = (p_0,p_1) \cdot (\sum_i(\omega_i,0) + \sum_{j=0}^{J}(\bar{y}_{j0},\bar{y}_{j1})) \geqq (p_0,p_1) \cdot (\sum_i(\omega_i,0) + \sum_{j=0}^{J}(y_{j0},y_{j1})) > 0$. Hence for some i , $p_0 \cdot \bar{x}_i > 0$, say for $i = 1$. By (I.11.4), (I.11.3) is true for $i = 1$.

Let $(p_0,p_1) = (p_{0c},p_{1c}) + (p_{0p},p_{1p})$ be the Yosida-Hewitt decomposition of (p_0,p_1) . (This is defined in the subsection of section I.7 dealing with the exclusion assumption.) By the Radon-Nikodyn theorem (Halmos [27], p. 128) , (P_{0c},P_{1c}) may be thought of as belonging to $\mathcal{L}_{1,L}(S,\mathcal{A}_0,P) \times \mathcal{L}_{1,L_p}(S,\mathcal{A}_1,P)$.

Because (I.11.3) is true for $i = 1$, it follows easily from the strong monotonicity assumption (I.2.16) that $p_{0c} \gg 0$. This in turn implies that if $\bar{x}_i > 0$, then $p_0 \cdot \bar{x}_i > 0$. Hence by I.II.4 , I.11.3 is true for all i such that $\bar{x}_i > 0$. If $\bar{x}_i = 0$, then $p_0 \cdot x_i > 0$ for any $x_i \in \mathcal{L}^{+}_{\infty,L_c}(S,\mathcal{A}_0,P)$ not equal to $\bar{x}_i$.

Hence, (I.11.3) is trivially true in this case. This proves (I.11.3) .

I.11.1 and I.11.3 together imply that $((\bar{x}_i,0), (\bar{y}_{j0}, \bar{y}_{j1})_{j=0}^{J}, (p_0,p_1))$ is a modified equilibrium for $\mathcal{E}_0$ with transfer payments. Theorem 3 of [9] implies that $((\bar{x}_i,0), (\bar{y}_{j0}, \bar{y}_{j1})_{j=0}^{J}, (p_{0c},p_{1c}))$ is an equilibrium for $\mathcal{E}_0$ with transfer payments. By the argument of section I.6, $((\overline{x}_i), (\overline{y}_{j0}, \overline{y}_{j1})_{j=1}^{J}, p_{0c})$ is a stationary equilibrium for $\mathcal{E}$ will transfer payments.

Q.E.D.

Part II: Price Stability

The proof that the permanent income hypothesis implies price stability is surprisingly simple. It turns out that market excess supply is the gradient of aggregate consumers' surplus. Therefore, the motion defined by the differential equation, $\frac{dp(t)}{dt}$ = excess demand, is a gradient process, which tends to minimize consumers' surplus. Consumers' surplus is a convex function of price, so that the gradient process converges.

In the next section, I describe the model. I list the assumptions and state the stability theorem in section II.2. In section II.3, I briefly discuss related literature. In section II.4, I sketch the idea of the proof. Sections II.5 and II.6 deal with technical facts needed in the formal proof, which is given in section II.7. The final section mentions a possible extension.

II.1 Definitions, Notation and the Model

The Economy

I deal with a pure trade economy with I consumers and L commodities. For $i = 1,\ldots,I$, $u_i = R^L_+ \rightarrow (-\infty,\infty)$ is the utility function of consumer i, and $\omega_i \in R^L_+$ is his endowment.

An allocation is a vector $(x_i)^I_{i=1}$ such that $x_i \in R^L_+$, for all i.

The allocation is feasible if $\sum_{i=1}^{I} (x_i - \omega_i) = 0$

Prices vary over int $R^L_+ = \{p \in R^L_+ \mid p >> 0\}$. Notice that prices are not normalized. The normalization is contained in the marginal utilities of money.

The marginal utility of money of each consumer is fixed. $\lambda_i > 0$ denotes that of consumer i, for each i.

The <u>short-run demand</u> of consumer i, given p, is $\xi_i(p)$, the set of solutions to the problem

$$\max \{u_i(x) - \lambda_i p \cdot x \mid x \in R^L_+\} . \qquad \text{II.1.1)}$$

$\xi_i(p)$ may be empty. I call the demand "short-run" because the marginal utility of money is fixed. Consumers do not necessarily satisfy their budget constraints. The budget constraint is assumed to be binding only in the long-run.

A <u>short-run equilibrium</u> is a vector $(p,(x_i))$, where

1) $p \in \text{int } R^L_+$,
2) (x_i) is a feasible allocation and
3) $x_i \in \xi_i(p)$, for all i.

Price Dynamics

Prices are a function of time. Time is denoted by t, and t varies over $(-\infty,\infty)$. $p(t)$ is the price vector at time t.

The fundamental differential equation determining the evolution of prices is the following.

$$\frac{dp(t)}{dt} = \sum_{i=1}^{I} (\xi_i(p(t)) - \omega_i). \qquad \text{II.1.2)}$$

Of course, in order that this equation makes sense, $\xi_i(p(t))$ must be well-defined, for all i. I will refer to the differential system II.1.2 as the <u>tâtonnement adjustment system</u>.

Stability

The <u>domain of definition of market demand</u> is $P = \{p \in \text{int } R^L_+ \mid \xi_i(p)$ is well-defined, for all $i\}$.

