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COMPETITIVE MARKET ALLOCATION MECHANISMS AND EX-ANTE OPTIMALITY*

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Comments are invited

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I. Introduction and Review of the Literature

In the past two decades there has grown an extensive literature on general equilibrium theory and welfare economics under uncertainty. The original work in this area is by Arrow (1964) and Debreu (1959) on ex-ante contingent claim markets. In their model each consumer has a budget constraint and a subjective probability distribution over a known, finite set of states of the world which might occur in the future. A state of the world is represented by a complete list of all events which would occur and parameters which would prevail simultaneously. Before the observation of the random variable a market in contingent commodity claims opens. Given the prices for these contingent claims, each consumer chooses a utility maximizing consumption bundle for each state of the world and purchases a portfolio of consumption claims, one of which can be redeemed for the appropriate consumption bundle when the uncertainty is resolved. After the random variable is observed all consumers and firms are assumed to be able to agree what state of the world exists. At that time no further markets open. Firms simply produce the amount contracted for ex-ante and distribute output to consumers according to the distribution of claims. Since all contracts are formalized ex-ante, all prices are known with certainty and there are therefore no state-dependent income effects.

Given this idealized market structure, they show that a competitive equilibrium in contingent claims achieves the same resource allocation in each state of the world as would be achieved if that state of the world were certain. Since competitive markets under certainty are Pareto optimal, the implication is that competitive ex-ante contingent claims markets are also Pareto optimal.

Despite the welfare attraction of this result, however, its usefulness is limited. Such complete contingent claims markets do not exist and probably will not come into being for at least three reasons. First, their operation requires all agents both to have and to assimilate more information about future states of the world than those agents can be expected either to have or to assimilate. Second, it may not be possible either to identify or to reach agreement about which state of the world has occurred in order to enforce contracts. Third, it is unlikely that agents would be willing to give up their rights to make spot trades after the uncertainty is resolved.

Research since these seminal works has focused on criticizing and extending Arrow's and Debreu's work. Starr (1973) criticized Arrow for ignoring the ambiguity implicit in his definition of Pareto optimality. While it is generally agreed what is meant by Pareto optimality when events are certain, the introduction of uncertainty allows a range of possible definitions of Pareto optimality and allocations which pass one test and may fail another. Arrow, for example, was only concerned in that model with ex-post optimality. Under this test, an allocation is ex-post Pareto optimal if, given a state of the world, it is not possible to reallocate resources in that state in such a way as to make one person better off without making someone else worse off. Alternatively, one might demand that an ex-ante set of consumption plans be Pareto optimal. Under that test, a set of consumption plans is ex-ante Pareto optimal if it is not possible ex-ante to increase one person's expected utility without reducing someone else's expected utility. This test is stronger than the ex-post test since all ex-ante optimal allocations in Arrow's model are ex-post optimal but the converse is not true.

Such conflicts among the different definitions of Pareto optimality arise whenever consumers have differing probability estimates over states. Since such differences of opinion exist, Starr's paper cautions researchers to specify what definitions of Pareto optimality they are invoking.

An important extension of the Arrow-Debreu model has been to study conditions which allow the existence and optimality of sequences of temporary spot market equilibria under uncertainty. Radnor (1968; 1972), Green (1973), and Jordan (1976) have studied conditions which allow sequences of plans and price expectations to be in equilibrium. Such equilibria are called rational expectations equilibria and are characterized by sets of price ratios for each state of the world on which all agents agree. Agents may have differing probability estimates over states of the world, but not over prices in each state. Hart (1975) has shown, however, that such equilibria will not in general yield ex-post optimal allocations of consumption goods.

This paper considers an extension of the literature on optimal allocation under uncertainty to an economy in which at least one production decision is made before any observation of the random variable and cannot be altered after it is observed. Examples include plant size and fixed price contracts. A particular application and extension of this model to water storage and delivery is discussed in a forthcoming paper (Hoffman, 1979). Random events are assumed to affect the production functions for goods and the utility functions of consumers.

The model developed in this paper describes an ex-ante optimal choice of plant size. Consumers are assumed to have differentiable, expected utility functions which are addively separable across states and firms are assumed to have differentiable profit functions and expected utility functions, which are addively separable across states. The issue of which firm managers are assumed to maximize is left open at first. The only restriction is that there be some differentiable, expected utility function over profits.

