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"The Optimum Quantity of Money" *

by

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1) Introduction

My purpose is to show that underlying what seems to be Milton Friedman's vision of reality, there is a rigorous model of competitive equilibrium which can serve as an alternative to the Arrow-Debreu model. More precisely, a careful analysis of Friedman's paper "The Optimum Quantity of Money" [28] leads naturally to a model in which an equilibrium is a stationary sequence of temporary equilibria and is also Pareto optimal. It is Pareto optimal even though there is no forward trading and there are no markets for contingent claim contracts.

I do not know whether Friedman would agree with my analysis, nor do I claim that he should agree. My aim is to synthesize his ideas and my own.

In "The Optimum Quantity of Money," Friedman argues that an economy cannot be economically efficient if any consumer economizes on cash balances. Consumers should be constrained by their average flow of income, but not by immediate shortages of cash. Wasteful economizing of cash would be avoided if money earned a real rate of interest equal to consumers' rate of time preference. The quantity of money which would be held by society in this situation is called the optimum quantity of money.

In explaining his ideas, Friedman uses a simple model, which he calls "a hypothetical simple society" ([28], p. 2). This is a stationary society with a constant population and with given tastes and resources. Consumers buy and sell services. Money is the only durable object which may be exchanged. There is no borrowing and lending. The nominal stock of money is fixed. Consumers are subject to random shocks. The shocks are such that "mean values do not (change)." I interpret this last assumption as meaning that the shocks form a stationary stochastic process.

I define a mathematically precise version of Friedman's model. I assume that the model is of an exchange economy, with no production, although Friedman is not clear on this point. The most important specification I make is that consumers live forever. Here, I follow Friedman. Friedman says that "it is simplest to regard the members of this society as being immortal and unchangeable." However, one could also interpret his model as an intergenerational model similar to Samuelson's consumption-loan model [61]. The random shocks in Friedman's model are assumed to form a Markov chain. Corresponding to each of the finitely many states of the chain, there is a Walrasian pure trade economy. Each consumer has an endowment and a concave utility function which is a function of the state. The endowments are not storable, but must be traded and consumed in the period in which they appear. Each consumer acts so as to maximize the expected value of the discounted infinite sum of his utilities from consumption. Consumers are assumed to have rational expectations in that they know the true probability distribution of future prices and of their own utility functions and endowments. Money has price one in every period. It is not needed to pay for purchases and does not enter anyone's utility function. It is useful to consumers only because it allows them to spend more than they earn sometimes. The only intertemporal aspect of the model is that consumers must decide each period how much money to save or dissave.

The assumption that consumers live foreover involves a delicate question of interpretation. What is involved is the interpretation of the time scale and of the nature of the random events. My model is designed to represent what one would see approximately in a grander model if one looked at how consumers reacted to everyday fluctuations over a short period of time. I think of random events as small events, which tend to average out after a year or so. Periods are days and a day two years hence is nearly infinitely far away. The infinite horizon should not be taken literally. It is simply a way to look at the consumer's life as a process rather than in totality.

My point of view seems to be roughly consistent with Friedman's. He specifically assumes that physical resources and "the state of the arts" are fixed. These assumptions would not be appropriate if he visualized a period of time spanning five to ten years, say. However, my point of view is not entirely consistent with what Friedman says. For he asserts that one "reason for holding money is as a reserve for future emergencies" ([28], p. 3). Emergencies are hardly the everyday events I have in mind. However, I do not find it appropriate to assume that ordinary consumers could ever hold sufficient assets to be able to handle major emergencies, even if money did pay interest. Most people are simply not that rich. The best they can do is to buy insurance against specific catastrophies.

I also part company with Friedman in not making money necessary for transactions. Friedman states that in his model the two motives for holding money are self-insurance and as a medium of exchange in order to circumvent the double coincidence of wants needed for barter. However, introducing transaction costs, limited information about trading possibilities, and so on, would only complicate my model. It would not change the conclusions.

