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STRATEGY-PROOF ALLOCATION MECHANISMS

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I. INTRODUCTION

Consider allocation mechanisms where each agents' strategy space is a set of a priori admissible utility functions. Such an allocation mechanism is strategy-proof if, for each agent, faithfully reporting his true utility function is a dominant strategy. The purpose of this paper is to characterize for the restricted domains associated with economic environments those strategy-proof allocation mechanisms that adjust marginally to small changes in agents' preferences. Our concern with the classical economic environment dictates a framework in which (a) the set of attainable alternatives is a subset of a ℓ-dimensional Euclidean space, (b) the domain of admissible preference n-tuples is restricted (utility functions may be required to satisfy such properties as continuity, monotonicity, and quasi-concavity), and (c) the standard representations of economies are admissible; in particular, the analysis applies to economies with and without production, with and without public goods, and with and without externalities.

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*Lemma 2 was reported by Satterthwaite [16]; it provided the starting point for the present analysis. We are greatly indebted to Donald Brown who suggested that the serial dictatorship of binary and Pareto Arrow social welfare functions (see Luce and Raiffa [10, p. 344]) might have a parallel in the present framework. This led to our formulation of our main result: Theorem 1. The proof of Lemma 3 was suggested by William Novshek, who pointed out that an early formal argument was inadequate. Carl Simon helped us to make the notation of a regular mechanism precise. Ehud Kalai pointed out that in the definition of regularity we could not assume the dimensionality of $B_j(u)$ to be ℓ-1. This committee was rounded out by Salvador Barbera, whose criticism and deep understanding helped to improve the exposition of the result.

Indeed, our goal has been to provide a result on strategy-proofness that is as basic for allocation mechanisms within economic environments as the Gibbard-Satterthwaite Theorem [4][15] is for voting procedures with unrestricted domain.

The transition from the social choice framework with its minimal mathematical structure on the attainable alternatives and admissible preferences to the economic environment with its considerable structure is usefully divided into two steps. For the first step, consider mechanisms that allocate public goods only and are regular. Consideration of mechanisms that also allocate private goods is deferred to the second step. Regularity means that the allocation changes smoothly as agents change their reported utility functions. We permit the set U of a priori admissible utility functions to be restricted to any (C^2) open set of utility functions. This requirement that U be open means that the mechanism is broadly applicable, which is to say that it must be defined for more than a "thin slice" of preferences such as those that are representable by additively separable or CES utility functions. With the addition of a few minor conditions, we prove that if an allocation mechanism is strategy-proof, regular, and allocates public goods only, then it is dictatorial. This corresponds precisely with the Gibbard-Satterthwaite Theorem that strategy-proofness in the social choice framework implies dictatorship.

For the second step, consider regular, broadly applicable mechanisms that allocate private goods. Whereas with "public goods only" each agent's utility evaluation of an allocation depends on every coordinate of the allocation vector (and broad applicability requires that the nature of this dependence be allowed to vary), with "private goods only" (and no externalities) each agent's evaluation of an allocation depends only on what he privately receives. The outcome of this step is fundamentally different than that of the first step where the validity of the Gibbard-Satterthwaite Theorem was confirmed for the public goods only economic environment. The following example demonstrates that the theorem fails and strategy-proof nondictatorial mechanisms do exist if the economy has private goods only.

There are two pure private goods, x and y, and three consumers. Commodity x is produced from y according to x = y. The economy begins with three units of y. Agent two and agent three share the first unit of y in proportions that depend on "the mean curvature" of one's utility function. If it is "very curved" two gets the unit of y and if it is "very flat," three gets the unit of y. The second unit of y is shared according to the rule obtained by replacing one by two, two by three, and three by one in the rule for sharing the first unit of y. Similarly, the third unit of y is shared by replacing one by three, two by one, and three by two in the first unit's rule. Each consumer is then assigned his utility maximal point on his budget line $x + y = \bar{y}$ where $\bar{y}$ is that share of the economy's initial endowment that he receives based on the mean curvatures of the other two agents' utility functions. This mechanism is strategy-proof because each agent's constraint set is exogenous to his own strategy and the mechanism automatically picks his utility maximal point on that set.[1]

[1] This mechanism is also efficient.

A striking feature of the preceding example is that the agent can maintain his bundle unchanged at the same time he causes changes in the bundles that the other agents receive. He does this by changing the mean curvature of his utility function while keeping its gradient constant at his current consumption bundle. We refer to mechanisms for which such action is possible as bossy and note that with public goods only (which means that everybody receives the identical bundle) bossiness is never possible. The above example shows that there exist bossy mechanisms that are strategy-proof and nondictatorial. Thus the Gibbard-Satterthwaite Theorem fails for private goods economies if bossy mechanisms are admitted.

If, however, bossy mechanisms are excluded from consideration, then we have been able to establish a Gibbard-Satterthwaite type theorem for private goods only and mixed public-private goods environments. It states that every nonbossy, regular mechanism that is strategy-proof and broadly applicable is a serial dictatorship. Serial dictatorship means that the mechanism consists of one or more hierarchies of dictators where the highest ranking agent in each hierarchy selects his allocation from a feasible set that is exogenously given, the second highest ranking agent selects his allocation from a feasible set that depends on the first agent's choice, the third highest ranking agent selects his allocation from a feasible set that depends on the first and second agents' choices, etc. With some minor conditions added, we are able to demonstrate that there is a single hierarchy.