In the first section we show that the following conditions are necessary for an ex-ante optimal choice of plant size: 1) subjective probability-weighted marginal rates of substitution across goods and states of the world are equal across consumers; and 2) each consumer's expected marginal rate of substitution between the good for which plant size is being chosen and all other goods is equal to the expected ratio of long run marginal costs. This result is consistent with the results on optimal capacity size obtained from studies of peak load pricing under uncertainty by Brown and Johnson (1974) and Crew and Kleindorfer (1976). It differs from their models, however, because it specifically takes account of possible differences in subjective probability distributions and utility functions among consumers. This is more the approach taken by McKay (1977).

Although the optimal plant size might be chosen ex-ante, that plant size will only be optimal by accident ex-post. This difference between ex-ante and ex-post optimality occurs because any one particular plant size will not be optimal under all demand conditions. For example, if demand is low, the equilibrium quantity will be relatively low and there will be excess capacity relative to the optimum and if demand is high, the variable inputs will be subject to diminishing returns to scale. Since ex-ante and ex-post optimality can only be made equal by accident in this economy, this paper will only be concerned with conditions which allow ex-ante optimality.

After outlining the necessary conditions for an ex-ante optimal choice of plant size and set of consumption plans we consider the question of whether and under what conditions such an ex-ante optimum is attainable as a competitive equilibrium. The following condition on consumer expected utility functions is found to be necessary for a competitive equilibrium to be ex-ante Pareto optimal. The ratios of consumer marginal expected utilities of income across states of the world must be constant across consumers. This condition arises because achieving an ex-ante Pareto optimum under competition requires that consumers give identical weights to price ratios across states of the world. Since incomes are random when all consumption and labor market decisions will be made ex-post, consumers weigh price ratios across states by the ratios of ex-ante marginal expected utilities of income when making consumption and labor market plans for different states.

This condition is also sufficient for an ex-ante Pareto optimum under competition if firms do not bear the risk associated with owning their own plants. The following example illustrates how and why ownership of plant makes a difference. Suppose there are two consumers A and B who are going to build a plant to produce some good. Suppose A thinks demand will be high and B thinks demand will be low. They agree what the price will be in each event, but they disagree about probability estimates. Clearly, in the absence of transactions costs, A would be willing to bribe B to agree to a larger plant. Thus ex-ante, gains from trade exist because consumers have different probability estimates when ex-ante fixed cost decisions are being made.

Now consider two different ways of organizing a competitive market producing plant. In the first model,firms building plant sell shares to consumers in a competitive market. They then build only as much plant as consumers will buy in shares at competitive prices. When the uncertainty is resolved, consumers rent infinitely divisible plant space back to firms which produce output. Consumers who think demand will be high will buy more shares than consumers who think demand will be low. The market becomes the means by which A bribes B. When the uncertainty is resolved, plant will rent for less than its long run marginal cost if there is excess capacity and for more than its long run marginal cost when demand exceeds the optimum quantity given the available plant.

In the second model, firms which produce the good which uses plant in its production process build and retain title to their own plants. Decisions about plant size will then be based on some individual or individuals in the firm's probability estimates and attitudes toward risk. In the absence of a market which allows an equilibrium of consumer expectations, some consumers will still wish the firms had built more plant and others will wish they had built less. Thus, the gains from trade remain unexhaused. On the other hand, if all consumers and firms owning plants have identical subjective probability distributions, consumers' ex-post marginal utilities of income are independent of the state of the world and firms owning plants strictly maximize expected profits instead of utility functions, then these gains from trade do not exist to begin with.

II. The Model: Ex-ante Optimality Conditions

Consider an economy in which they are n consumers, indexed $i = 1,\ldots,n$ and S(finite) possible states of the world. These states of the world are distributed according to discrete subjective probabilities α_s^i indexed $s = 1,\ldots,S$. The economy produces two goods for final consumption in each state of the world. x_s^i is i's consumption of x in state s and y_s^i is i's consumption of y in state s. x uses both plant and labor in its production process and y only uses labor. The differentiable production function for x in s is $x_s(P(L_o),L_{xs})$, where L_{xs} is labor devoted to the production of x in s and $P(L_o)$ is the ex-ante production function for plant. L_o is labor devoted to building plant ex-ante $\sum_{i=1}^{n} x_s^i \leq (P(L_o),L_{xs}) \cdot y_s(L_{ys})$ is the differentiable production function for y in S, where L_{ys} is labor devoted to producing y in s. $\sum_{i=1}^{n} y_s^i \leq y_s(L_{ys})$.