I now return to what I do. I define a <u>monetary equilibrium</u> to be an infinite sequence of random temporary equilibria such that the price of money is always one and such that all prices are uniformly bounded away from zero and infinity. By assuming that prices are bounded, I exclude the inflationary equilibria which apparently may occur in almost any model with an infinite horizon and rational expectations. (See, for example, Gale [29], p. 24.)

The sequence of random price vectors in a monetary equilibrium do not necessarily obey a stationary probability law. This fact is a weakness of the concept of monetary equilibrium. For it makes little sense to assume rational expectations if the probability distributions involved are not stationary.

(Observation does not reveal the distribution law of a non-stationary distribution). I suspect that in a wide class of cases, there exist no stationary monetary equilibria. I hope to turn to this question in a later paper.

I make two strong assumptions about endowments and utility functions, which guarantee that consumers need money in order to compensate for fluctuations in their incomes and needs. I then prove that there exists a monetary equilibrium provided that the interest earned on money is less than every consumer's rate of time preference. I also prove that in this case, a monetary equilibrium need not be Pareto optimal. More precisely, if each consumer always consumes something, then the equilibrium is not Pareto optimal. This is in accord with Friedman's argument. Every consumer economizes on money balances to some extent, since his rate of time preference exceeds the rate of interest. Since consumers economize, the equilibrium cannot be Pareto optimal.

A consumer would not economize on money balances if the rate of interest equaled his rate of time preference. He would accumulate money balances until he was fully self-insured. For self-insurance would be costless, since the trade-off between present and future expenditure would be the same when measured in terms of money or utility. So following Friedman, I assume that all consumers have the same rate of time preference and that money earns interest at this rate. I prove that in this case, there exists no monetary equilibrium for almost every choice of consumers' random endowments. ("Almost every" means "for all endowments except those belonging to a set of Lebesque measure zero.") In proving this result, I make a special assumption which guarantees that the underlying stochastic fluctuation is sufficiently random. The idea of the proof is that a monetary equilibrium can exist only if the pattern of net expenditures of each consumer is periodic and not random. Periodicity can be destroyed by small perturbations in endowments. (Such periodicity is illustrated by the example given in section 13).

If the pattern of net expenditures of some consumer were not periodic, he would need an infinite quantity of money in order to insure himself completely. For he would have to protect himself against an arbitrarily long run of bad luck. In short, I prove that almost surely, the optimum quantity of money is infinite.

I express the infiniteness of the optimum quantity of money in another way. I show that for almost every choice of consumers' endowments the following is true. The real stock of money in a monetary equilibrium may be made arbitrarily large by paying interest on money at a rate which is sufficiently close to the common rate of time preference.

One might interpret these results as a criticism of Friedman's notion of an optimum quantity of money. I make his model precise and reduce the idea to an absurdity. However, this would not be a valid interpretation of my work. It seems fair to say that Friedman's primary interest was in economic policy. From a practical point of view, the idea that the optimum quantity of money is infinite is perhaps just silly. This idea becomes important only when one tries to use a precise model of general equilibrium in order to express Friedman's ideas.

It might seem that the infiniteness of the optimum quantity of money is simply an artifact of my model. It is, of course, a consequence of the bizarre assumption that consumers never die. However, the theoretical problem cannot be resolved by assuming that consumers do die. For if consumers do die, the optimum rate of interest may not lead to a Pareto optimal allocation. Imagine for the moment, a version of my model with mortal consumers. Suppose that each consumer lives many periods and is replaced at his death by a new consumer. Suppose also that there is no inheritance and that each consumer knows when he is going to die. Then, each consumer would spend all his money during his last period of life. Also toward the end of his life, he would tend to decumulate money. As a result, he might at some point be caught without enough cash. Such illiquidity would cause economic inefficiency. Clearly, no matter what the rate of interest, there would exist a monetary equilibrium and the real quantity of money would be finite. Hence given a social welfare function, there would exist an optimum real rate of interest. However, there is no reason to believe that this rate of interest would equal the common rate of time preference. Monetary equilibrium with an optimum rate of interest would not necessarily give rise to a Pareto optimal allocation. I do not pursue this line of thought in this paper.