Our work builds on and complements a long list of previous contributions. These are conveniently classified by whether they originated in the incentive compatibility literature or within the social choice literature. Samuelson in a classic paper [14] stated that in an economy with public goods it would be in an individual's interest to misrepresent his preferences. Hurwicz [8] showed that even in a standard (finite number of agents) private goods, perfectly competitive economy, an individual can gain by misrepresenting his preferences; thus, gain from misrepresentation does not by itself distinguish private goods economies from economies with public goods. In the same paper he proved that for two-person, two-good exchange economies there exists no strategy-proof mechanism that, (a) always generates Pareto optimal outcomes, (b) is individually rational, and (c) works for all economies in which agents have convex indifference curves. Green and Laffont [5] considered incentive compatibility within the context of an economy having one or more public goods and a single, private good. Within this specific context and under the strong restriction on the set U of admissible utility functions that each agents' utility be linear in the private good (i.e., utility is transferrable), they showed that every strategy-proof mechanism is necessarily a Groves mechanism (see Clarke [1], Groves [6], and Groves and Loeb [7]).

Gibbard [4] and Satterthwaite [15], for the case of unrestricted domain and in the context of the social choice literature, showed that no strategy-proof voting procedure exists that is nondictatorial and has a range of at least three alternatives. Kalai and Muller [9] and Maskin [11][12] have asked to what degree the set U of admissible utility functions must be reduced in order to obtain a possibility result instead of Gibbard and Satterthwaite's impossibility theorem. They independently derived necessary and sufficient conditions for the set U of admissible utility functions to admit the construction of nondictatorial strategy-proof mechanisms that are derivable from an Arrow social welfare function. Since requiring that economic allocation mechanisms be rationalizable by an Arrow social welfare function is unnaturally restrictive, these results are not satisfactory in the present context. Dasgupta, Hammond, and Maskin [2] used results derived in the context of social choice theory to show that for pure exchange economies there exists no allocation mechanism that is (a) nondictatorial, (b) always achieves Pareto optimality, and (c) works for all economies in which agents have strictly convex and strictly monotone preferences.

The relationship of our results to these preceding results has at least two aspects that deserve further comment. First, our results show that moving from the public goods only case to the inclusion of private goods changes the problem's nature and identifies the important role that bossiness plays in the possibility of constructing broadly applicable, strategy-proof mechanisms. Second, short of successfully deriving necessary and sufficient conditions for U to admit the construction of nondictatorial, strategy-proof allocation mechanisms, our requirement that a mechanism be broadly applicable appears to allow U to be restricted about as tightly as possible and still obtain an impossibility result, i.e. broadness appears to be akin to a necessary and sufficient condition. The evidence for this comes from two sides. On one side, Green and Laffont's possibility result depends crucially on their assumption of transferrable utility, an assumption that implies that their set U of admissible utility functions is not broadly applicable. On the other side, Hurwicz and Dasgupta, Hammond, and Maskin in their impossibility results require U to be larger than what broadness at its most stringent requires.

Three sections follow. Section II presents the model and formally states the theorems. Section III discusses the key assumptions. Section IV is devoted to proofs.

II. THE MODEL AND THEOREMS

For simplicity assume that all agents have the same consumption set $X \subset R^{\ell}$ $(\ell \geq 2)$, which is compact, convex, and has a non-empty interior. The class of admissible utility functions on X is denoted by U. We assume throughout that U is a convex subset of a linear function space and is endowed with a C^2 topology. The i^{th} agent is defined by his utility function $u_i \in U$. An allocation mechanism for an n agent economy is a function $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n): U^n \to X^n$. The allocation mechanism σ is manipulable by i at $u \in U^n$ if there exists $\bar{u}_i$ such that $u_i\sigma_i(u\backslash\bar{u}_i) > u_i\sigma_i(u)$, where $(u\backslash\bar{u}_i)$ denotes the vector u with i^{th} coordinate replaced by $\bar{u}_i$. The allocation mechanism σ is strategy-proof on U^n if it is not manipulable by any i at any $u \in U^n$. The allocation mechanism σ is broadly applicable (BA) if U is open.

The allocation mechanism σ is non-bossy (NB) if for all $u \in U^n$, for all i,j, and for all u_j^1, u_j^2, $[\sigma_j(u\backslash u_j^1) = \sigma_j(u\backslash u_j^2) \Rightarrow \sigma_i(u\backslash u_j^1) = \sigma_i(u\backslash u_j^2)]$. Let $C^2(X)$ denote all C^2 functions that are defined on X. The strategy-proof mechanism σ is regular at $u = (u_1, u_2, \ldots, u_n)$ if:

a. σ is continuously differentiable in u; in particular, for all $v \in [C^2(X)]^n$, the derivative $D_v\sigma(u) = \lim_{\lambda \to 0}[\sigma(u+\lambda v) - \sigma(u)]/\lambda$ exists with the standard properties that for all $c,d \in R$

and for all $v,w \in [C^2(X)]^n, D_{cv+dw} = cD_v\sigma(u)$ $+ dD_w\sigma(u)$.

b. For all i, $B_i(u) = \{x^i \in X | \text{a } u_i' \text{ exists with } x^i = \sigma_i(u\backslash u_i')\}$ is a m_i-dimensional, $0 \leq m(i) \leq \ell - 1$, smooth manifold in a neighborhood of $\sigma_i(u)$ and is continuously differentiable in u. Formally, there exists $f: X \times U^n \to R^{\ell-m(i)}$ such that f is C^2 in both variables and $B_i(u) = \{x \in X | f(x,u) = 0\}$.

c. For all i, $\sigma_i(u)$ is the unique and regular maximizer of u_i on $B_i(u)$.[3]

Let $R \subset U^n$ denote the set of all regular points. We speak of a regular mechanism to suggest the smoothness and nondegeneracy assumptions on σ that guarantee (a), (b), and (c). Whenever we refer to strategy-proof allocation mechanisms, we restrict attention to those for which the regular points $R \subset U^n$ form an open set.

Let (v_i) denote the vector in $[C^2(X)]^n$ that is zero in all coordinates except the i^{th} and is v_i in that coordinate. Agent i <u>affects</u> agent j at $u \in U^n$ if $i \neq j$ and $\exists\, v_i \in C^2(X)$ exists such that $D_{(v_i)}\sigma_j(u) \neq 0$.