The dynamic system II.1.2 is said to be <u>globally stable</u> if the following are true.

i) There exists a unique short-term equilibrum price vector, $\bar{p}$. (II.1.3)

ii) If $p(0) = q \in P$, then $p(t)$ is defined for all $t > 0$ and $\lim_{t\to\infty} p(t) = \bar{p}$, where $p(t)$ is the solution to (I.1.2) with initial conditions $p(0) = q$.

II.2 <u>The Stability Theorem</u>

I make the following regularity assumptions.

II.2.1) (Differentiability) For each i, $u_i = R_+^L \to (-\infty,\infty)$ is twice continuously differentiable on all of its domain.

II.2.2) (Monotonicity) For each i, $Du_i(x) \gg 0$, for all x, where $Du_i(x)$ is the vector of first derivatives of u_i at x.

II.2.3) (Concavity) For each i, $D^2u_i(x)$ is negative definite, for all x, where $D^2u_i(x)$ is the matrix of second derivatives of u_i at x.

II.2.4) For all i, $\sum_{i=1}^{I} \omega_i \gg 0$.

II.2.5) <u>Theorem</u> If assumptions II.2.1 - II.2.4 apply, then the <u>tâtonnement</u> adjustment system (II.1.2) is globally stable.

<u>Remark</u> Christian von Weizsäcker has pointed out to me that this

theorem depends on the implicit assumption that utility is additively separable with respect to time. I have not investigated what happens if the current flow of utility depends on current and past consumption.

Interpretation

It is hard to reconcile the model of the stability theorem with that of stationary equilibrium. One possibility is to think of the price adjustment process discussed here as converging to an equilibrium in a time period which is short relative to the time period used in the model of stationary equilibrium. Thus, if the time periods of the stationary equilibrium model are weeks , then the price adjustment process discussed here should come to equilibrium in half a day. Another approach is to give up the assumption that prices in stationary equilibrium clear markets. One can imagine that prices adjust from period to period in reaction to the excess demand of the previous period. The stability result proved here leads one to suppose that such a lagged adjustment process would be stable in the sense that it would be consistent with a stationary probability distribution of prices. I have not explored this point of view in detail. In my opinion, it is not worthwhile to pursue this sort of thinking without first specifying who sets prices and to what end.

II.3) Review of the Literature

From an intuitive point of view, I simply show that intertemporal substitution leads to price stability. The idea that this should be so is old. For instance, Hicks mentions in Value and Capital [28], p. 249, that substitution across time tends to stabilize prices, as long as the elasticity of price expectations is less than one. (I owe this reference to Arrow.)

As far as I know, Hicks' idea has never been expressed rigorously.

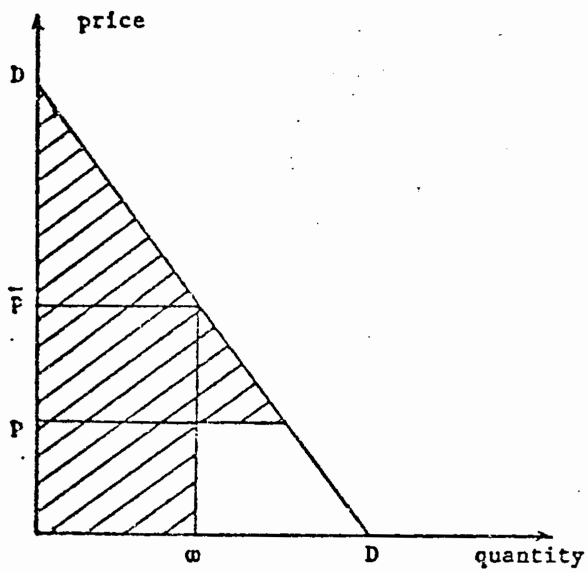
Most stability theory has been in terms of the usual excess demand functions of general equilibrium theory. That is, demand is defined by maximization of utility on a budget set. Several conditions have been found on aggregate excess demand functions which guarantee stability. This literature is explained in Arrow and Hahn [6], chapters 11 and 12.

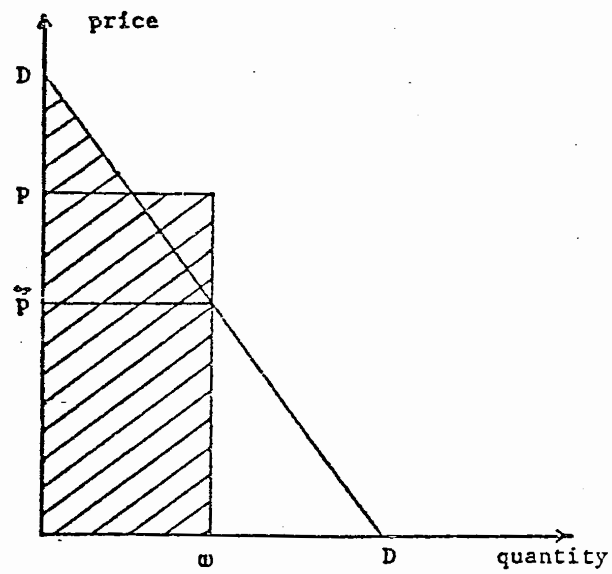
Another branch of literature has shown that there are probably no reasonable assumptions on individual utility functions or endowments which guarantee stability. I have already mentioned this work in the introduction. It includes the examples of Gale [24] and Scarf [43] and the results of Sonnenschein [46, 47], Mantel [33, 34], Debreu [18] and McFadden, et al. [35], which show that almost any excess demand function is possible.