I return now to the idea that the optimum quantity of money is infinite in the model of this paper. I interpret the infiniteness of the optimum quantity of money as expressing the idea that the optimum quantity of money in a more realistic model would be so large that consumers would rarely be constrained in their day to day lives by lack of cash. That is, they would be able to insure themselves effectively against small fluctuations.

In the theory of the consumer, self-insurance is expressed as constancy of the marginal utility of money. It stays constant over time, even as prices, and current needs and income fluctuate. I call this consumption the <u>permanent income hypothesis</u>. This is a notion I have explained before, using a model of a single consumer [9]. Here, I express the idea in a general equilibrium framework, and relate it to Friedman's ideas on the optimum quantity of money.

Ideally, I would like to have proved that if the rate of interest paid on money were sufficiently close to the common rate of time preference, then in a monetary equilibrium each consumer's marginal utility of money would be nearly constant. Unfortunately, I could not prove this, and it may not be true. If a monetary equilibrium is not stationary, one can say little about marginal utilities of money. As I have said, it is not clear whether stationary monetary equilibria exist.

The permanent income hypothesis leads naturally to a new version of equilibrium theory. In this theory, each consumer's demand function is defined by the assumption that the marginal utility of money is constant. He simply spends money on each good until the utility gained from consuming the quantity bought with the last dollar equals the fixed marginal utility of money. The consumer's budget constraint is that his long-term average expenditure per period not exceed his long-run average income per period. These long-run averages are computed using the true distribution of future prices, for the consumer is assumed to know this distribution. The consumer adjusts his marginal utility of money so as to bring his average expenditures into line with his average income.

Assume that each consumer's demand is defined in this fashion. I define a <u>stationary equilibrium</u> to be a stationary distribution of prices such that aggregate excess demand is always zero. I prove that in the model of this paper, a stationary equilibrium exists and is Pareto optimal. (I have discussed stationary equilibrium in three unpublished papers, [6,7,8].)

Stationary equilibrium is a way of describing the world when the quantity of money is optimal. This might seem confusing, for money plays no role in stationary equilibrium. But since the optimum quantity of money is infinite, it cannot play a role. In fact, the absence of money is an advantage from my point of view, for I seek a simple model of general equilibrium.

The fact that money disappears from the model expresses Frank Hahn's criticism of Friedman's theory of the optimum quantity of money [36]. Hahn expresses his main criticism as follows. "The necessary conditions for Pareto efficiency in a world of uncertainty with inter-temporal choice will in general be fulfilled by a market economy only if money plays no role." Hahn elaborates this point in three other papers [37,38,39]. An allocation is Pareto optimal only if it can be generated by equilibrium in Arrow-Debreu markets for forward and contingent claims. But in such an equilibrium, money plays no role. Hence, money is "inessential" in any system which generates Pareto optimal allocations.

In order to reconcile Hahn's and Friedman's ideas, one may think of Friedman's optimum quantity of money as optimal only in some asymptotic or approximate sense. One can think of money as present but nearly irrelevant from the point of view of equilibrium theory. Since cash rarely constrains consumers, it may be ignored.

This point of view also helps to reconcile Hahn's views with those of Starrett [70]. Starrett argued that Pareto inefficiency arises in Hahn's model of equilibrium with transactions costs only because of the lack of an intertemporal unit of account. Starrett is careful to point out that this unit of account would not be real money. In his model, consumers have unlimited ability to borrow and lend the unit of account. The only restriction is that debt be repaid in the last period of life. The point of my work is that in a model much like Hahn's, real money resembles Starrett's intertemporal unit of account asymptotically as the rate of interest approaches the common rate of time preference.

The concept of stationary equilibrium involves many notions that are commonly associated with modern, politically conservative, economic thinking. Not only is stationary equilibrium related to Friedman's optimum quantity of money and to his permanent income hypothesis, but it is based on the idea of rational expectations. I view stationary equilibrium as expressing rigorously the conservative vision of Walrasian equilibrium.

