

Discussion Paper No. 392R

AN ALGORITHM FOR COMPUTING EQUILIBRIA IN A
LINEAR MONETARY ECONOMY

by

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August, 1979

Abstract An algorithm is presented for computing equilibria in a linear monetary economy, that is, an exchange economy in which all individuals have linear utility functions and in which goods are bought and sold only in exchange for money. The algorithm computes the equilibrium prices by solving a finite sequence of linear programming problems.

Acknowledgements. The author is indebted to Andrew Daughety, Nimrod Megiddo, and Arie Tamir for helpful suggestions and advice.

Key words: equilibrium, linear monetary economy, linear programming.

Abbreviated title: Computing Equilibria.

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1. Equilibria in a linear monetary economy

In this paper, we develop a method to compute price equilibria for linear monetary economies. By linear monetary economy, we mean a pure exchange economy in which goods can be bought and sold only in exchange for money, and in which all individuals have linear utility functions for goods and money. An equilibrium of such an economy should be interpreted as a short-run equilibrium or a one-period temporary equilibrium in a dynamic economy, so that the individuals' utility for money is derived from their expected use of money in future periods.

Temporary monetary equilibria of this form, with a "cash in advance" constraint on individuals' demand, have been proposed by Clower [1]; see also Myerson [3].

Our algorithm may be used in simulating such models.

Many algorithms have been suggested for computing economic equilibria; see Scarf [4]. Eaves [2] has shown that Lemke's algorithm can be used to solve general linear exchange economies in finitely many steps. We shall show here that linear monetary economies have enough special structure so that computing their equilibria can be reduced to solving a finite sequence of linear programming problems. In fact, when one has a good first estimate of what the equilibrium prices might be (as would be the case in simulations of a dynamic model, where last period's prices could be used), our algorithm could require only one linear program to converge.

Let I denote the number of individuals in the economy, with i denoting a typical individual, so that i always ranges

over $i=1, \dots, I$. Let J denote the number of goods and other non-money assets, with j denoting a typical non-money asset, so that j always ranges over $j=1, \dots, J$. We may think of money as asset #0.

Let X_{ij} denote the quantity of asset j which individual i has available to sell this period, and let X_{i0} denote the quantity of money which i has available to spend this period. We assume that each individual i has a linear utility function for assets, and so let U_{ij} denote i 's marginal utility for asset j . We shall assume that all $U_{ij} > 0$, which will guarantee that any equilibrium will have all positive prices. Without loss of generality (rescaling the individual's utility functions if necessary), we assume that each individual has unit marginal utility for money, so that $U_{i0} = 1$.

Our problem is to compute prices, supply, and demand in a market equilibrium. We shall let p_j denote the price of asset j . Our supply and demand variables will be expressed in terms of the market value of the quantities supplied and demanded (evaluated at the p_j prices), rather than in physical units. That is, s_{ij} will denote the value of the quantity of asset j which individual i supplies to the market, and d_{ij} will denote the value of the quantity of j which i demands from the market. Thus, the actual quantities of j supplied and demanded by i would be s_{ij}/p_j and d_{ij}/p_j respectively.

Given the market prices p_j , each individual i wants to choose his s_{ij} and d_{ij} quantities so as to maximize

$$(1) \quad \sum_{j=1}^J \left(\frac{U_{ij}}{p_j} - 1 \right) (d_{ij} - s_{ij})$$

subject to

$$(2) \quad s_{ij}/p_j \leq X_{ij} \quad (\forall j=1, \dots, J),$$

$$(3) \quad \sum_{j=1}^J d_{ij} \leq X_{i0},$$

$$(4) \quad s_{ij} \geq 0, \quad \text{and} \quad d_{ij} \geq 0 \quad (\forall j=1, \dots, J).$$