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[3]Since σ is straightforward, $\sigma_i(u)$ maximizes u_i on $B_i(u)$. For $\sigma_i(u)$ to be a regular maximizer or u_i on $B_i(u)$ we must have (i), $\forall i$, the gradient of u_i evaluated at $\sigma_i(u)$ does not vanish and (ii) its relevant bordered Hessian does not vanish.

Agent i affects j's utility at $u \in U^n$ if $i \neq j$ and a $v_i \in C^2(X)$ exists such that $D_{(v_i)} u_j \sigma_j(u) \neq 0$. If i affects j at u, we write iA(u)j, and if i affects j's utility at u we write $i\tilde{A}(u)j$. Finally, define $S^{ij} = \{u \in R: iA(u)j\}$ and $\tilde{S}^{ij} = \{u \in R: i\tilde{A}(u)j\}$.

The following result establishes that if σ is strategy-proof and satisfies NB and BA, then for each regular point u ε R, the affects relation A is an acyclic relation. Since the sets S^{ij} are open (Lemma 1), it follows that, for each u ε R, there are a collection of serial dictatorships that are fixed in a neighborhood of u. The fact that serial dictatorship rather than dictatorship obtains is analogous to Luce and Raiffa's observation [10, p. 344] that a serially dictatorial social welfare function (where agents who are low on the "pecking order" get to determine the rank only of the alternatives those above them are indifferent among) is consistent with all of Arrow's conditions except nondictatorship.

Theorem 1. If σ be strategy-proof and satisfies NB and BA, then, for all u ε R, A(u) is acyclic.

We say that σ is everywhere total if A(u) is total for each u ε R.[4] For strategy-proof mechanisms that satisfy NB and BA, Theorem 2 asserts that if the set R of regular points is connected (an assumption that appears to hold for standard allocation mechanisms and is related to Debreu's result [3] that a finite number of equilibria is a generic property of standard economies) and if at each regular point one agent

[4] A relation Q on S is total if for all s, t ε S either sQt, tQs, or s=t.

in each pair can affect the other agent, then there is a single serial dictatorship. In particular, this means that one person always receives his most preferred bundle in a fixed constraint set.

> Theorem 2. Let σ be strategy-proof and satisfy NB and BA. If in addition $\mathcal{R}$ is connected and A is everywhere total, then there exists a permutation $Q:\{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$ such that, for all $u \in \mathcal{R}$, $iA(u)j$ if and only if $Q(i) > Q(j)$.

For economies with pure public goods only property NB is automatically satisfied and serial dictatorship reduces to dictatorship, so that for this case Theorem 2 reads as follows.

> Theorem 3. Let σ be strategy-proof and satisfy BA. In addition, let $\mathcal{R}$ be connected and A be everywhere total. If σ is defined for public goods only, then σ is dictatorial.

For σ to be defined for public goods only means that, for all i,j and for all $u \in U^n$, $\sigma^i(u) = \sigma^j(u)$. Dictatorship means formally that an agent i exists whose constraint set B_i is independent of $(u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_n)$; furthermore, for each $u \in R$, the allocation that all the agents collectively receive is i's most preferred point on B_i.

III. DISCUSSION OF THE MODEL

1. Our specification of the mechanism σ is flexible enough to accommodate pure exchange economies, pure public good economies, economies with production, and mixed public-private good economies. For the case of pure public goods economies, the n components of the mechanism $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ are constrained to be identical. For the case of a purely private good economy, no restrictions apply across σ's components except those that constrain the outcome to be feasible. For the case of mixed public and private goods, only the public goods components of the functions $\sigma_i = (\sigma_{i1}, \sigma_{i2}, \ldots, \sigma_{i\ell})$, i=1, 2,...,n, are restricted to be identical across agents.

2. Specification of each agents strategy space to be the set U of a priori admissible utility functions is not restrictive because we are concerned in this paper with dominant strategy mechanisms. This assertion follows from Gibbard's observation [4] that within a given environment a dominant strategy mechanism exists if and only if an equivalent strategy-proof mechanism with strategy space U^n exists.[5] Gibbard's observation means that because we are interested in dominant strategy mechanisms we can, as a matter of convenience, study those mechanisms that limit each agent's strategy space to a set U of admissible utility functions.

3. The requirement that a mechanism be broadly applicable follows from the observation that while preferences within an economic environment may have considerable a priori structure such as

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[5] This can be restated in Gibbard's terminology: a straightforward game form exists within a given environment if and only if a nonmanipulable voting scheme exists within the environment.

strict convexity, preferences are not naturally limited to any particular parametric form. Here, through our assumption on the openness of u, we assert that if u_i is an admissible utility function describing agent i's preferences, then all those utility functions u_i that are sufficiently near u_i (in the sense of the C^2 topology) should also be a priori admissible.

4. The example in the introduction illustrates that mechanisms exist for the allocation of private goods that are strategy-proof, but not dictatorial. Because of this, we introduced the notion of bossiness and showed that a Gibbard-Satterthwaite type theorem obtains when attention is restricted to nonbossy mechanisms. It is therefore appropriate to consider whether nonbossiness is a reasonable or desirable condition to require of mechanisms.[6]

While we have not exhaustively considered this question, we have identified some considerations that bear on nonbossiness's reasonableness and desirability. The first and most intuitive reason for requiring nonbossiness is that it precludes agent i from moving agent j's allocation without incurring a real cost in the specific sense of an altered allocation for himself. Otherwise agent i would have the power to blackmail agent j, a power that might be considered normatively

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[6] As we have already noted, nonbossiness is trivially satisfied for the case of a pure public goods economy since a change in any one agent's allocation means by definition an identical change in every other agent's allocation. Thus, for example, nonbossiness is present in practically all of the literature on social choice theory.

unacceptable within an allocation mechanism.[7]

We now consider a second justification for nonbossiness, one that relates to simplicity of design. Most allocation mechanism, including the competitive mechanism, have equilibria that can simply and naturally be defined in terms of an adding up condition and some marginal equalities arising from the several agents' first order conditions. Such mechanisms might appropriately be called first order. They necessarily have the property that if several agents change their preferences but maintain their initial marginal rates of substitution at their initial allocations, then the initial equilibrium is retained unchanged because the changes in preference leave the adding up condition and marginal equalities intact. This, however, means that a bossy mechanism cannot be a first order mechanism.[8] Thus the simplicity of first order mechanisms can only be purchased at the cost of excluding bossy mechanisms from consideration.