Each consumer is assumed to have a differentiable expected utility function of the form $EU^i = U_o^i(L_o^i) + \sum_{s=1}^{S} \alpha_s^i U_s^i(x_s^i, y_s^i, L_s^i)$, where L_o^i and L_s^i are labor supplied by i ex-ante and in state s, respectively. Without loss of generality, we assume no consumption takes place ex-ante. $L_o \leq \sum_{i=1}^{n} L_o^i$ and $L_{xs} + L_{ys} \leq \sum_{i=1}^{n} L_s^i$. The following assumptions are also invoked to ensure that x_s^i, y_s^i, L_s^i, and L_o^i are always positive and that i is not satiated at any point. $\lim_{x_s^i \to 0} \frac{\partial U_s^i}{\partial x_s^i} = \lim_{y_s^i \to 0} \frac{\partial U_s^i}{\partial y_s^i} = +\infty \ \forall i,s;$

$$\lim_{x_s^i \to \infty} \frac{\partial U_s^i}{\partial x_s^i} > 0 \ \forall i,s; \quad \lim_{y_s^i \to \infty} \frac{\partial U_s^i}{\partial y_s^i} > 0 \ \forall_i,s; \quad \lim_{\left[L_s^i \to 24\text{hrs/day}\right]} \frac{\partial U_s^i}{\partial L_s^i} = \lim_{\left[L_o^i \to 24\text{hr/day}\right]} \frac{\partial U_o^i}{\partial L_o^i} = -\infty \ \forall i,s;$$

$\lim_{L_s^i \to 0} \frac{\partial U_s^i}{\partial L_s^i} < 0 \ \forall i,s;$ and $\lim_{L_o^i \to 0} \frac{\partial U_o^i}{\partial L_o^i} < 0 \ \forall i.$ These assumptions allow us to express an expected welfare maximum as a weighted sum of individual expected utilities. Let

$$EW = \sum_{i=1}^{n} \beta^i \; EU^i$$ where

$$\beta^i = i\text{'s welfare weight}$$

The welfare problem is to maximize EW, subject to the following constraints:

$$L_o \leq \sum_{i=1}^{n} L_o^i$$

$$L_{xs} + L_{ys} \leq \sum_{i=1}^{n} L_s^i \qquad \forall s$$

$$\sum_{i=1}^{n} x_s^i \leq x_s(P(L_o), L_{xs}) \qquad \forall_s$$

$$\sum_{i=1}^{n} y_s^i \leq y_s(L_{ys}) \qquad \forall_s$$

The first order conditions for a constrained maximum, which are summarized in the appendix, reduce to the following necessary conditions for an ex-ante optimum.

$$\frac{\alpha_s^i \dfrac{\partial U_s^i}{\partial x_s^i}}{\alpha_r^i \dfrac{\partial U_r^i}{\partial x_r^i}} = \frac{\alpha_s^h \dfrac{\partial U_s^h}{\partial x_s^h}}{\alpha_r^h \dfrac{\partial U_r^h}{\partial x_r^h}} = \frac{\alpha_s^i \dfrac{\partial U_s^i/\partial L_s^i}{\partial x_s/\partial L_{xs}}}{\alpha_r^i \dfrac{\partial U_r^i/\partial L_r^i}{\partial x_r/\partial L_{xr}}} = \frac{\alpha_s^h \dfrac{\partial U_s^h/\partial L_s^h}{\partial x_s/\partial L_{xs}}}{\alpha_r^h \dfrac{\partial U_r^h/\partial L_r^h}{\partial x_r/\partial L_{xr}}} \quad \forall i,h,r,s \quad (1)$$

$$\frac{\sum_{s=1}^{S}\left\{\alpha_s^i \frac{\partial U_s^i}{\partial x_s^i}\frac{\partial x_s}{\partial P}\right\}}{\alpha_r^i \frac{\partial U_r^i}{\partial y_r^i}} = \frac{\sum_{s=1}^{S}\left\{\alpha_s^h \frac{\partial U_s^h}{\partial x_s^h}\frac{\partial x_s}{\partial P}\right\}}{\alpha_r^h \frac{\partial U_r^h}{\partial y_r^h}} = \frac{\frac{\partial U_o^i/\partial L_o^i}{\partial P/\partial L_o}}{\alpha_r^i \frac{\partial U_r^i/\partial L_r^i}{\partial y_r/\partial L_{yr}}} = \frac{\frac{\partial U_o^h/\partial L_o^h}{\partial P/\partial L_o}}{\alpha_r^h \frac{\partial U_r^h/\partial L_r^h}{\partial y_r/\partial L_{yr}}}$$