To see where the coefficients in (1) come from, observe that every dollar (or unit of money) spent to buy j brings in $1/p_j$ units of j , each of which contributes U_{ij} units of utility, while the loss of the dollar spent reduces i 's utility by 1 (since $U_{i0} = 1$). Thus $(U_{ij}/p_j - 1)$ is the coefficient of d_{ij} in i 's utility formula. Similarly, every dollar's worth of j sold contributes one unit of utility for the money brought in, but costs U_{ij}/p_j units of utility for the $1/p_j$ units of j sent out. So $(1 - U_{ij}/p_j)$ is the coefficient of s_{ij} in (1). Constraint (2) asserts that the quantity of j sold by i cannot exceed i 's available endowment of j . Constraint (3) asserts that i 's total spending cannot exceed his available money balances. Constraint (4) gives the obvious non-negativity constraints. (This interpretation of (1)-(4) is in the spirit of [1], but differs slightly from [3]. It is easy to see that the individual's decision problems in [3] are mathematically equivalent to (1)-(4), with a simple translation of notation.¹⁾

Our problem is to find prices (p_1, \dots, p_J) such that, for all $j=1, \dots, J$:

$$(5) \quad \sum_{i=1}^I s_{ij} = \sum_{i=1}^I d_{ij}$$

when the individuals choose the s_{ij} and d_{ij} to maximize (1) subject to (2)-(4).

It will be more convenient to work with the complementary slackness conditions generated by the optimization problem (1)-(4). Thus, we shall need the following fact.

Lemma. Given any individual i and prices p_j , suppose that the supplies s_{ij} and demands d_{ij} satisfy (2)-(4). Then the s_{ij} and d_{ij} maximize (1) subject to (2)-(4) if and only if there exists some number $q_i \geq 0$ such that for every j

$$(6) \quad q_i \leq p_j / U_{ij}, \text{ and}$$

$$(7) \quad q_i \leq 1,$$

and such that the following complementarity conditions are satisfied, for every j :

$$(8) \quad s_{ij} = 0 \text{ or } p_j \geq U_{ij};$$

$$(9) \quad s_{ij} = X_{ij} p_j \text{ or } p_j \leq U_{ij};$$

$$(10) \quad d_{ij} = 0 \text{ or } q_i = p_j / U_{ij}; \text{ and}$$

$$(11) \quad d_{i0} = 0 \quad \text{or} \quad q_i = 1;$$

where

$$(12) \quad d_{i0} = X_{i0} - \sum_{j=1}^J d_{ij} .$$

Proof. (8) and (9) simply assert that i sells no j if $p_j < U_{ij}$ and sells all his j if $p_j > U_{ij}$. Interpreting U_{ij} as i 's personal reservation price for j , we can easily see that (8) and (9) are the conditions for optimal sales.

On the demand side, i wants to allocate all of his money to demanding those assets for which U_{ij}/p_j is maximal and greater than 1. But this is equivalent to allocating all demand to those assets for which p_j/U_{ij} is minimal and less than one. Conditions (6), (7), and (10)-(12) imply that

$$(13) \quad q_i = \min \{1, \min_j (p_j/U_{ij})\}$$

and that no asset is demanded unless $q_i = p_j/U_{ij}$. Thus $1/q_i$ is i 's marginal utility of income, and (6)-(7) and (10)-(12) are the conditions for optimal demand. Q.E.D.

2. The algorithm

The problem of computing a price equilibrium is equivalent to the following optimization problem, from which we will derive our algorithm:

$$(14) \quad \text{Minimize} \quad z_0 + \sum_{j=1}^J z_j$$

subject to the complementarity constraints (8)-(11) (for all i and j) and:

$$(15) \quad z_j - \sum_{i=1}^I (s_{ij} - d_{ij}) \geq 0 \quad (\forall j) ;$$

$$(16) \quad z_j - \sum_{i=1}^I (d_{ij} - s_{ij}) \geq 0 \quad (\forall j) ;$$

$$(17) \quad z_0 - z_j \geq 0 \quad (\forall j) ;$$

$$(18) \quad X_{ij} p_j - s_{ij} \geq 0 \quad (\forall i, \forall j) ;$$

$$(19) \quad d_{i0} + \sum_{j=1}^J d_{ij} = X_{i0} \quad (\forall i) ;$$

$$(20) \quad p_j - U_{ij} q_i \geq 0 \quad (\forall i, \forall j) ;$$

$$(21) \quad q_i \leq 1 \quad (\forall i) ;$$

$$(22) \quad s_{ij} \geq 0, d_{i0} \geq 0, d_{ij} \geq 0 \quad (\forall i, \forall j) ;$$

$$(23) \quad p_j \geq \text{minimum} \{U_{ij} \mid X_{ij} > 0\} .$$

Since (18)-(22) merely restate (2)-(4) and (6)-(7), it should be clear that any solution to the above optimization problem achieving a value of zero in the objective function must be a market equilibrium. Constraint (23) merely states that no asset's price can drop below every potential seller's

reservation price for the asset, a condition which must hold in equilibrium. (The roles of z_0 and of constraint (23) may seem redundant here, but they will be needed to guarantee convergence of our algorithm.) However, this optimization problem is not quite a linear program, because of the alternative conditions in the constraints (8)-(11). Our algorithm will search through a series of linear programs generated by considering restricted versions of these constraints.

We shall consider two ways of restricting (8)-(11): the method of price-based constraints, and the method of quantity-based constraints. In each method, we begin with some given reference values p_j^0 , q_i^0 , s_{ij}^0 , d_{i0}^0 , d_{ij}^0 , satisfying (18)-(23) and (8)-(11), generated at a previous stage in the algorithm.

In the case of price-based constraints, we replace (8)-(11) by the following constraints, for every i and j :

$$(24) \quad s_{ij} = 0 \quad \text{and} \quad p_j \leq U_{ij}, \quad \text{if} \quad p_j^0 < U_{ij};$$

$$(25) \quad X_{ij}p_j - s_{ij} = 0 \quad \text{and} \quad p_j \geq U_{ij}, \quad \text{if} \quad p_j^0 > U_{ij};$$

$$(26) \quad p_j = U_{ij}, \quad \text{if} \quad p_j^0 = U_{ij};$$

$$(27) \quad d_{ij} = 0, \quad \text{if} \quad p_j^0 - U_{ij}q_i^0 > 0;$$

$$(28) \quad p_j - U_{ij}q_i = 0, \quad \text{if} \quad p_j^0 - U_{ij}q_i^0 = 0;$$

$$(29) \quad d_{i0} = 0, \quad \text{if} \quad q_i^0 < 1;$$

$$(30) \quad q_i = 1, \quad \text{if} \quad q_i^0 = 1.$$

In the case of quantity-based constraints, we replace (8)-(11) by the following constraints, for every i and j :

$$(31) \quad s_{ij} = 0 \quad \text{and} \quad p_j \leq U_{ij}, \quad \text{if} \quad s_{ij}^0 = 0 ;$$

$$(32) \quad X_{ij}p_j - s_{ij} = 0 \quad \text{and} \quad p_j \geq U_{ij}, \quad \text{if} \quad s_{ij}^0 = X_{ij}p_j^0 ;$$

$$(33) \quad p_j = U_{ij}, \quad \text{if} \quad 0 < s_{ij}^0 < X_{ij}p_j^0 ;$$

$$(34) \quad d_{ij} = 0, \quad \text{if} \quad d_{ij}^0 = 0 ;$$

$$(35) \quad p_j - U_{ij}q_i = 0, \quad \text{if} \quad d_{ij}^0 > 0 ;$$

$$(36) \quad d_{i0} = 0, \quad \text{if} \quad d_{i0}^0 = 0 ;$$

$$(37) \quad q_i = 1, \quad \text{if} \quad d_{i0}^0 > 0.$$

It is straightforward to check that (24)-(30) do imply (8)-(11), as do (31)-(37).