The notion of stationary equilibrium can serve as an alternative to the Arrow-Debreu model. By an alternative, I mean that each model is appropriate in certain settings. (The Arrow-Debreu model is defined in Arrow [1] and in Chapter 7 of Debreu [21].) Stationary equilibrium has the obvious advantage that trading takes place all the time, not exclusively in some ethereal initial period. However, in my opinion, stationary equilibrium can be thought of as applying only in short run contexts where random shocks are never severe. If the context is not short run, then stationarity does not make sense. The world changes over long periods of time. If random shocks are severe, then external insurance is needed. External insurance is formalized by contingent claims contracts.

The notion of stationary equilibrium strikes one as intriguing, in spite of its limitations. For instance, it provides a partial solution to a problem posed by Arrow [2]. The problem is to explain why we do not in reality observe complete markets for contingent claims. I intend to develop this point in a later paper.

Turning to another matter, I give a new solution in this paper to a problem posed by Hahn [35]. The problem is how to prove the existence of a competitive equilibrium in which money has a positive price. My monetary equilibrium is such an equilibrium. The device I use to give money value is the infinite horizon (together with the need for insurance). This is, of course, an artificial device, though perhaps more elegant than others that have appeared in the literature.

I wish to emphasize that I do not view this existence result as in any way explaining why money exists, nor as providing a basis for monetary theory. The existence result is simply a convenient way to describe precisely an aspect of reality which interests me here. Much of what is monetary about money is excluded from my model.

The following three sections contain formal definitions, assumptions and statements of results. In section 5, I attempt to relate my work to the vast literature on the optimal quantity of money and on the link between general equilibrium and monetary theory. The body of the paper gives formal proofs. The basic idea of the proof that the optimum quantity of money is infinite may be found in Schechtman [66]. The last section contains an example.

2) Definition, Notation, and the Model

Notation

R^L denotes L-dimensional Euclidean space. Let x and y belong to R^L. "$x \geqq y$" means "$x_k \geqq y_k$, for all k." "$x > y$" means "$x \geqq y$ and $x \neq y$." "$x >> y$" means "$x_k > y_k$, for all k." R_+^L denotes $\{x \epsilon R^L \mid x \geqq 0\}$. int R_+^L denotes $\{x \epsilon R^L \mid x >> 0\}$.

Let $f: U \to (-\infty, \infty)$ be twice differentiable, where U is an open subset of R^L. $Df(x)$ denotes the vector of first derivatives of f at x. $D^2 f(x)$ denotes the matrix of second order partial derivatives at x.

Prob $[A \mid B]$ denotes the conditional probability of A given B, where A and B are formulas describing events. Prob [A] denotes the probability of A. $E(x \mid B)$ denotes the expectation of the random variable x given the event B. Ex denotes the expectation of the random variable x.

The Underlying Stochastic Process

Exogenous fluctuations are governed by a stochastic process $\{s_n\}_{n=-\infty}^{\infty}$. The random variable s_n take their values in a set A. A is called the set of states of the <u>environment</u>. I assume that $\{s_n\}$ is a Markov chain. That is, $\{s_n\}$ is a Markov process with stationary probabilities and A is a finite set. If a and b belong to A, the P_{ab} denotes the transition probability, $\text{Prob}[s_{n+1} = b \mid s_n = a]$, for any n. Similarly, $P_{ab}^{(k)} = \text{Prob}[s_{n+k} = b \mid s_n = a]$, for $k \geqq 1$. I also assume that $\{s_n\}$ is ergodic with no transient states. That is, there is a positive integer n such that $P_{ab}^{(n)} > 0$, for all a and b.

Since $\{s_n\}$ is ergodic, there exists a unique stationary probability distribution on A, $(\pi_a)_{a \epsilon A}$. π satisfies $\pi_b = \sum_{a \epsilon A} \pi_a P_{ab}$, for all $b \; \epsilon \; A$. Since $\{s_n\}$ has no transient states, $\pi_a > 0$ for all $a \; \epsilon \; A$.

I will always assume that s_1 is distributed according to π. That is, the probability distribution of $(s_1, s_2, \ldots\ldots)$ is determined by the unique stationary distribution for the process $\{s_n\}$.