Equation (1) states that an optimal set of ex-ante consumption plans for x and y for different states of the world equates the following ratios: 1) the ratio of marginal expected utilities between x_s and y_s for each consumer in each state of the world; 2) the marginal rate of transformation between x_s and y_s as evaluated by each consumer; and 3) the ratio of marginal expected utilities between x_s and y_s and the subjective marginal rate of transformation between x_s and y_s for each consumer in each state of the world. Equation (2) states that at such an optimum the following expectations are equated: 1) the expected marginal rate of substitution between plant and y_r for each consumer in each state of the world; 2) the expected marginal rate of tansformation between plant and y_r as evaluated by each consumer in each state of the world; and 3) the expected marginal rate of substitution and the expected marginal rate of transformation between plant and y_r for each consumer in each state of the world.

III. A Competitive Model in which Plant is Chosen Ex-ante and Consumption Goods are Produced Ex-post

In this economy, assume that consumers and firm managers know what prices will prevail in each state of the world, but may disagree about probabilities to attach to states. A firm manager may either maximize profits or some individual's expected utility of profits. Ex-ante, each consumer plans how much labor to supply in each state and makes a set of consumption plans contingent on the observation of the random variable. The object is to maximize expected utility subject to a set of random budget constraints, one of which will apply after the uncertainty is resolved. Firm x produces plant ex-ante and makes a set of production plans contingent on the observation of the random variable. Depending on the market arrangement, firm x may either sell shares to plant ex-ante and then rent back space ex-post or retain title to the plant. Firm y simply makes a set of production plans ex-ante and then produces the appropriate amount ex-post.

Before continuing, it is important to consider whether the competitive equilibrium in the market for plant and a set of equilibrium consumption plans might exist. Radner (1972), Green (1973), and Jordan (1976) have examined conditions which guarantee the existence of a sequence of equilibrium consumption plans and we assume their conditions apply. It is probably not unreasonable to assume that a market for plant would operate as a stock market as long as the expected returns were non-negative. Dréze (1974) outlines conditions which guarantee an equilibrium in the stock market. This rules out technology which is subject to long run increasing returns to scale, however, because the expected return on capacity would be negative and capacity rights would be worthless to consumers. We assume, therefore, that the range of increasing returns to scale for the good produced with plant capacity is less than or equal to the expected demand for that good for some group of consumers at the price corresponding to minimum efficient scale.

Now let us consider the maximization problems under the two alternative market arrangements. Consumer i's problems are:

1) consumers own plant space:

Max EU^i as on page 8 — Multipliers

$$\text{s.t.}\quad q_o P^i \leq w_o L_o^i + \theta_x^i \Pi_{xo} \qquad \lambda_o^i$$

$$p_{xs} x_s^i + p_{ys} y_s^i \leq q_s P^i + w_s L_s^i + \theta_x^i \Pi_{xs} + \theta_y^i \Pi_{ys} \quad \forall_s, \qquad \lambda_s^i$$

where q_o = ex-ante price per unit of plant space;

q_s = price per unit of plant space in s;

P^i = amount of plant space owned by i;

p_{xs} = ex-post price of x in s;

w_s = ex-post wage in s;

Π_{xo}, Π_{xs} = firm x's profits ex-ante and in state s; and

Π_{ys} = firm y's profits in state s.

2) Firm x owns plant space:

Max EU^i as above

$$\text{s.t.}\quad p_{xs} x_s^i + p_{ys} y_s^i \leq w_o L_o^i + w_s L_s^i + \theta_x^i \Pi_{xs} + \theta_y^i \Pi_{ys} \qquad \delta_s^i$$

Firm x's problems are:

1) consumers own plant space:

$$\text{Max } E\phi_x(\Pi_x) = q_o P(L_o) - w_o L_o + \sum_{s=1}^{S} \alpha_{xs}\, \phi_x(p_{xs} x_s(P_s, L_{xs}) - w_s L_{xs} - q_s P_s),$$

where ϕ_x = x's differentiable utility function over profits (Π_x);

α_{xs} = x's subjective probability of being in s; and

P_s = rental of plant space in s.

2. Firm x owns plant space:

$$\text{Max } E\phi_x(\Pi_x) = -w_o L_o + \sum_{s=1}^{S} \alpha_{xs}\phi_x(p_{xs}x_s(P(L_o),L_{xs}) - w_s L_{xs})$$

Firm y's problem is the same in both cases:

$$\text{Max } E\phi_y(\Pi_y) = \sum_{s=1}^{S} \alpha_{ys}\ \phi_y(p_{ys}y_s(L_{ys}) - w_s L_{ys}), \text{ where}$$

ϕ_y = y's differentiable utility function over profits (Π_y); and

α_{ys} = y's subjective probability of being in s.