One final comment concerning bossiness and nonbossiness is appropriate. Bossiness only has meaning within the context of pure private

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[7]If agent i's bossing of agent j could only take the form of moving j along his indifference curve, then i could not blackmail j. This situation of bossiness without blackmail, however, is exceptional in the sense that if agent j could transform himself into agent j' by changing his preferences an arbitrarily small amount in an appropriate manner, then agent i when he bossed agent j' would affect the utility of j', i.e. agent i would have the power to blackmail agent j'. See Lemma 3 below for proof of this.

[8]The reason is simple. If a mechanism is first order, then the only way agent i can change agent j's allocation is to change his marginal rate of substitution, which for first order mechanisms means his own allocation also changes. But bossiness requires that i be able to change j's allocation without changing his own allocation. Therefore bossiness is incompatible with being a first order mechanism.

goods where each agent cares not at all about what allocations other agents receive. If agents are allowed to care even slightly about other agents' allocations, Theorem 3 applies, and nonbossiness is no longer needed as an assumption to establish that strategy-proofness implies serial dictatorships.

5. As Hurwicz [8] observed, the competitive mechanism is not a strategy-proof means for allocating private goods among a finite set of consumers. It is instructive to see how this obtains as an application of our theorems for arbitrary regular mechanism. Suppose, contrary to the assertion, that the competitive mechanism is strategy-proof for a finite set of consumers. At any regular point $u \in U^n$ the competitive mechanism is first order; therefore it is nonbossy. Theorem 1 consequently applies and states that the A relation must be acyclic. This, however, is not true for the competitive mechanism. Each agent faces a constraint set B_i (offer curve) that varies with every other agents' preferences. This means that each pair of agents reciprocally affects each other, which is to say that the A relationship is cyclic, not acyclic as the assumption that it is strategy-proof necessarily implies. Therefore the competitive mechanism is not strategy-proof.

6. The results do not apply to situations where an allocation mechanism is discontinuous. So-called pivotal mechanisms (see Vickery's [18] second price auction and Moulin's [13] analysis of strategy-proofness under single-peakedness) have discontinuities and one can interpret the present analysis as supporting the importance of discontinuity in the construction of satisfactory strategy-proof mechanisms.

7. Serial dictatorship is unattractive not only because it implies a very nonsymmetric distribution of power, but also because it generally generates allocations that are not Pareto optimal. To see this consider an economy in which there are at least two private goods and strictly convex production possibilities. Assume the mechanism σ is strategy-proof and serially dictatorial. The agent who is at the bottom of a hierarchy of the dictators faces a strictly convex feasible set because he gets to choose among the residual production possibilities after everyone else has chosen. **Bottom** ranking agents cannot affect the choices of the higher ranking agents; therefore the marginal rates of substitution (for private goods) of the higher ranking agents at their allocations may be taken as fixed and given relative to the strategies of the bottom ranking agents. Moreover, because σ generates optimal outcomes, the marginal rates of substitution of those higher ranking agents are all equal. Consequently, for optimality of the outcome to be preserved, the bottom ranking agent must choose that unique point on his strictly convex feasible set that results in him also having the same marginal rate of substitution for private goods. But generally his preferences will be such that he chooses a different point. This destroys optimality and contradicts the assumption that σ always generates optimal outcomes. Therefore serial dictatorship generally violates optimality. [9]

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[9] Strategy-proofness and Pareto optimality appear to be inconsistent requirements at a quite fundamental level. For example, Walker [19] has shown that even if utility is transferrable (i.e. broad applicability is violated), then Pareto optimality and strategy-proofness are inconsistent requirements to require of a mechanism.

IV. PROOFS

Lemma 1. Let σ be strategy-proof. Then, for all $i \neq j$, (i) $\tilde{S}^{ij} \subset S^{ij}$, (ii) S^{ij} is open, and (iii) $\tilde{S}^{ij}$ is open.

Proof. The first claim follows from the fact $i\tilde{A}(u)j$ implies $iA(u)j$. The latter two are a consequence of (a) in the definition of regular.

Lemma 2. Let σ be strategy-proof and satisfy NB and BA. Then, for all $i \neq j$, $\tilde{S}^{ij} \cap \tilde{S}^{ji} = \emptyset$.

Proof. We assume $1\tilde{A}(\bar{u})2$ at the regular utility n-tuple $\bar{u}$ and show that $2\tilde{A}(\bar{u})1$ is impossible. With the understanding that $\bar{u}_3, \bar{u}_4, \ldots, \bar{u}_n$ are fixed, we imagine that σ depends only on u_1 and u_2. The assumption $1\tilde{A}(\bar{u})2$ means there exists $v_1 \in C^2(X)$ such that $D_{(v_1)}\bar{u}_2\sigma_2(\bar{u}) > 0$. Let $v_2 \in C^2(X)$ be arbitrary. We will show $D_{(v_2)}\bar{u}_1\sigma_1(\bar{u}) \neq 0$ leads to a contradiction.