A <u>history</u> for the process $\{s_n\}$ from time n to time n + m is a finite sequence $a_n, a_{n+1}, \ldots, a_{n+m}$ such that $a_k \varepsilon A$, for all k, and Prob$[s_n = a_n, \ldots, s_{n+m} = a_{n+m}] > 0$. Histories will be denoted by $a_n, \ldots, a_{n+m}$. A <u>history following</u> a_n is a finite sequence $a_{n+1}, \ldots, a_{n+m}$ such that $a_n, a_{n+1}, \ldots, a_{n+m}$ forms a history.

The Economy

The economy is a pure trade economy with no production. Initial endowments and utility functions fluctuate in response to fluctuations in s_n.

There are L commodities and I consumers, where L and I are positive integers. The <u>endowment</u> of consumer i is determined by $\omega_i : A \to R_+^L$, for $i = 1, \ldots, I$. $\omega_i(s_n)$ is the endowment vector of the consumer in period n. The utility function of consumer i is $u_i : R_+^I \times A \to (-\infty, \infty)$. His utility function at time n is $u_i(., s_n) : R_+^L \to (-\infty, \infty)$. I assume that for all i, $u_i(., a)$ is everywhere twice differentiable. Also, $Du_i(x, a) >> 0$ and $D^2U_i(x, a)$ is negative definite, for all x. In other words, $u_i(\cdot, a)$ is differentiably strictly monotone and strictly concave.

A <u>consumption plan</u> for a consumer is of the form $x = (x_n(a_i \ldots, a_n))$, where $n = 1, 2, \ldots$ and $a_1, \ldots, a_n$ varies over histories and where each $x_n(a_1, \ldots, a_n)$ belongs to R_+^L. $x_n(s_1, \ldots, s_n)$ is the consumer's consumption bundle at time n.

Consumer i discounts utility at the rate δ_i, where $0 < \delta_i \leqq 1$. $\delta_i^{-1} - 1$ is his <u>rate of time preference</u>.

The expected value of the utility to consumer i of a consumption plan x is $U_i(x) = E[\sum_{n=0}^{\infty} \delta_i^{n-1} u_i(x_n(s_1, \ldots, s_n), s_n)]$, where E denotes the expected value operator. $U_i(x)$ is well-defined as long as $\delta_i < 1$ and the $x_n(s_1,\ldots,s_n)$ are uniformly bounded.

An allocation is a set of consumption plans $(x_i) = (x_1,\ldots,x_I)$, where x_i is the consumption plan of consumer i. The allocation is said to be feasible if $\sum_{i=1}^{I}(x_{in}(a_1,\ldots,a_n) - \omega_i(a_n)) = 0$, for all n and all histories $a_1,\ldots, a_n$.

If $\delta_i < 1$, for all i, then a feasible allocation (x_i) is said to be Pareto optimal if there exists no other feasible allocation $(\bar{x}_i)$ such that $U_i(\bar{x}_i) \geq U_i(x_i)$, for all i and $U_i(\bar{x}_i) > U_i(x_i)$, for some i. This definition makes no sense if $\delta_i = 1$, for some i. Suppose that $\delta_i = 1$, for all i. Then, the feasible allocation (x_i) is said to be Pareto optimal if there exists no feasible allocation $(\bar{x}_i)$ such that

2.1) $$E[\sum_{n=1}^{N} u_i(\bar{x}_{in}(s_1,\ldots,s_n),s_n)] \geqq E[\sum_{n=1}^{N} u_i(x_{in}(s_1,\ldots,s_n)s_n)],$$ for all i and N, with inequality for some i and N.

Monetary Equilibrium

A price system is of the form $p = (p_n(a_1,\ldots,a_n))$, where the $p_n(a_1,\ldots,a_n)$ belong to R_+^L. $p_n(a_1,\ldots,a_n)$ is the price vector at time n.