The equilibrium conditions for these markets are:

1. Consumers own plant space:

$$\sum_{i=1}^{n} L_o^i = L_o \tag{3}$$

$$\sum_{i=1}^{n} P^i = P(L_o) = P_s \tag{4}$$

$$\sum_{i=1}^{n} L_s^i = L_{xs} + L_{ys} \qquad \forall s \tag{5}$$

$$\sum_{i=1}^{n} x_s^i = x_s(P_s,L_{xs}) = x_s(P(L_o),L_{xs}) \qquad \forall s \tag{6}$$

$$\sum_{i=1}^{n} y_s^i = y_s(L_{ys}) \qquad \forall s \tag{7}$$

2. Firm x owns plant space:

(3) and (5)-(7)

The first order conditions for firm utility maximization and constrained consumer utility maximization, which are summarized in the appendix, reduce to the following equations describing the competitive equilibria.

1) Consumers own plant space:

$$\frac{\frac{\alpha_s^i}{\lambda_s^i}\frac{\partial U_s^i}{\partial x_s^i}}{\frac{\alpha_r^i}{\lambda_r^i}\frac{\partial U_r^i}{\partial y_r^i}} = \frac{\frac{\alpha_s^h}{\lambda_s^h}\frac{\partial U_s^h}{\partial x_s^h}}{\frac{\alpha_r^h}{\lambda_r^h}\frac{\partial U_r^h}{\partial y_r^h}} = \frac{\frac{\alpha_s^i}{\lambda_s^i}\frac{\partial U_s^i/\partial L_s^i}{\partial x_s/\partial L_{xs}}}{\frac{\alpha_r^i}{\lambda_r^i}\frac{\partial U_r^i/\partial L_r^i}{\partial y_r/\partial L_{yr}}} = \frac{\frac{\alpha_s^h}{\lambda_s^h}\frac{\partial U_s^h/\partial L_s^h}{\partial x_s/\partial L_{xs}}}{\frac{\alpha_r^h}{\lambda_r^h}\frac{\partial U_r^h/\partial L_r^h}{\partial y_r/\partial L_{yr}}} \quad \forall i,h,r,s \qquad (8)$$

$$\frac{\sum_{s=1}^{S}\left\{\frac{\alpha_s^i}{\lambda_o^i}\frac{\partial U_s^i}{\partial x_s^i}\frac{\partial x_s}{\partial P_s}\right\}}{\frac{\alpha_r^i}{\lambda_r^i}\frac{\partial U_r^i}{\partial y_r^i}} = \frac{\sum_{s=1}^{S}\left\{\frac{\alpha_s^h}{\lambda_o^h}\frac{\partial U_s^h}{\partial x_s^h}\frac{\partial x_s}{\partial P_s}\right\}}{\frac{\alpha_r^h}{\lambda_r^h}\frac{\partial U_r^h}{\partial y_r^h}} = \frac{\frac{1}{\lambda_o^i}\frac{\partial U_o^i/\partial L_o^i}{\partial P/\partial L_o}}{\frac{\alpha_r^i}{\lambda_r^i}\frac{\partial U_r^i/\partial L_r^i}{\partial y_r/\partial L_{yr}}} = \frac{\frac{1}{\lambda_o^h}\frac{\partial U_o^h/\partial L_o^h}{\partial P/\partial L_o}}{\frac{\alpha_r^h}{\lambda_r^h}\frac{\partial U_r^h/\partial L_r^h}{\partial y_r/\partial L_{yr}}} \quad \forall i,h,r \qquad (9)$$

2) Firm x owns plant space:

(8) m substituting δ_s^i for λ_s^i $\forall i,s$ and the following:

$$\frac{\sum_{s=1}^{S}\left\{\alpha_{xs}\,\phi_x'\,\frac{\alpha_s^i}{\lambda_s^i}\frac{\partial U_s^i}{\partial x_s^i}\frac{\partial x_s}{\partial P}\right\}}{\frac{\alpha_r^i}{\lambda_r^i}\frac{\partial U_r^i}{\partial y_r^i}} = \frac{\sum_{s=1}^{S}\left\{\alpha_{xs}\,\phi_x'\,\frac{\alpha_s^h}{\lambda_s^h}\frac{\partial U_s^h}{\partial x_s^h}\frac{\partial x_s}{\partial P}\right\}}{\frac{\alpha_r^h}{\lambda_r^h}\frac{\partial U_r^h}{\partial y_r^h}} =$$