Define $B(\lambda) = B_2(\bar{u}_1 + \lambda v_1, \bar{u}_2)$.[10] By regularity, $B(\lambda)$ is a smooth manifold in a neighborhood of $\sigma_2(\bar{u})$, provided that λ is sufficiently small. By strategy-proofness and regularity, $\sigma_2(\bar{u}_1 + \lambda v_1, \bar{u}_2)$ is the unique maximizer of $\bar{u}_2$ on $B(\lambda)$ (again provided that λ is small) and by hypothesis $D_{(v_1)}\bar{u}_2\sigma_2(\bar{u}) > 0$. Similarly, for λ small enough, $\sigma_2(\bar{u}_1, \bar{u}_2 + \lambda v_2)$ is the unique maximizer of $(\bar{u}_2 + \lambda v_2) \in U$ on $B(0)$. Since σ satisfies BA, for any λ sufficiently small, there exists $w \in U$ such that $\sigma_2(\bar{u}_1, \bar{u}_2 + \lambda v_2)$

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[10] Throughout the proof the arguments $\bar{u}_3, \ldots, \bar{u}_n$ are suppressed because they remain constant.

is the unique maximizer of w on B(0) and $\sigma_2(\bar{u}_1 + \lambda v_1, \bar{u}_2)$ is the unique maximizer of w on $B(\lambda)$.[11]

Since σ is strategy-proof $\sigma_2(\bar{u}_1, w) = \sigma_2(\bar{u}_1, \bar{u}_2 + \lambda v_2)$ and $\sigma_2(\bar{u}_1 + \lambda v_1, w) = \sigma_2(\bar{u}_1 + \lambda v_1, \bar{u}_2)$. By NB $\bar{u}_1\sigma_1(\bar{u}_1, \bar{u}_2 + \lambda v_2) = \bar{u}_1\sigma_1(\bar{u}_1, w)$ and $\bar{u}_1\sigma_1(\bar{u}_1 + \lambda v_1, w) = \bar{u}_1\sigma_1(\bar{u}_1 + \lambda v_1, \bar{u}_2)$. But by strategy-proofness $\bar{u}_1\sigma_1(\bar{u}_1, w) \geq \bar{u}_1\sigma_1(\bar{u}_1 + \lambda v_1, w)$. Therefore

$$(1) \qquad \bar{u}_1\sigma_1(\bar{u}_1, \bar{u}_2 + \lambda v_2) \geq \bar{u}_1\sigma_1(\bar{u}_1 + \lambda v_1, \bar{u}_2)$$

for all λ sufficiently small. We may assume (if necessary replace v_2 with $-v_2$) that there exists a sequence $\lambda_n \to 0$ such that for all n

$$(2) \qquad \bar{u}_1\sigma_1(\bar{u}_1, \bar{u}_2) \geq \bar{u}_1\sigma_1(\bar{u}_1, \bar{u}_2 + \lambda_n v_2)$$

From (1) and (2),

$$(3) \qquad 0 \geqq \frac{\bar{u}_1\sigma_1(\bar{u}_1, \bar{u}_2 + \lambda_n v_2) - \bar{u}_1\sigma_1(\bar{u}_1, \bar{u}_2)}{\lambda_n} \geqq \frac{\bar{u}_1\sigma_1(\bar{u}_1 + \lambda_n v_1, \bar{u}_2) - \bar{u}_1\sigma_1(\bar{u}_1, \bar{u}_2)}{\lambda_n}$$

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[11] The utility function $w \in U$ may be constructed by picking a function $v_w \in C^2(X)$ with the properties

$$\nabla(\bar{u}_2 + v_w) = \nabla(\bar{u}_2 + \lambda v_2)$$

when evaluated at $\sigma_2(\bar{u}_1, \bar{u}_2 + \lambda v_2)$ and

$$\nabla(\bar{u}_2 + v_w) = \nabla \bar{u}_2$$

when evaluated at $\sigma_2(\bar{u}_1 + \lambda v_1, \bar{u}_2)$ and then defining $w \equiv \bar{u}_2 + v_w$. For λ small enough and judicious choice of v_w the resulting w will be an element of U since U is open.

As λ_n approaches zero (3) becomes

(4) $$0 \geq D_{(v_2)} \bar{u}_1 \sigma_1(\bar{u}) \geq D_{(v_1)} \bar{u}_1 \sigma_1(\bar{u}) .$$

The right hand side of (4) must be zero because σ is strategy-proof; if it were not zero, then the first order condition for $\bar{u}_1$ being agent one's dominant strategy would not be met. Therefore $0 \geq D_{(v_2)} \bar{u}_1 \sigma_1(\bar{u}) \geq 0$, which is to say that $D_{(v_2)} \bar{u}_1 \sigma_1(\bar{u}) = 0$. Thus agent two cannot affect agent one's utility at $\bar{u}$.

<u>Lemma 3.</u> If σ is strategy-proof and satisfies BA, then, for all $i \neq j$, $\bar{\tilde{S}}^{ij} \supset S^{ij}$ where $\bar{\tilde{S}}^{ij}$ denotes the closure of $\tilde{S}^{ij}$.

Proof. Suppose the lemma is false. A $u \in R$ therefore exists such that; (a) $i \; A(u) \; j$ and not $i \; \tilde{A}(u) \; j$ and (b) a neighborhood $N(u) = N(u_1) \times N(u_2) \times \ldots \times N(u_n) \subset R$ exists for which $u' \in N(u)$ implies $i \; A(u') \; j$ and not $j \; \tilde{A}(u') \; i$. Regularity and BA imply that we may select a neighborhood $\bar{N} = N(\sigma_j(u)) \subset X$ so that (a) corresponding to each x in $B_j(u) \cap \bar{N}$ is an admissible utility function $u_j^x \in N(u_j)$ that has its maximum on $B_j(u)$ at x and (b), for all $u' \in N(u)$, the manifold $B_j(u') \cap \bar{N}$ is smooth, continuously differentiable in u, and m-dimensional where $0 \leq m \leq \ell-1$. Note that because σ is strategy-proof $\sigma_j(u \backslash u_j^x) = x$. Because the proof for the general case where the dimensionality m of $B_j(u)$ may have any value between 0 and $\ell-1$ is lengthy, we present here a proof for the special case where $m = \ell-1$. Proof for the general case is presented in the technical memorandum [17].