It should be noticed that a short-run equilibrium is an equilibrium with transferable utility. The effect of the permanent income hypothesis is to make utility transferable. It is well-known that equilibria with transferable utility maximize a weighted sum of utilities over the set of feasible allocations. (See Shapley and Shubik [44] and Aumann and Shapley [8], pp. 184-6.) This idea is used in the proof of theorem II.2.1. Many people seem to be aware of the fact that equilibrium is globally stable when utility is transferable, though I have found no reference. The literature on economies with transferable utility deals with the relation between game theory and general equilibrium theory.

II.4 The Idea of the Proof

I have already mentioned that the *tâtonnement* system tends to minimize aggregate surplus. The surplus referred to is the sum of consumers' surplus, as it is usually defined, and the value of the initial endowment. It is easy to visualize what is going on in terms of usual supply and demand scissors diagram. Suppose that there is only one good. In the diagrams below, DD is the demand curve and ω is the fixed market supply. $\bar{p}$ is the equilibrium price vector. At the price p , the total surplus is the shaded area. This surplus is reduced by moving p toward $\bar{p}$ and it is minimized at $\bar{p}$, provided the demand curve slopes downward. The permanent income hypothesis implies that the shaded areas in the diagrams do represent the surplus and that the demand curve does slope downward.





II.5) Properties of Short-Run Demand

In this section, I discuss some elementary properties of short-run demand functions. Let $u: R^L_+ \to (-\infty,\infty)$ satisfy assumptions II.2.1 - II.2.3

and let ξ be the short-run demand function determined by u and some marginal utility of money, $\lambda > 0$. That is, for $p \in \text{int } R^L_+$, $\xi(p)$ is defined by II.1.1, with u and λ substituted for u_i and λ_i, respectively. Since u is strictly concave, ξ is a continuous function on its domain of definition. Because u is strictly monotone, $\xi(p)$ is not defined if some component of p is zero. It is not hard to show that

ξ is defined on a non-empty, open and convex subset of $\text{int } R^L_+$. (II.5.1)

Also,

if $\bar{p}$ belongs to the boundary of $\{p \mid \xi(p) \text{ is defined}\}$, (II.5.2)
then $\lim_{p \to \bar{p}} \|\xi(p)\| = \infty$, where the limit is through points p at which $\xi(p)$ is defined.

The usual theorem on the uniqueness of solutions to differential equations requires that the right hand side be Lipschitz. The only reason I assume that utility functions are twice differentiable is to obtain this Lipschitz property. So, I now prove that

ξ is Lipschitz. (II.5.3)

That is, for each compact set C in the domain of definition of ξ, there exists a positive number K such that $\|\xi(p) - \xi(q)\| \leq K\|p - q\|$, for all p and q in C.

Clearly, if $\xi(p)$ is defined, then $\frac{\partial u(\xi(p))}{\partial x_k} \leq \lambda p_k$, for $k = 1,\dots,L$, with equality if $\xi_k(p) > 0$. These inequalities define $\xi(p)$. For each subset $A \subset \{1,\dots,L\}$ and each $p \in \text{int } R^L_+$, let $f_A(p)$ be the unique vector, $y \in R_+^L$ (if one exists) such that

$$\frac{\partial u(y)}{\partial x_k} = \lambda p_k , \quad \text{for} \quad k \notin A \tag{II.5.4}$$

$$y_k = 0, \quad \text{if} \quad k \in A.$$

Since the matrix of second partial derivatives of u, D^2u, is negative definite, it follows that the matrix of derivatives of the left-side of (II.5.4) is non-singular. Hence by the implicit function theorem, f_A is continuously differentiable on its domain of definition. Since f_A is continuously differentiable, it is Lipschitz. (In order to apply the implicit function theorem, it is necessary to extend u to a twice differentiable function defined on an open set containing R_+^L.)

Clearly, ξ is equal, whenever it is defined, to one of the finitely many functions f_A. Also, the set on which ξ is equal to one of these functions is closed in the domain of definition of ξ. It follows at once that ξ is Lipschitz.

I now turn to a discussion of consumers' surplus. First of all, I need a definition. If h is a real-valued function defined on a convex subset of R^L, then $z \in R^L$ is said to be a <u>subgradient</u> of h at $p \in R^L$ if $z \cdot p - h(p) \geqq z \cdot q - h(q)$, for all q in the domain of h. That is, z is a subgradient of h at p, if the linear functional $(z, -1)$ reaches a maximum on the graph of h at the point $(p, h(p))$. Clearly, if h has a subgradient at every point in its domain, then h is a convex function. (Rockafellar discusses subgradients in [41] on p. 214 and following.)

I define the surplus of a consumer with utility function u and marginal utility of money λ to be $h(p) = \lambda^{-1}u(\xi(p)) - p \cdot \xi(p)$. The definition of ξ (II.1.1) implies that

$$-\xi(p) \text{ is a subgradient of } h \text{ at } p. \tag{II.5.5}$$

For by (I.1.1), $u(\xi(p)) - \lambda q \cdot \xi(p) \leq u(\xi(q)) - \lambda q \cdot \xi(q)$. Hence, $-\xi(p)' p - (\lambda^{-1}u(\xi(p)) - p \cdot \xi(p)) = -\lambda^{-1}u(\xi(p)) \geqq -\xi(p)\cdot q - (\lambda^{-1}u(\xi(q)) - q \cdot \xi(q))$, for all q in the domain definition of ξ. Since h has a subgradient at every point,

$$h \text{ is a convex function.} \tag{II.5.6}$$

II.6) Properties of Aggregate Consumers' Surplus

Let $u_1,\ldots,u_I$ and $\lambda_1,\ldots,\lambda_I$ be as in section II.1 and suppose that assumptions II.2.1 - II.2.4 apply. The aggregate surplus is defined to be $H(p) = \sum_{i=1}^{I} [\lambda_i^{-1} u_i(\xi_i(p)) - p\cdot \xi_i(p) + p\cdot \omega_i]$. $Z(p)$ denotes the market excess demand. That is, $Z(p) = \sum_{i=1}^{I} [\xi_i(p) - \omega_i]$.