$r \geq 0$ denotes the nominal interest rate paid on money. Interest payments are financed by a lump-sum tax. Let τ_i be the tax payments paid by consumer i each period. $M_{in}(p,x;\ a_1,\ldots,a_n)$ denotes the money holdings of consumer i at the end of period n, given the price system p, his consumption program x and the history $a_1,\ldots, a_n$. $M_{in}(p,x;\ a_1,\ldots, a_n)$ is defined inductively as follows.

2.2) $M_{i0}(p,x) \equiv M_{i0}$ is given and $M_{in}(p,x;a_1,\ldots,a_n) = (1 + r)M_{i,n-1}(p,x;$
$a_1,\ldots,a_{n-1}) \quad + p_n(a_1,\ldots,a_n)\cdot(\omega_i(a_n) - x_{in}(a_1,\ldots,a_n)) -$
τ_i, for all $n \geqq 1$ and for all histories $a_1,\ldots,a_n$.

I will assume that $\Sigma M_{i0} = 1$. In order to assure that the nominal supply of money never changes, I assume that $\sum_{i=1}^{I} \tau_i = r$.

The <u>budget set</u> of consumer i, given a price system p, is $\beta_i(p) = \{x | x$ is a consumption program such that $M_{in}(p,x; a_1,\ldots,a_n) \geq 0$, for all n and for all histories $a_1,\ldots,a_n\}$.

In order to guarantee that $\beta_i(p)$ be non-empty, I assume that $\tau_i = r\, M_{i0}$, for all i. If τ_i exceeded rM_{i0}, the consumer might have no way to avoid being driven to bankruptcy by tax payments.

If $\delta_i < 1$, then consumer i's maximization problem is the following.

2.3) $$\max \{E\, [\sum_{n=1}^{\infty} \delta_i^{n-1} u_i(x_n(s_1,\ldots,s_n),s_n)]) \,|\, x \in \beta_i(p)\}.$$

$\xi_i(p)$ denotes the solution to this problem, if it exists. $\xi_i(p)$ does indeed exist, provided that the components of p are uniformly bounded away from zero and infinity. It is not necessary to prove this fact for the purposes of this paper. The strict concavity of the functions $u_i(\cdot, a)$ guarantees that the solution of (2.3) is unique.

If $\delta_i = 1$, then (2.3) makes no sense. However, one may still obtain a plausible definition of $\xi_i(p)$, though $\xi_i(p)$ is no longer necessarily unique. I will return to this matter in a subsection below.

A <u>monetary equilibrium</u> is a vector $(p,(x_i))$, where p is a price system, (x_i) is an allocation and both satisfy the following conditions.

2.4) (x_i) is a feasible allocation.

2.5) The components of $p_n(a_1,\ldots,a_n)$ are uniformly bounded away from zero and infinity as n and $a_1,\ldots,a_n$ vary.

2.6) $x_i \in \xi_i(p)$, for all i.

Remark: Given a monetary equilibrium $(p,(x_i))$ with positive interest rate, r, it is possible to define an equivalent deflationary equilibrium $(\bar{p},(x_i))$ with no interest payments. One simply deflates the taxes and prices at rate r. $\bar{p}$ is defined by $\bar{p}_n(a_1,\ldots,a_n) = (1+r)^{-n+1}p_n(a_1,\ldots,a_n)$. The tax payments of consumer i in period n are $\bar{\tau}_{in} = (1+r)^{-n+1}\tau_i$. His holdings of money at the beginning of period n turn out to be $\bar{M}_{in}(\bar{p},x_i;\ s_1,\ldots,s_n) = (1+r)^{-n+1}M_{in}(p,x_i;s_1,\ldots,s_n)$.

Marginal Utilities of Money, When $\delta_i < 1$

There are marginal utilities of money associated with any monetary equilibrium $(p,(x_i))$, provided that $r \leqq \delta_i^{-1} - 1$. I will always assume that $r \leqq \delta_i^{-1} - 1$. The marginal utilities of money are simply the multipliers associated with the consumers' budget constraints. The marginal utility of money of consumer i is an infinite vector $\lambda_i = (\lambda_{in}(a_1,\ldots,a_n))$, where each $\lambda_{in}(a_1,\ldots,a_n)$ is a positive number. The marginal utility of money must be distinguished from the marginal utility of expenditure. The vector of marginal utilities of expenditure associated with x_i and p is always denoted by $\alpha_i = (\alpha_{in}\ (a_1,\ldots,a_n))$, where again each $\alpha_{in}(a_1,\ldots,a_n)$ is a positive number. $\alpha_{in}(a_1,\ldots,a_n)$ is defined as follows.