Let $\perp(z)$ denote for any $z \in \overline{X}$ the normal to $B_j(u)$ that passes through z. Establish a new coordinate system for the neighborhood $\overline{N}$. Let a point that is z in the original system become the point (x,y) in the new system where (i) $x = \perp(z) \cap B_j(u)$ and (ii) y is the Euclidean distance (up) from x to z. Figure 1 illustrates this transformation.

Let $v_i \in C^2(X)$ be arbitrary and for each $(x,0) \in \overline{N}$, define $f^x(\lambda)$ to be the second component of the unique point $\perp(x,0) \cap B_j(\lambda)$ where $B_j(\lambda) = B_j(u\backslash u_i + \lambda v_i)$. Thus $\perp(x,0) \cap B_j(\lambda) \equiv (x, f^x(\lambda))$. By assumption, $D_{(v_1)} u_j^x \sigma_j(u\backslash u_j^x) = 0$ for all $(x,0) \in \overline{N}$. Therefore $df^x(0)/d\lambda = 0$ for all $(x,0) \in \overline{N}$ because if otherwise, then changing λ would cause j's constraint set to move in the direction $\perp(x,0)$, thus making feasible points that j prefers to $\sigma_j(u\backslash u_j^x)$. Figure 2 illustrates this.

Define for all $(x,y) \in \overline{N}$, $F(x,y;\lambda) = y - f^x(\lambda)$. Note that within $\overline{N}$ $B_j(\lambda)$ is represented by the solution to $F(x,y;\lambda) = 0$; by (b) of the definition of regularity F is C^2. Therefore, since σ is strategy-proof, the point $\sigma_j(u\backslash u_i + \lambda v_i)$ maximizes u_j subject to $F(x,y;\lambda) = 0$. Regularity guarantees that the derivative $D_{(v_i)} \sigma_j(u)$ exists. Finally, the definition of F and the result $df^x(0)/d\lambda = 0$ together imply that for all points $(x,0) \in B_j(0) \cap \overline{N}$

(5) $$\frac{dF(x,0;0)}{d\lambda} = -\frac{df^x(0)}{d\lambda} = 0,$$

which is to say that j's constraint set $B_j(\lambda)$ is fixed and does not move as λ varies about 0. Therefore $D_{(v_i)}\sigma_j(u) = 0$ because the only way i can affect j is by moving j's constraint set $B_j(\lambda)$. This contradicts the hypothesis that iA(u)j, which completes the proof.

Figure 1

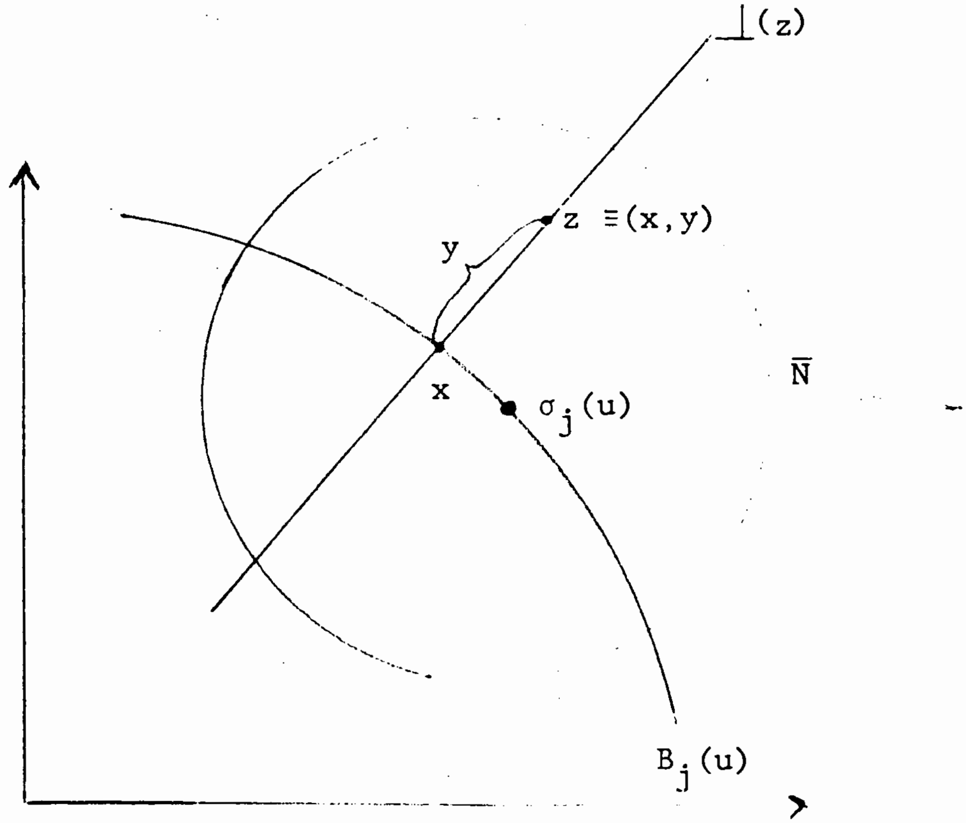
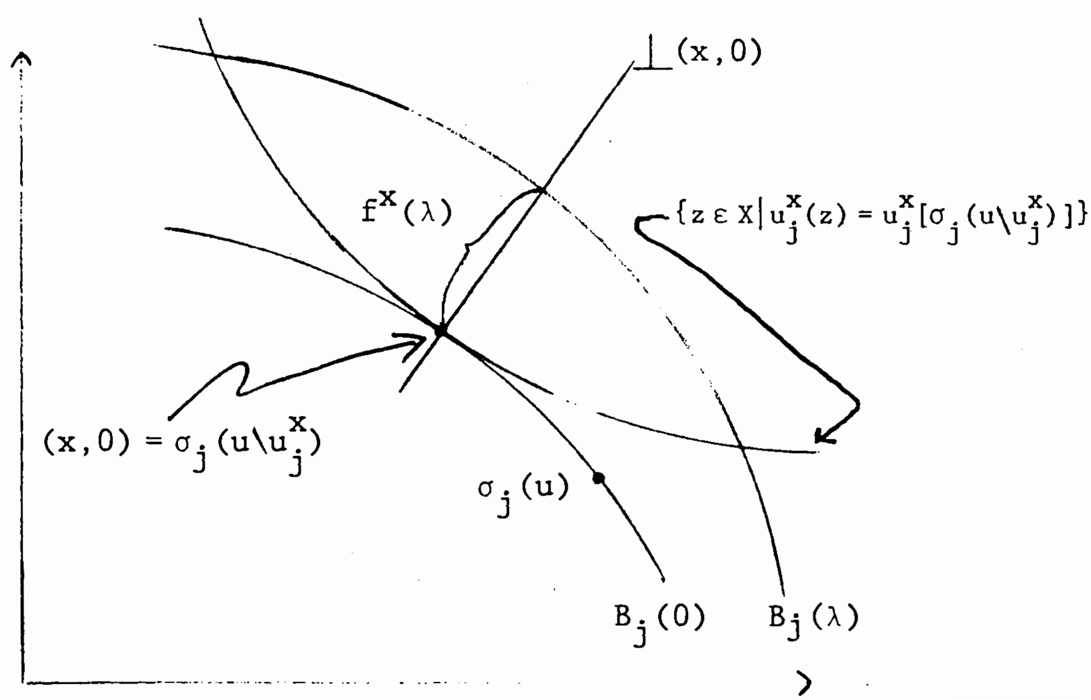