(II.5.1), (II.5.5) and (II.5.6) imply the following.

Z and H are defined on an open convex set $P \subset \text{int } R^L_+$. (II.6.1)

H is a convex function and $-Z(p)$ is a subgradient of H at p, for every $p \in P$.

I now show that

there exists a unique $\bar{p} \in P$ at which $Z(\bar{p}) = 0$. (II.6.2)

$H(p)$ achieves a minimum at $\bar{p}$.

$\overline{p}$ is the unique short-term equilibrium vector.

The function $W(x_1,\ldots,x_I) = \sum_{i=1}^{I} \lambda_i^{-1} u_i(\overline{x}_i)$ is well-defined, since $\lambda_i > 0$, for all i. W may be thought of as a social welfare function defined on the set of feasible allocations for the economy. Since the set of feasible allocations is compact, W achieves a maximum, say at $(\overline{x}_i)$. Since W is strictly concave and the set of feasible allocations is convex, this maximum is unique.

Since $\sum_{i=1}^{I} \omega_i >> 0$, it follows that for each good k, there is a consumer, say consumer j, for whom $\overline{x}_{jk} > 0$. Let $\overline{p}_k = \lambda_j^{-1} \frac{\partial u_i(\overline{x}_i)}{\partial y_k}$, where $\frac{\partial u_i}{\partial y_k}$ is the $k^{\underline{th}}$ partial derivative of u_i. Clearly, $\xi_i(\overline{p}) = \overline{x}_i$, for all i, so that $Z(\overline{p}) = 0$. Since $-Z(\overline{p})$ is a subgradient of H at $\overline{p}$, H achieves a minimum at $\overline{p}$.

I now show that $\overline{p}$ is the only zero of Z. Let p be such that $Z(p) = 0$. Then, $(\xi_i(p))_{i=1}^{I}$ is a feasible allocation which satisfies the first order conditions for a maximum of W. Since W is strictly concave and twice continuously differentiable, $(\xi_i(p))$ is a constrained maximum of W. Hence, $(\xi_i(p)) = (\overline{x}_i)$. It follows at once that $p = \overline{p}$. This completes the proof of (II.6.2).

II.7) Proof of Theorem II.2.5

I have already proved that there exists a unique short-run equilibrium price vector, $\overline{p}$. I must now prove that solutions of the differential equation $\frac{dp(t)}{dt} = Z(p(t))$ converge to $\overline{p}$.

It follows immediately from (II.5.3) that Z is a Lipschitz function. Hence by the existence theorem for solutions to ordinary differential equations

([14], pp. 6,10 and 15), if $q \in P$, then there exists $\bar{t} \in (0,\infty]$ and a unique function $p = [0,\bar{t}) \to R_{+}^{L}$ such that $\frac{dp(t)}{dt} = Z(p(t))$, for all t, and

$$\bar{t} = \infty \text{ or } \{Z(p(t)) \mid \leqq t < \bar{t}\} \text{ is unbounded or} \tag{II.7.1}$$

$$\lim_{t \to \bar{t}} Z(p(t)) \text{ exists and belongs to the boundary of } P .$$

I now prove that $\bar{t} = \infty$. First of all, I show that

$$\|Z(p(t))\| \text{ is a non-increasing function of } t. \tag{II.7.2}$$

In order to prove II.7.2, it is enough to show that

$$\limsup_{\varepsilon \to 0} \varepsilon^{-1}[Z(p(t+\varepsilon)) \cdot Z(p(t+\varepsilon)) - Z(p(t)) \cdot Z(p(t))] \leqq 0. \tag{II.7.3}$$

First, observe that by the subgradient property of $-Z$,

$$-p(t) \cdot Z(p(t)) - H(p(t)) \geqq -p(t+\varepsilon) \cdot Z(p(t)) - H(p(t+\varepsilon)) \tag{II.7.4}$$

and

$$-p(t+\varepsilon) \cdot Z(p(t+\varepsilon)) - H(p(t+\varepsilon)) \geqq -p(t) \cdot Z(p(t+\varepsilon)) - H(p(t)). \tag{II.7.5}$$

Subtracting II.7.5 from II.7.4 and collecting terms, I obtain

$$\varepsilon^{-2}[p(t+\varepsilon) - p(t)] \cdot [Z(p(t+\varepsilon)) - Z(p(t))] \leqq 0. \tag{II.7.6}$$

This implies that

$$\varepsilon^{-1}[Z(p(t)) + O(\varepsilon)] \cdot [Z(p(t+\varepsilon)) - Z(p(t))] \leqq 0. \qquad \text{(II.7.7)}$$

where $\lim_{\varepsilon\to 0} O(\varepsilon) = 0$. Since Z is Lipschitz, the expression $\varepsilon^{-1}[Z(p(t+\varepsilon)) - Z(p(t))]$ is bounded, Hence, (II.7.7) implies that

$$\limsup_{\varepsilon\to 0} \varepsilon^{-1} Z(p(t)) \cdot [Z(p(t+\varepsilon)) - Z(p(t))] \leqq 0. \qquad \text{(II.7.8)}$$

Similarly since Z is Lipschitz, (II.7.6) implies that

$$\limsup_{\varepsilon\to 0} \varepsilon^{-1} Z(p(t+\varepsilon)) \cdot [Z(p(t+\varepsilon) - Z(p(t))] \leqq 0. \qquad \text{(II.7.9)}$$

Adding (II.7.8) and (II.7.9), one obtains (II.7.3). This completes the proof of (II.7.2).