2.7) $\alpha_{in}(a_1,\ldots,a_n)$ is the smallest number t such that

$$\frac{\partial u_i(x_{in}(a_1,\ldots,a_n),a_n)}{\partial z_k} \leqq t\ p_{nk}(a_1,\ldots,a_n)\ ,$$

for all k , with equality if $x_{ink}(a_1,\ldots,a_n) > 0$.

In this formula, $\dfrac{\partial u_i(x,a)}{\partial z_k}$ denotes the partial derivative of $u_i(x,a)$ with respect to the k^{th} component of x .

λ_i and α_i satisfy the following conditions. These apply for all i, n and $a_1,\ldots,a_n$.

2.8) $$\lambda_{in}(a_1,\ldots,a_n) = \max\ [\alpha_{in}(a_1,\ldots,a_n), \delta_i(1+r)E[\lambda_{i,n+1}(a_1,\ldots,a_n, s_{n+1}) \mid s_n = a_n)]$$

2.9) $$\lambda_{in}(a_1,\ldots,a_n) > \delta_i(1+r)E[\lambda_{i,n+1}(a_1,\ldots,a_n,s_{n+1}) \mid s_n = a_n]$$ only if $M_{in}(p,x_i;\ a_1,\ldots,a_n) = 0$.

2.10) $$\lambda_{in}(a_1,\ldots,a_n) > \alpha_{in}(a_1,\ldots,a_n) \text{ only if } x_{in}(a_1,\ldots,a_n) = 0.$$

The marginal utilities of money are uniformly bounded above and away from zero. Before stating this fact, I define some key bounds.

2.11) Let $\bar{\omega} \in R^L_+$ be such that $\sum_{i=1}^{I} \omega_i(a) \ll \bar{\omega}$, for all $a \in A$. There exist $\underline{q}$ and $\bar{q}$ such that $0 \ll \underline{q} \ll Du_i(x,a) \ll \bar{q}$, for all $a \in A$ and for all $x \in R^L_+$ such that $x \leqq \bar{\omega}$.

The existence of $\underline{q}$ and $\bar{q}$ follows from the strict monotonicity of the functions $u_i(\cdot,a)$ and from the continuity of their derivatives. By (2.5), there exist vectors $\underline{p}$ and $\bar{p}$ in R^L_+ such that $0 \ll \underline{p} \leqq p_n(a_1,\ldots,a_n) \leqq \bar{p}$, for all n and $a_1,\ldots,a_n$. The bounds on the marginal utilities of money are as follows.

2.12) $$\min_k \bar{p}_k^{-1}\ \underline{q}_k \leqq \lambda_{in}(a_1,\ldots,a_n) \leqq \max_k \underline{p}_k^{-1}\ \bar{q}_k \text{ , for all } i, n \text{ and } a_1,\ldots,a_n .$$

I now sketch the proof that if $\delta_i < 1$, then marginal utilities of money exist and satisfy (2.8)-(2.10) and (2.12). I here use the methods of Schechtman [66] or of my own paper [9]. Let $\lambda_i^N = (\lambda_{in}^N(a_1,\ldots,a_n))$ be the vector of marginal utilities of money associated with the solution of the problem

$$\max \{E[\sum_{n=1}^{N} \delta_i^{n-1} u_i(x_n(s_1,\ldots,s_n),s_n)] \mid x \in \beta_i(p) \}$$