Figure 2



Lemma 4. If σ is strategy-proof and satisfies NB and BA, then for all $i \neq j$, $S^{ij} \cap S^{ji} = \emptyset$.

Proof. By Lemma 1, $\tilde{S}^{ij}$ and $\tilde{S}^{ij}$ are open and by Lemma 2 $\tilde{S}^{ij} \cap \tilde{S}^{ji} = \emptyset$. Therefore $\tilde{\tilde{S}}^{ij} \cap \tilde{S}^{ji} = \emptyset$ and so $S^{ij} \cap \tilde{S}^{ji} = \emptyset$ follows from Lemma 3. Since S^{ij} and $\tilde{S}^{ji}$ are open and disjoint, $S^{ij} \cap \tilde{\tilde{S}}^{ji} = \emptyset$. Applying Lemma 3 once again gives $S^{ij} \cap S^{ji} = \emptyset$, which is the required result.

Lemma 5. If σ is strategy-proof and satisfies NB and BA, then, for all distinct i,j, and k, $\tilde{S}^{ij} \cap \tilde{S}^{jk} \subset \tilde{\tilde{S}}^{ik}$.

Proof. Let $\bar{u} \in \tilde{S}^{ij} \cap \tilde{S}^{jk}$ and $N(\bar{u}) \subset \tilde{S}^{ij} \cap \tilde{S}^{jk}$ be arbitrary. We will show that a $u \in N(\bar{u})$ exists such that $u \in \tilde{S}^{ik}$. Assume without loss of generality that $i=1$, $j=2$, and $k=3=n$. Since $\bar{u} \in \tilde{S}^{12}$ there exists v_1 such that $D_{(v_1)}\bar{u}_2\sigma_2(\bar{u}) > 0$, and since $\bar{u} \in \tilde{S}^{23}$ there exists v_2 such that $D_{(\alpha v_2)}\bar{u}_3\sigma_3(\bar{u}) > 0$ for every scalar $\alpha > 0$. BA and regularity imply a $\tilde{u}_2$, a $\bar{\lambda} \in (0,1]$, and a $\alpha > 0$ exist such that $(\bar{u}_1,\tilde{u}_2,\bar{u}_3) \in N(\bar{u})$ and $\tilde{u}_2$ attains a unique maximum on $B_2(\bar{u}_1 + \lambda v_1,\bar{u}_2,\bar{u}_3)$ at $\sigma_2(\bar{u}_1 + \lambda v_1,\bar{u}_2 + \alpha\lambda v_2, \bar{u}_3)$ for all $\lambda \in [0, \bar{\lambda}]$. Therefore, for all $\lambda \in [0, \bar{\lambda}]$, $\sigma_2(\bar{u}_1 + \lambda v_1,\tilde{u}_2,\bar{u}_3)$ $= \sigma_2(\bar{u}_1 + \lambda v_1,\bar{u}_2 + \alpha\lambda v_2,\bar{u}_3)$ because σ is strategy-proof and $\sigma_3(\bar{u}_1 + \lambda v_1,\tilde{u}_2,\bar{u}_3)$ $= \sigma_3(\bar{u}_1 + \lambda v_1,\bar{u}_2 + \alpha\lambda v_2,\bar{u}_3)$ because σ satisfies NB. Therefore

$$D_{(v_1)}\bar{u}_3\sigma_3(\bar{u}_1,\tilde{u}_2,\bar{u}_3) = D_{(v_1,\alpha v_2,0)}\bar{u}_3\sigma_3(\bar{u}_1,\bar{u}_2,\bar{u}_3) = D_{(v_1)}\bar{u}_3\sigma_3(\bar{u}) + D_{(\alpha v_2)}\bar{u}_3\sigma_3(\bar{u})$$

where the second equality follows from the linearity of the D operator. If this expression is nonzero, then $(\bar{u}_1,\tilde{u}_2,\bar{u}_3) \in \tilde{\tilde{S}}_{13}$. Suppose, on the other hand, it is zero. $D_{(\alpha v_2)}\bar{u}_3\sigma_3(\bar{u}) \neq 0$ by hypothesis; therefore $D_{(v_1)}\bar{u}_3\sigma_3(\bar{u}) \neq 0$ and, thus, $\bar{u} \in \tilde{S}_{13}$. Consequently, in either of the possible cases, there exists a $u \in N(\bar{u})$ such that $u \in \tilde{S}^{13}$, which completes the proof.