(II.7.2) implies that $\bar{t} = \infty$. For, (II.7.2) implies that $\{Z(p(t)) \mid 0 \leqq t < \bar{t}\}$ is bounded. Also, if $\lim_{t\to\bar{t}} p(t)$ belongs to the boundary of P, then it follows from (II.5.2) that $\lim_{t\to\bar{t}} \|Z(p(t))\| = \infty$. Hence by (II.7.1), $\bar{t} = \infty$.

I next show that

$$\limsup_{\varepsilon\to 0} \varepsilon^{-1}[H(p(t+\varepsilon)) - H(p(t))] \leqq - \|Z(p(t))\|^2, \text{ for} \qquad \text{(II.7.10)}$$

all $t < 0$.

Observe that by II,7.5,

$$\varepsilon^{-1}[H(p(t+\varepsilon)) - H(p(t))] \leqq - \varepsilon^{-1} Z(p(t+\varepsilon)) \cdot [p(t+\varepsilon) - p(t)].$$

Since Z is a continuous function, (II.7.10) follows.

(II.7.2), (II.7.10) and the fact that H has a finite minimum imply that

$$\lim_{t\to\infty} \|Z(p(t))\| = 0. \tag{II.7.11}$$

Also, (II.7.10) implies that the point $\bar{p}$ at which $Z(\bar{p}) = 0$ is the unique point at which H achieves the minimum. For suppose that $q \neq \bar{p}$ and $H(q) = H(\bar{p})$. By II.6.2, $Z(q) \neq 0$. Hence, if $p(0) = q$, then by (II.7.10), $H(p(t)) < H(q)$, for $t > 0$, contrary to the hypothesis about q.

Because H is convex and achieves its minimum at a unique point, it follows that

$$\text{for every real number } r, \{p \mid H(p) \leq r\} \text{ is bounded.} \tag{II.7.12}$$

Hence, in order to prove that $\lim_{t\to\infty} p(t) = \bar{p}$, it is enough to show that $\bar{p}$ is the only cluster point of $p(t)$ as $t \to \infty$. Let q be any such cluster point. By II.7.11, $Z(q) = 0$, and so by II.6.2, $q = \bar{p}$.

This completes the proof that $\lim_{t\to\infty} p(t) = \bar{p}$ and hence the proof of II.2.5.

Q.E.D.

II.8) A Possible Extension

It may be possible to extend the stability theorem to the case in which the functions u_i are simply concave. More precisely, it may be possible to drop the assumptions that the u_i are differentiable (II.2.1) and that $D^2u_i(x)$ is always negative definite (II.2.3) and replace them by the assumptions that the u_i are continuous and concave. In this case, the functions Z and H are still well-defined, H is convex and Z is a subgradient of H. These facts give some hope that it may be possible to generalize the proof given above. In fact, it may be sufficient simply to employ the results and methods of Brézis [12]. He studies differential systems in which the right-hand side is not a function, but is a set of subgradients of a convex function. That is, the velocity vector is required only to be some subgradient. Solutions exist ([12], p. 54, theorem 3.1). They converge if a compactness condition is made, which is similar to (II.7.12) ([12],p.90, theorem 3.11). I owe knowledge of Brézis to Bernard Cornet.

Part III: Consumers' Surplus

The result proved in this section is probably almost self-evident to someone well-versed in the theory of consumers' surplus. There are, however, some slight technical difficulties in the proof.

Let $u_i = R_+^L \to (-\infty,\infty)$, λ_i and ξ_i, for $i = 1,\ldots,I$, be as in the previous section. Let $p(t)$, where $0 \leq t \leq 1$, be a continuously differentiable path of prices in $\text{int } R_+^L$, such that $\xi_i(t)$ is well-defined for all t and i. The measured change in consumers' surplus along this path is $-\int_0^1 \sum_{i=1}^I \xi_i(p(t)) \cdot p'(t)dt$, where $p'(t)$ denotes the derivative of the function p.

III.1) Theorem Assume that each u_i satisfies assumptions II.2.1 - II.2.3. Then, $-\int_0^1 \sum_{i=1}^I \xi_i(p(t)) \cdot p'(t)\,dt + \sum_{i=1}^I [p(1) \cdot \xi_i(p(1)) - p(0) \cdot \xi_i(p(0))] = \sum_{i=1}^I \lambda_i^{-1}[u_i(\xi_i(p(1)) - u_i(\xi_i(p(0)))]$

Proof First of all, assume that $\xi_i(p(t))$ is piecewise differentiable as a function of t. Then for each $i = 1,\ldots,I$,

$$-\int_0^1 \xi_i(p(t)) \cdot p'(t)dt + [p(1) \cdot \xi_i(p(1)) - p(0) \cdot \xi_i(p(0))]$$

$$= \int_0^1 p(t) \cdot \frac{d}{dt} \xi_i(p(t))dt$$

$$= \int_0^1 \lambda_i^{-1} Du_i(\xi_i(p(t)) \cdot \frac{d}{dt} \xi_i(p(t))dt$$

$$= \lambda_i^{-1} \int_0^1 \frac{d}{dt} u_i(\xi_i(p(t))dt = \lambda_i^{-1}[u_i(\xi_i(p(1)) - u_i(\xi_i(p(0))].$$