$$\frac{\frac{1}{\sum_{s=1}^{S}\delta_s^i}\frac{\partial U_o^i/\partial L_o^i}{\partial P/\partial L_o}}{\frac{\alpha_r^i}{\delta_r^i}\frac{\partial U_r^i/\partial L_r^i}{\partial y_r/\partial L_{yr}}} = \frac{\frac{1}{\sum_{s=1}^{S}\delta_s^h}\frac{\partial U_o^h/\partial L_o^h}{\partial P/\partial L_o}}{\frac{\alpha_r^h}{\delta_r^h}\frac{\partial U_r^h/\partial L_r^h}{\partial y_r/\partial L_{yr}}} \quad \forall i,h,r,s \qquad (10)$$

Note first that whichever group owns plant space, an ex-ante equilibrium set of consumption and labor market plans does not equate the ratios of marginal expected utilities in different states of the world across consumers. Rather, each consumer weights his marginal expected utility from consumption or labor in each state of the world by the reciprocal of his marginal expected utility of income in that state of the world (λ_s^i) when forming consumption and labor market plans. In general, therefore, ex-ante competitive equilibrium consumption and labor market plans will not be ex-ante Pareto optimal when consumer incomes in each state of the world are random. By the same argument, capacity size chosen competitively ex-ante will not in general be optimal if consumers face random incomes.

However, while it is true that the competitive solution will not in general be ex-ante Pareto optimal, much less restrictive assumptions about expected utility functions will guarantee ex-ante optimality when consumers own plant space.

Theorem 1: In the context of the model derived above, a necessary and sufficient condition for a competitive equilibrium to be ex-ante Pareto optimal when consumers own plant space is for the ratios of marginal expected utilities of income in different states of the world to be equal across consumers.

Proof: We wish to show that (8) = (1) $\Leftrightarrow \frac{\lambda_s^i}{\lambda_r^i} = \frac{\lambda_s^h}{\lambda_r^h}$ $\forall i,h,r,s$; and that (9) = (2) $\Leftrightarrow \frac{\lambda_o^i}{\lambda_s^i} = \frac{\lambda_o^h}{\lambda_s^h}$ $\forall i,h,s$. Sufficiency is trivial. To show it is necessary, note that if it were not true, we could not cancel the ratios of marginal expected utilities from equations (8) and (9). Q.E.D.

Theorem 2: In the context of the model derived above, the following conditions are sufficient to guarantee that a competitive equilibrium will be ex-ante Pareto optimal if firms own plant space:

1) All consumers and firms owning plant have identical subjective probability distributions;

2) each consumer's marginal utility of expected income in each state of the world is a constant function of his subjective probability associated with that state of the world; (i.e. ex-post, each consumer has a constant marginal utility of income); and

3) firms owning plant strictly maximize expected profits.

Note that assumption (2) trivally satisfies the necessary and sufficient condition of Theorem 1.

<u>Proof</u>: Let α_s = agreed upon probability of being in s;

C^i = i's ex-post marginal utility of income; and

ϕ_x^1 = 1

substitute α_s, c^i, and ϕ_x^1 in (8) and (10):

(8) becomes:

$$\frac{\dfrac{\alpha_s}{C^i\alpha_s}\dfrac{\partial U_s^i}{\partial x_s^i}}{\dfrac{\alpha_r}{C^i\alpha_r}\dfrac{\partial U^i}{\partial y_r^i}} = \frac{\dfrac{\alpha_s}{C^h\alpha_s}\dfrac{\partial U_s^h}{\partial x_s^h}}{\dfrac{\alpha_r}{C^i\alpha_r}\dfrac{\partial U_r^h}{\partial y_r^h}} = \frac{\dfrac{\alpha_s}{C^i\alpha_s}\dfrac{\partial U_s^i/\partial L_s^i}{\partial x_s/\partial L_{xs}}}{\dfrac{\alpha_r}{C^i\alpha_r}\dfrac{\partial U_r^i/\partial L_r^i}{\partial y_r/\partial L_{yr}}} = \frac{\dfrac{\alpha_s}{C^h\alpha_s}\dfrac{\partial U_s^h/\partial L_s^h}{\partial x_s/\partial L_{xs}}}{\dfrac{\alpha_r}{C^h\alpha_r}\dfrac{\partial U_r^h/\partial L_r^h}{\partial y_r/\partial L_{yr}}} \quad \forall i,h,r,s \qquad (11),$$

which reduces to (1) given the restrictive assumptions.