We can now describe our algorithm. At the first iteration, choose any vector of reference prices satisfying (24). (In a dynamic model, we could generally use the previous period's equilibrium prices.) Then define q_i^0 by

$$q_i^0 = \min \{1, \min_j (p_j^0 / U_{ij})\} .$$

Now solve the linear programming problem of minimizing (14) subject to (15)-(23) and (24)-(30). It is straightforward to check that this linear program does have feasible solutions. For example, all the constraints are satisfied by letting each

$p_j = p_j^0$ and $q_i = q_i^0$, and letting the d_{ij} and s_{ij} be any optimal demand and supply quantities for individual i at these prices.

At the second and every subsequent iteration, let the new reference values $(p_i^0, s_{ij}^0, d_{ij}^0, q_i^0)$ be the optimal solution from the linear program solved at the preceding iteration. At every even iteration (second, fourth, etc.), solve the linear programming problem of minimizing (14) subject to (15)-(23) and (31)-(37) (using the quantity-based constraints). At every odd iteration (third, fifth, etc.), solve the linear programming problem of minimizing (14) subject to (15)-(23) and (24)-30) (using the price-based constraints). At any iteration, if the objective function cannot be reduced then the preceding iteration's optimal solution should be retained. The algorithm terminates at a market equilibrium when some linear program achieves an optimal solution with $z_0 = 0$.

3. Convergence of the algorithm

Theorem. The algorithm must terminate at a market equilibrium within a finite number of iterations.

Proof. At every iteration, the optimal solution from the preceding iteration is still feasible for the new linear programming problem, by the way the new constraints were constructed ((24)-(30) or (31)-(37)). Thus, we know that all our linear programs are feasible, and the value of the objective function can never increase at any iteration.

At each iteration, we construct a linear programming problem by selecting one of the two alternatives in each of

if i is willing to buy j , and we have a two-way arc between i and j if i is actually buying j , in the market solution represented by the reference values.

To interpret this graph, observe that a one-way arc represents a direction in which demand could be increased without violating the price-based constants (27)-(30). A two-way arc represents a direction in which demand could be increased or decreased without violating (27)-(30). Thus, in the price-based problem solved at the odd iteration, it must be possible to shift some positive amount of demand along any directed path in the graph which goes from one asset-vertex to another.

Given this directed graph, we say that vertex α reaches vertex β iff there exists some directed path in the graph from α to β . (Any vertex reaches itself.) Let R_1 be the set of all vertices which do not reach the vertex 0 or any asset-vertex j such that $\sum_{i=1}^I (s_{ij}^0 - d_{ij}^0) \geq 0$. Let R_2 be the set of all vertices which are not reached by the vertex 0 or by any asset-vertex j such that $\sum_{i=1}^I (d_{ij}^0 - s_{ij}^0) \geq 0$.

These sets R_1 and R_2 can contain both asset-vertices and individual-vertices. Every asset in R_1 must have excess demand; and every asset in R_2 must have excess supply. (There may be assets with excess demand or excess supply which are not in R_1 or R_2 , however.) Every individual in R_1 must be only willing to buy assets in R_1 ; and every individual in R_2 must be actually spending all of his money on assets in R_2 .

Observe that R_1 and R_2 cannot both be empty. If they were both empty then, in the price-based linear program solved in the odd iteration, it would be possible to shift some demand

from every asset with excess demand to an asset without excess demand, and it would be possible to shift some demand to every asset with excess supply from some asset without excess supply. These shifts could strictly reduce

the maximum market imbalance z_0 without increasing the sum of market imbalances $\sum_{j=1}^J z_j$. Such demand shifts would thus

contradict our assumption that the reference values were optimal for the linear program at an odd iteration.

(This is why we have included the z_0 term in the objective function (14).)