Proof of Theorem 1. Without loss of generality, suppose that $S^{12} \cap S^{23} \cap \ldots \cap S^{(J-1)J} \neq \emptyset$. We will show that necessarily $S^{12} \cap S^{23} \cap \ldots \cap S^{(J-1)J} \cap S^{J1} = \emptyset$. Because S is open, a u and N(u) exist such that $u \in N(u) \subset S^{12} \cap S^{23} \cap \ldots \cap S^{(J-1)J}$. Lemma 3 states that $S^{ij} \subset \tilde{\tilde{S}}^{ij}$; therefore $u \in N(u) \subset \tilde{\tilde{S}}^{12} \cap \tilde{\tilde{S}}^{23} \ldots \cap \tilde{\tilde{S}}^{(J-1)J}$. This implies that a $\bar{u} \in N(u)$ exists such that $\bar{u} \in \tilde{S}^{12} \cap \tilde{S}^{23} \cap \ldots \cap \tilde{S}^{(J-1)J}$ because every neighborhood of a point contained in the closure of a set must meet the set. Consequently $\tilde{S}^{12} \cap \tilde{S}^{23} \cap \ldots \cap \tilde{S}^{(J-1)J} \neq \emptyset$.

Pick any $\bar{u}$ and neighborhood $N(\bar{u})$ such that $\bar{u} \in N(\bar{u}) \subset \tilde{S}^{12} \cap \tilde{S}^{23} \cap \ldots \cap \tilde{S}^{(J-1)J}$. By Lemma 5, a $\bar{u}'$ exists such that $\bar{u}' \in N(\bar{u}) \cap \tilde{S}^{13}$, which is equivalent to saying that $\bar{u} \in \tilde{\tilde{S}}^{13}$. We now show that $\bar{u} \in \tilde{\tilde{S}}^{14}$. Pick a neighborhood $N(\bar{u}')$ such that $N(\bar{u}') \subset N(\bar{u}) \cap \tilde{S}^{13}$. Since $N(\bar{u}) \subset \tilde{S}^{12} \cap \tilde{S}^{23} \cap \tilde{S}^{34} \cap \ldots \cap \tilde{S}^{(J-1)J}$, $N(\bar{u}') \subset \tilde{S}^{34}$ also. Thus $N(\bar{u}') \subset \tilde{S}^{13} \cap \tilde{S}^{34}$ and, by Lemma 5, a $\bar{u}'' \in N(\bar{u}') \cap \tilde{S}^{14}$ exists. Since $\bar{u}'' \in N(\bar{u}') \subset N(\bar{u})$, $\bar{u}'' \in N(\bar{u})$, which implies that $\bar{u} \in \tilde{\tilde{S}}^{14}$. Repeated application of this argument leads to the conclusion $\bar{u} \in \tilde{\tilde{S}}^{1J}$. Since $\bar{u}$ was picked arbitrarily from $\tilde{S}^{12} \cap \tilde{S}^{23} \cap \ldots \cap \tilde{S}^{(J-1)J}$, it follows that $\tilde{S}^{12} \cap \tilde{S}^{23} \cap \ldots \cap \tilde{S}^{(J-1)J} \subset \tilde{\tilde{S}}^{1J}$.

$\tilde{\tilde{S}}^{1J} \cap S^{J1} = \emptyset$ because $\tilde{S}^{1J} \subset S^{J1}$, $\tilde{S}^{1J} \cap S^{J1} = \emptyset$, implies $\tilde{\tilde{S}}^{1J} \cap S^{J1} = \emptyset$, and Lemma 4 states that $S^{1J} \cap S^{J1} = \emptyset$. Therefore $\tilde{\tilde{S}}^{1J} \subset \mathcal{R}\setminus S^{J1}$ where $\mathcal{R}\setminus S^{J1}$ is the complement of S^{J1} relative to $\mathcal{R}$. This means that $\tilde{S}^{12} \cap \tilde{S}^{23} \cap \ldots \cap S^{(J-1)J} \subset \tilde{\tilde{S}}^{1J} \subset \mathcal{R}\setminus S^{J1}$ or, equivalently,

$$\tilde{S}^{12} \cap \tilde{S}^{23} \cap \ldots \cap \tilde{S}^{(J-1)J} \cap S^{J1} = \emptyset . \tag{6}$$

Since each of the J sets on the left hand side of ( ) are open in $\mathcal{R}$, alternating applications of (a) the rule that if $A \cap B = \emptyset$, then $\overline{A} \cap B = \emptyset$ and (b) Lemma 3 gives:

$$\begin{aligned}
&\tilde{\tilde{S}}^{12} \cap (\tilde{S}^{23} \cap \ldots \cap S^{J1}) = \emptyset \\
&S^{12} \cap (\tilde{S}^{23} \cap \ldots \cap S^{J1}) = \emptyset \\
&\tilde{\tilde{S}}^{23} \cap (S^{12} \cap \tilde{S}^{34} \cap \ldots \cap S^{J1}) = \emptyset \qquad (7) \\
&S^{23} \cap (S^{12} \cap \tilde{S}^{34} \cap \ldots \cap S^{J1}) = \emptyset \\
&\qquad\vdots \\
&S^{12} \cap S^{23} \cap \ldots \cap S^{J1} = \emptyset ,
\end{aligned}$$

which is the required result.

Theorems 2 and 3 follow immediately from Theorem 1.

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