(10) becomes:

$$\frac{\sum_{s=1}^{S}\left\{\alpha_s \frac{\alpha_s}{C^i\alpha_s}\frac{\partial U_s^i}{\partial x_s^i}\frac{\partial x_s}{\partial P}\right\}}{\frac{\alpha_r}{C^i\alpha_r}\frac{\partial U_r^i}{\partial y_r^i}} = \frac{\sum_{s=1}^{S}\left\{\alpha_s \frac{\alpha_s}{C^h\alpha_s}\frac{\partial U_s^h}{\partial x_s^h}\frac{\partial x_s}{\partial P}\right\}}{\frac{\alpha_r}{C^h\alpha_r}\frac{\partial U_r^h}{\partial y_r^h}} =$$

$$\frac{\frac{1}{C^i\sum_{s=1}^{S}\alpha_s}\;\frac{\partial U_o^i/\partial L_o^i}{\partial P/\partial L_o}}{\frac{\alpha_r}{C^i\alpha_r}\;\frac{\partial U_r^i/\partial L_r^i}{\partial y_r/\partial L_{yr}}} = \frac{\frac{1}{C^i\sum_{s=1}^{S}\alpha_s}\;\frac{\partial U_o^h/\partial L_o^h}{\partial P/\partial L_o}}{\frac{\alpha_r}{C^h\alpha_r}\;\frac{\partial U_r^h/\partial L_r^h}{\partial y_r/\partial L_{yr}}} \qquad \forall i,h,r \qquad (12),$$

which reduces to (2) given the restrictive assumptions and the fact that $\sum_{s=1}^{S}\alpha_s = 1$. Q.E.D.

While the necessary and sufficient condition in Theorem 1 is without question a strong restriction to place on consumer utility functions, it is not without precedent in the literature. For example, it has been invoked to derive stockholder unanimity theorems (Forsythe, 1976). Forsythe (1976) interprets the condition that λ_o^i/λ_s^i be equal for all consumers for each s as an assumption that there exists a set of contingent claim price ratios for consumption ex-ante and consumption in each possible state of the world. The condition that $\frac{\lambda_s^i}{\lambda_r^i}$ be equal for all consumers for each pair of states of the world assumes that such contingent claim price ratios also exist across states of the world.

The sufficient conditions outlined in Theorem 2 are far more restrictive, however. While they are not necessary, deviations from the sufficient conditions outlined in Theorem 2 will generally be similarly restrictive as compared to those outlined in Theorem 1. This suggests a tentative conclusion that more frequently, an efficient choice of capacity size can be made as the outcome of a competitive market in capacity rights than as a management decision within the firm. That is, the market in capacity rights more frequently exhausts the gains from trade created by uncertainty about capacity utilization than does a market only in the final output.

Concluding Remarks

The results outlined in this paper suggest three general conjectures about ex-ante optimality of decisions under uncertainty in the context of an economy characterized by rational expectations. First, the objective function used by firms in risky situations may not matter if firms can somehow redistribute all the risk back to consumers. A second related conjecture is that in general, risk taking by firms probably makes the attainment of ex-ante optimality very difficult in a decentralized economy. This suggests that policies directed towards developing markets which allow consumers to assume more risk, such as a market in capacity rights, might lead to improved resource allocation. Finally, even when expectations are rational, income effects may lead to deviations from ex-ante optimality of sequences of allocation plans.

Appendix

The lagrange equation for constrained welfare maximization is:

$$\mathcal{L} = \sum_{i=1}^{n} \beta^i [U_o^i(L_o^i) + \sum_{s=1}^{S} \alpha_s^i U_s^i(x_s^i, y_s^i, L_s^i)] - \gamma(L_o - \sum_{i=1}^{n} L_o^i) -$$

$$\sum_{s=1}^{S} \lambda_{1s}(L_{xs} + L_{ys} - \sum_{i=1}^{n} L_s^i) - \sum_{s=1}^{S} \lambda_{2s}(\sum_{i=1}^{n} x_s^i - x_s(P(L_o), L_{xs})) -$$

$$\sum_{s=1}^{S} \lambda_{3s}(\sum_{i=1}^{n} y_s^i - y_s(L_{ys}))$$

The first order necessary conditions are:

$$\beta^i \frac{\partial U_s^i}{\partial L_o^i} + \gamma = 0 \qquad \forall i$$

$$\beta^i \alpha_s^i \frac{\partial U_s^i}{\partial L_s^i} + \lambda_{1s} = 0 \qquad \forall i,s$$