For any asset j in R_1 , and for any individual i , if $s_{ij}^0 < p_j^0 X_{ij}$ then $p_j^0 < U_{ij}$. Similarly, for any asset j in R_2 , and for any individual i , if $s_{ij}^0 > 0$ then $p_j^0 > U_{ij}$. These facts must hold because otherwise supplies would have been increased in R_1 or decreased in R_2 during the odd iteration, since such supply changes would have reduced the total market imbalance (excess demand in R_1 , excess supply in R_2) without violating (24) or (25).

For any asset j in R_1 or R_2 , we must have $\sum_{i=1}^I s_{ij}^0 > 0$.

In R_1 , this fact follows from the first sentence of the preceding paragraph together with constraint (23). In R_2 , this fact is obvious, since all assets in R_2 have excess supply.

Using the observations from the preceding three paragraphs, we can now show that our reference values could not be optimal for the quantity-based problem solved at the even iteration. First consider R_1 , in which all assets have excess demand. Suppose that we try to increase p_j for every asset j in R_1 , q_i for every individual i in R_1 , and s_{ij} for every asset j in R_1 and every individual i . If these

increases above the reference values are kept proportionate to each other, then small strict increases are possible without violating any of the constraints (18)-(23) and (31)-(37). Small increases in q_i would be blocked by (20) only if there were some asset j not in R_1 such that $p_j^0 = U_{ij}q_i^0$, but this would imply that the arc $i \rightarrow j$ is in the graph, so that i could not be in R_1 . (By definition of R_1 , there cannot be any path reaching from inside R_1 to outside R_1 .) Similarly, (21) could not be a binding constraint for any i in R_1 , since this would imply $i \rightarrow 0$ would be in the graph. Constraints (32) and (35) will never be violated by these increases, since in each case both variables are being increased in the same proportion. (Recall $d_{ij}^0 > 0$ implies that $i \leftrightarrow j$ is in the graph.) Constraints (31) and (33) can never block small price increases in R_1 , because $s_{ij}^0 < X_{ij}p_j^0$ implies $p_j^0 < U_{ij}$. Thus we can strictly increase all prices in R_1 and (because $\sum_i s_{ij}^0 > 0$ for j in R_1) all supplies of assets in R_1 . So if R_1 is non-empty, then the objective function (14) must decrease at the even iteration.

If R_1 is empty then we must check R_2 , where there is excess supply. Thus, we want to decrease the following variables: p_j for every asset j in R_2 , q_i for every individual i in R_2 , and s_{ij} for every asset j in R_2 and every individual i . As before, if these decreases below the reference values are kept proportionate to each other, then sufficiently small strict decreases are possible without violating

any of the constraints (18)-(23) and (31)-(37). Constraint (20) could not block small price decreases for j in R_2 , because either $p_j^0 > U_{ij}q_i^0$ or else i is an individual in R_2 as well. Constraints (32) and (33) can never block small price decreases because $s_{ij}^0 > 0$ implies $p_j^0 > U_{ij}$. And (23) could not be binding, or else supplies would have been reduced at the odd iteration. So we can strictly decrease all prices and supplies of assets in R_2 , which must reduce market imbalance. Since $R_1 \cup R_2 \neq \emptyset$, the objective function (14) must decrease at the even iteration, if it did not decrease at the preceding odd iteration.

Q.E.D.

Footnotes

¹The "marginal value" U_{ij} in this paper corresponds to the ratio $v_{ij}(t)/v_{i0}(t)$ in [3], and the "endowments" X_{ij} and X_{i0} in this paper correspond to the maximum transaction rates $n_{ij}x_{ij}(t)$ and $n_{i0}x_{i0}(t)$ in [3]. What we have labelled s_{ij} and d_{ij} in this paper would correspond to $p_j s_{ij}(t)$ and $p_j d_{ij}(t)$ in the notation of [3], where the supply and demand variables are measured in physical units.

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