The first inequality above follows from integration by parts. The second equality follows from the fact that $\frac{\partial u_i(\xi_i(p(t))}{\partial x_k} \leqq \lambda_i p_k(t)$, for $k = 1,\ldots,L$, with equality if $\xi_{ik}(p(t)) > 0$. Hence,

$\frac{d}{dt}\xi_{ik}(p(t)) = 0$ if $p_k(t) > \lambda_i^{-1}\frac{\partial u_i(\xi_i(p(t))}{\partial x_k}$. This proves the theorem if each function $\xi_i(p(t))$ is piecewise differentiable.

In order to complete the proof, it is enough to show that the path $p(t)$ may be approximated arbitrarily closely by paths $q(t)$ such that $\xi_i(q(t))$ is piecewise differentiable. q must be such that $q(0) = p(0)$, $q(1) = p(1)$, and $\sup_{0\leq t\leq 1} \|q'(t) - p'(t)\|$ is small. Since the theorem would be true for any such q, it would be true for p.

For each subset $A \subset \{1,\ldots,L\}$, let $f_{iA}(p)$ be defined by (II.5.4), with u_i substituted for u. Recall that f_{iA} is continuously differentiable, so that $f_{iA}(p(t))$ is continuously differentiable as a function of t. Hence, it is sufficient to perturb p so that the path $p(t)$ leaves each region of form $\{q \in \text{int } R_+^L \mid \xi_i(q) = f_{iA}(q)\}$ only finitely often. The boundary between two such regions is of the form $C(i,A,B) = \{q \in \text{int } R_+^L \mid \xi_{ik}(q) = 0$, if and only if $k \in A$, $\frac{\partial u_i(\xi_i(q)}{\partial y_k} = \lambda_i p_k$, if $k \in B$, and $\frac{\partial u_i(\xi_i(q))}{\partial y_k} > \lambda_i p_k$, if $k \notin B \}$, where A and B are subsets of $\{1,\ldots L\}$, $A \cap B \neq \emptyset$ and $A \cup B = \{1,\ldots,L\}$.

I will show that

$C(i,A,B)$ is contained in a submanifold of R^L of dimension less than L and is closed in the domain of definition ξ_i, for all i, A and B. (III.2)

It will then follow by a standard application of the transversality theorem that a perturbation of p exists which intersects each of the sets $C(i,A,B)$ only finitely often.

u_i has a twice continuously differentiable extension, $\hat{u}_i$, to an open set containing R^L_+. The existence of such an extension is, in fact, the usual definition of differentiability for a function defined on a closed set. Since assumption II.2.3 is an open condition, I may assume that it applies to $\hat{u}_i$. That is, I assume that $D^2\hat{u}_i(x)$ is everywhere negative definite.

Let $f_{i,A\setminus B}$ be defined as follows. For $q \in \text{int } R^L_+$, $\hat{f}_{i,A\setminus B}(q) = y$ is the solution (if it exists) of the following equations:

$$\frac{\partial \hat{u}_i(y)}{\partial x_k} = \lambda_i q_k, \text{ if } k \in B \text{ and } y_k = 0, \text{ if } k \notin B$$

Let $\hat{C}(i,A,B) = \{q \in \text{int } R^L_+ \mid \hat{f}_{i,A\setminus B,k}(q) = 0, \text{ for } k \in A \cap B\}$. $C(i,A,B) \subset \hat{C}(i,A,B)$.

I claim that

$$\hat{C}(i,A,B) \text{ is a submanifold of } R^L \text{ of dimension } L - |A \cap B| \qquad \text{III.3)}$$

and is closed in the domain of ξ_i,

where $|A \cap B|$ denotes the cardinality of $A \cap B$. Clearly, (III.3) implies (III.2).

Observe that $\hat{f}_{i,A\setminus B}$ maps into the linear space $V = \{x \in R^L | x_k = 0, \text{ for } k \notin B\}$ and that $\hat{C}(i,A,B)$ is the inverse image under $\hat{f}_{i,A\setminus B}$ of the linear subspace $\{x \in V \mid x_k = 0, \text{ if } k \in A \cap B\}$. Hence by the implicit function theorem, $\hat{C}(i,A,B)$ is a submanifold if the derivative of $\hat{f}_{i,A\setminus B}$ everywhere has full rank.

In order to see that $D\hat{f}_{i,A\setminus B}(q)$ has full rank, consider the $|B| \times |B|$ matrix

$$M = \left(\frac{\partial \hat{f}_{i,A\setminus B,j}(q)}{\partial x_k} \right), \text{ where } j \text{ and } k \text{ vary over } B.$$

It is easy to see that $M = \lambda_i N^{-1}$, where

$$N = \left(\frac{\partial^2 \hat{u}_i(\hat{f}_{i,A\setminus B}(q))}{\partial x_j \partial x_k} \right), \text{ where } j \text{ and } k \text{ vary over } B.$$

Since $D^2\hat{u}_i(y)$ is always negative definite, N^{-1} is negative definite and hence invertible. This proves that $D\hat{f}_{i,A\setminus B}(q)$ has full rank, and hence proves (III.3) and so (III.2).