$$\beta^i \alpha_s^i \frac{\partial U_s^i}{\partial x_s^i} - \lambda_{2s} = 0 \qquad \forall i,s$$

$$\beta^i \alpha_s^i \frac{\partial U_s^i}{\partial y_s^i} - \lambda_{3s} = 0 \qquad \forall i,s$$

$$-\gamma + \sum_{s=1}^{S} \lambda_{2s} \frac{\partial x_s}{\partial P} \frac{\partial P}{\partial L_o} = 0$$

$$-\lambda_{1s} + \lambda_{2s} \frac{\partial x_s}{\partial L_{xs}} = 0 \qquad \forall s$$

$$-\lambda_{1s} + \lambda_{3s} \frac{\partial y_s}{\partial L_{ys}} = 0 \qquad \forall s$$

The lagrange equations for constrained consumer utility maximization are as follows.

1. Consumers own plant space:

$$\mathcal{L} = U_o^i(L_o^i) + \sum_{s=1}^{S} \alpha_s^i \, U_s^i(x_s^i, y_s^i, L_s^i) - \lambda_o^i(q_o P^i - w_o L_o^i - \theta_x^i \Pi_{xo}) -$$

$$\sum_{s=1}^{S} \lambda_s^i(p_{xs} x_s^i + p_{ys} \, y_s^i - q \, P^i - w_s L_s^i - \theta_x^i \Pi_{xs} - \theta_y^i \Pi_{ys})$$

2. Firm x owns plant space

$$\mathcal{L} = U_o^i(L_o^i) + \sum_{s=1}^{S} \alpha_s^i \, U_s^i(x_s^i, y_s^i, L_s^i) - \sum_{s=1}^{S} \delta_s^i(p_{xs} x_s^i + p_{ys} y_s^i - w_o L_o^i - w_s L_s^i - \theta_x^i \Pi_{xs} - \theta_y^i \Pi_{ys})$$

Combining these with the firm utility maximization problems outlined in the text, the first order necessary conditions are as follows.

1) consumers own plant space

$$\frac{\partial U_s^i}{\partial L_o^i} + \lambda_o^i w_o = 0 \qquad \forall i \qquad (1A)$$

$$\alpha_s^i \frac{\partial U_s^i}{\partial L_s^i} + \lambda_s^i w_s = 0 \qquad \forall i, s \qquad (2A)$$

$$\alpha_s^i \frac{\partial U_s^i}{\partial x_s^i} - \lambda_s^i p_{xs} = 0 \qquad \forall i, s \qquad (3A)$$

$$\alpha_s^i \frac{\partial U_s^i}{\partial y_s^i} - \lambda_s^i \, p_{ys} = 0 \qquad \forall i, s \qquad (4A)$$

$$- q_o\lambda_o^i + \sum_{s=1}^{S} \lambda_s^i q_s = 0 \qquad \forall i \qquad (5A)$$

$$q_o \frac{\partial P}{\partial L_o} - w_o = 0 \qquad (6A)$$

$$\alpha_{xs}\, \phi_x'(p_{xs} \frac{\partial x_s}{\partial P_s} - q_s) = 0 \qquad \forall s \qquad (7A)$$

$$\alpha_{xs}\phi_x'(p_{xs} \frac{\partial x_s}{\partial L_{xs}} - w_s) = 0 \qquad \forall s \qquad (8A)$$

$$\alpha_{ys}\phi_y'(p_{ys} \frac{\partial y_s}{\partial L_{ys}} - w_s) = 0 \qquad \forall s \qquad (9A)$$

2) Firm x owns plant space:

(8A),(9A) and the following:

$$\frac{\partial U_s^i}{\partial L_o^i} + w_o \sum_{s=1}^{S} \delta_s^i = 0 \qquad \forall i,s \qquad (10A)$$

$$\alpha_s^i \frac{\partial U_s^i}{\partial L_s^i} + \delta_s^i w_s^i = 0 \qquad \forall i,s \qquad (11A)$$

$$\alpha_s^i \frac{\partial U_s^i}{\partial x_s^i} - \delta_s^i p_{xs} = 0 \qquad \forall i,s \qquad (12A)$$

$$\alpha_s^i \frac{\partial U_s^i}{\partial y_s^i} - \delta_s^i \, p_{ys} = 0 \qquad \forall i,s \qquad (13A)$$

$$- w_o + \sum_{s=1}^{S} \alpha_{xs} \phi_x' p_{xs} \frac{\partial x_s}{\partial P} \frac{\partial P}{\partial L_o} = 0 \qquad (14A)$$

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