I do not spell out how one uses the transversality theorem. (Transversality theory is described in Abraham and Robbin [1], chapter 4.) The idea is as follows. Since the path $p(t)$ is of dimension 1, it can be perturbed so as to avoid all sets $\hat{C}(i,A\cap B)$ of dimension less than $L-1$, except at its endpoints. There are IL sets $\hat{C}(i,A\cap B)$ of dimension $L-1$. Order them and call them $\hat{C}(1),\ldots,\hat{C}(IL)$. Perturb $p(t)$ so that it intersects $\hat{C}(1)$ transversally (that is, so that it is never tangent to $\hat{C}(1)$). If the perturbation is small enough, $p(t)$ will still miss all the sets $\hat{C}(i,A,B)$ of dimension less than $L-1$. Now perturb $p(t)$ so that it intersects $\hat{C}(2)$ transversally. If the perturbation is small enough, $p(t)$ remains transverse to $\hat{C}(1)$ and still intersects no set of dimension less than $L-1$. Continuing inductively, one obtains a path $p(t)$ which intersects all the sets $\hat{C}(j)$ transversally and intersects no set of dimension less than $L-1$. Since it intersects each set $\hat{C}(j)$ transversally, it does so finitely often.

Q.E.D.

Consumers' surplus has been the subject of controversy for more than a century. Most of the controversy has centered around the fact that consumers' surplus is a doubtful measure of welfare when the marginal utility of money is variable. It is well-known that in the usual theory of consumer demand, the marginal utility of money is constant in only very special cases. (See Samuelson [42].) The history of consumers' surplus is reviewed briefly in Willig [50], p. 589, footnote 1. Willig himself gives estimable bounds for errors occurring when the marginal utility of money does vary.

Acknowledgement I worked out the connection between the permanent income hypothesis and consumers' surplus together with Hal Varian. Also, together we found that consumers' surplus was the potential function for the tâtonnement price adjustment process. I had earlier shown only that there existed some potential function.

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Part III: Consumers' Surplus

The result proved in this section is probably almost self-evident to someone well-versed in the theory of consumers' surplus. There are, however, some slight technical difficulties in the proof.

Let $u_i = R_+^L \to (-\infty,\infty)$, λ_i and ξ_i, for $i = 1,\ldots,I$, be as in the previous section. Let $p(t)$, where $0 \leq t \leq 1$, be a continuously differentiable path of prices in int R_+^L, such that $\xi_i(t)$ is well-defined for all t and i. The measured change in consumers' surplus along this path is $-\int_0^1 \sum_{i=1}^I \xi_i(p(t)) \cdot p'(t)dt$, where $p'(t)$ denotes the derivative of the function p.

III.1) Theorem Assume that each u_i satisfies assumptions II.2.1 - II.2.3. Then, $-\int_0^1 \sum_{i=1}^I \xi_i(p(t)) \cdot p'(t)\,dt + \sum_{i=1}^I [p(1) \cdot \xi_i(p(1)) - p(0) \cdot \xi_i(p(0))] = \sum_{i=1}^I \lambda_i^{-1}[u_i(\xi_i(p(1)) - u_i(\xi_i(p(0)))]$

Proof First of all, assume that $\xi_i(p(t))$ is piecewise differentiable as a function of t. Then for each $i = 1,\ldots,I$,

$$-\int_0^1 \xi_i(p(t)) \cdot p'(t)dt + [p(1) \cdot \xi_i(p(1)) - p(0) \cdot \xi_i(p(0))]$$

$$= \int_0^1 p(t) \cdot \frac{d}{dt} \xi_i(p(t))dt$$

$$= \int_0^1 \lambda_i^{-1} Du_i(\xi_i(p(t)) \cdot \frac{d}{dt} \xi_i(p(t))dt$$

$$= \lambda_i^{-1} \int_0^1 \frac{d}{dt} u_i(\xi_i(p(t))dt = \lambda_i^{-1}[u_i(\xi_i(p(1)) - u_i(\xi_i(p(0))].$$

The first inequality above follows from integration by parts. The second equality follows from the fact that $\frac{\partial u_i(\xi_i(p(t))}{\partial x_k} \leq \lambda_i p_k(t)$, for $k = 1,\ldots,L$, with equality if $\xi_{ik}(p(t)) > 0$. Hence, $\frac{d}{dt}\xi_{ik}(p(t)) = 0$ if $p_k(t) > \lambda_i^{-1} \frac{\partial u_i(\xi_i(p(t))}{\partial x_k}$. This proves the theorem if each function $\xi_i(p(t))$ is piecewise differentiable.

In order to complete the proof, it is enough to show that the path $p(t)$ may be approximated arbitrarily closely by paths $q(t)$ such that $\xi_i(q(t))$ is piecewise differentiable. q must be such that $q(0) = p(0)$, $q(1) = p(1)$, and $\sup_{0 \leq t \leq 1} \|q'(t) - p'(t)\|$ is small. Since the theorem would be true for any such q, it would be true for p.

For each subset $A \subset \{1,\ldots,L\}$, let $f_{iA}(p)$ be defined by (II.5.4), with u_i substituted for u. Recall that f_{iA} is continuously differentiable, so that $f_{iA}(p(t))$ is continuously differentiable as a function of t. Hence, it is sufficient to perturb p so that the path $p(t)$ leaves each region of form $\{q \in \text{int } R_+^L \mid \xi_i(q) = f_{iA}(q)\}$ only finitely often. The boundary between two such regions is of the form $C(i,A,B) = \{q \in \text{int } R_+^L \mid \xi_{ik}(q) = 0$, if and only if $k \in A$, $\frac{\partial u_i(\xi_i(q)}{\partial y_k} = \lambda_i p_k$, if $k \in B$, and $\frac{\partial u_i(\xi_i(q))}{\partial y_k} > \lambda_i p_k$, if $k \notin B$ $\}$, where A and B are subsets of $\{1,\ldots L\}$, $A \cap B \neq \emptyset$ and $A \cup B = \{1,\ldots,L\}$.

I will show that

$C(i,A,B)$ is contained in a submanifold of R^L of dimension less than L and is closed in the domain of definition ξ_i, for all i, A and B. (III.2)

It will then follow by a standard application of the transversality theorem that a perturbation of p exists which intersects each of the sets $C(i,A,B)$ only finitely often.

u_i has a twice continuously differentiable extension, $\hat{u}_i$, to an open set containing R^L_+. The existence of such an extension is, in fact, the usual definition of differentiability for a function defined on a closed set. Since assumption II.2.3 is an open condition, I may assume that it applies to $\hat{u}_i$. That is, I assume that $D^2\hat{u}_i(x)$ is everywhere negative definite.

Let $f_{i,A\setminus B}$ be defined as follows. For $q \in \text{int } R^L_+$, $\hat{f}_{i,A\setminus B}(q) = y$ is the solution (if it exists) of the following equations:

$$\frac{\partial \hat{u}_i(y)}{\partial x_k} = \lambda_i q_k, \quad \text{if } k \in B \text{ and } y_k = 0, \quad \text{if } k \notin B$$

Let $\hat{C}(i,A,B) = \{q \in \text{int } R^L_+ \mid \hat{f}_{i,A\setminus B,k}(q) = 0, \text{ for } k \in A \cap B\}$. $C(i,A,B) \subset \hat{C}(i,A,B)$.

I claim that

$$\hat{C}(i,A,B) \text{ is a submanifold of } R^L \text{ of dimension } L - |A \cap B| \qquad \text{III.3)}$$
$$\text{and is closed in the domain of } \xi_i,$$

where $|A\cap B|$ denotes the cardinality of $A \cap B$. Clearly, (III.3) implies (III.2).

Observe that $\hat{f}_{i,A\setminus B}$ maps into the linear space $V = \{x \in R^L | x_k = 0,$ for $k \notin B\}$ and that $\hat{C}(i,A,B)$ is the inverse image under $\hat{f}_{i,A\setminus B}$ of the linear subspace $\{x \in V \mid x_k = 0, \text{ if } k \in A\cap B\}$. Hence by the implicit function theorem, $\hat{C}(i,A,B)$ is a submanifold if the derivative of $\hat{f}_{i,A\setminus B}$ everywhere has full rank.

In order to see that $D\hat{f}_{i,A\setminus B}(q)$ has full rank, consider the $|B| \times |B|$ matrix

$$M = \left(\frac{\partial \hat{f}_{i,A\setminus B,j}(q)}{\partial x_k}\right), \quad \text{where } j \text{ and } k \text{ vary over } B.$$

It is easy to see that $M = \lambda_i N^{-1}$, where

$$N = \left(\frac{\partial^2 \hat{u}_i(\hat{f}_{i,A\setminus B}(q))}{\partial x_j \partial x_k}\right), \quad \text{where } j \text{ and } k \text{ vary over } B.$$

Since $D^2\hat{u}_i(y)$ is always negative definite, N^{-1} is negative definite and hence invertible. This proves that $D\hat{f}_{i,A\setminus B}(q)$ has full rank, and hence proves (III.3) and so (III.2).

I do not spell out how one uses the transversality theorem. (Transversality theory is described in Abraham and Robbin [1], chapter 4.) The idea is as follows. Since the path $p(t)$ is of dimension 1, it can be perturbed so as to avoid all sets $\hat{C}(i,A\cap B)$ of dimension less than $L-1$, except at its endpoints. There are IL sets $\hat{C}(i,A\cap B)$ of dimension $L-1$. Order them and call them $\hat{C}(1),\ldots,\hat{C}(IL)$. Perturb $p(t)$ so that it intersects $\hat{C}(1)$ transversally (that is, so that it is never tangent to $\hat{C}(1)$). If the perturbation is small enough, $p(t)$ will still miss all the sets $\hat{C}(i,A,B)$ of dimension less than $L-1$. Now perturb $p(t)$ so that it intersects $\hat{C}(2)$ transversally. If the perturbation is small enough, $p(t)$ remains transverse to $\hat{C}(1)$ and still intersects no set of dimension less than $L-1$. Continuing inductively, one obtains a path $p(t)$ which intersects all the sets $\hat{C}(j)$ transversally and intersects no set of dimension less than $L-1$. Since it intersects each set $\hat{C}(j)$ transversally, it does so finitely often.

Q.E.D.

Consumers' surplus has been the subject of controversy for more than a century. Most of the controversy has centered around the fact that consumers' surplus is a doubtful measure of welfare when the marginal utility of money is variable. It is well-known that in the usual theory of consumer demand, the marginal utility of money is constant in only very special cases. (See Samuelson [42].) The history of consumers' surplus is reviewed briefly in Willig [50], p. 589, footnote 1. Willig himself gives estimable bounds for errors occurring when the marginal utility of money does vary.

Acknowledgement I worked out the connection between the permanent income hypothesis and consumers' surplus together with Hal Varian. Also, together we found that consumers' surplus was the potential function for the *tâtonnement* price adjustment process. I had earlier shown only that there existed some potential function.

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