This problem clearly has a solution, and the λ_i^N satisfy (2.8) - (2.10) and (2.12) . It is not hard to show that the numbers $\lambda_{in}^N(a_1,\ldots,a_n)$ are non-decreasing in N . (See Schechtman [66], p. 224, theorem 1.7, or [9], p. 270, lemma 5.1.) The $\lambda_{in}^N(a_1,\ldots,a_n)$ are uniformly bounded above. This follows from the following facts: $r \leq \delta_i^{-1} - 1$; prices are uniformly bounded away from zero and bounded above (2.5); and utility functions are concave and have continuous finite derivatives. $\lambda_{in}(a_1,\ldots,a_n)$ is simply $\lim_{N\to\infty} \lambda_{in}^N(a_1,\ldots,a_n)$. Passage to the limit in (2.8) - (2.10) and (2.12) proves that λ_i satisfies the same conditions.

Demand When $\delta_i = 1$

I now turn to the question of the definition of demand when $\delta_i = 1$ (and $r = 0$). A program x belongs to $\xi_i(p)$ if $x \in \beta_i(p)$ and if there exists a vector of marginal utilities of money, $\lambda_i = (\lambda_{in}(a_1,\ldots,a_n))$, such that x, λ_i and p satisfy (2.8)-(2.10), with $\delta_i(1+r) = 1$, and also satisfy (2.13) and (2.14) below.

2.13) There exists $\bar{M} > 0$ such that $M_{in}(p,x;a_1,\ldots,a_n) \leqq \bar{M}$, for all n and $a_1,\ldots,a_n$.

2.14) There exist positive numbers $\underline{\lambda}$ and $\bar{\lambda}$ such that $\underline{\lambda} \leqq \lambda_{in}(a_1,\ldots,a_n \leqq \bar{\lambda}$, for all n and $a_1,\ldots,a_n$.

Programs in $\xi_i(p)$ are optimal in a long-run average sense. In fact, if $\bar{x} = (\bar{x}_n(a_1,\ldots,a_n)) \in \xi_i(p)$, then $\bar{x}$ solves the problem

$$2.15) \quad \max\left\{\liminf_{N\to\infty} N^{-1}E[\sum_{n=1}^{N} u_i(x_n(s_1,\ldots,s_n),s_n)] \mid x_i \in \beta_i(p)\right\}.$$

First of all, observe that $\max\{\liminf_{N\to\infty} N^{-1}E[\sum_{n=1}^{N} u_i(x_n(s_1,\ldots,s_n),s_n)] \mid x_i \in \beta_i(p)\} \leqq \liminf_{N\to\infty} N^{-1}\max\{E[\sum_{n=1}^{N} u_i(x_n(s_1,\ldots,s_n),s_n)] \mid x_i \in \beta_i(p)\}$. It follows that it is sufficient to prove that there is a constant B such that

$$2.16) \quad \max\{E[\sum_{n=1}^{N} u_i(x_n(s_1,\ldots,s_n),s_n) \mid x_i \in \beta_i(p)\}$$
$$\leqq E[\sum_{n=1}^{N} u_i(\bar{x}_n(s_1,\ldots,s_n),s_n)] + B, \text{ for all } N.$$

I now prove (2.16). Let λ_i be the vector of marginal utilities of money associated with $\bar{x}$. Clearly, $\bar{x}$ solves the problem
$$\max\{E[\sum_{n=1}^{N} u_i(x_n(s_1,\ldots,s_n),s_n) + \lambda_{iN}(s_1,\ldots,s_N)M_{iN}(p,x;s_1,\ldots,s_N)] \mid x \in \beta_i(p)\}.$$
For this is a finite dimensional maximization problem with a concave objective function. Hence, it is sufficient to satisfy the first order conditions. But these conditions are given by (2.8)-(2.10).

By (2.13) and (2.14), $E[\lambda_{iN}(s_1,\ldots,s_N)M_{iN}(p,x;s_1,\ldots,s_N)] \leqq \bar{\lambda}\,\bar{M}$. It follows that (2.16) is true with $B = \bar{\lambda}\,\bar{M}$. This completes the proof that $\bar{x}$ solves (2.15).

Stationary Equilibrium

Stationary equilibrium is the concept of equilibrium appropriate when the rate of interest equals the common rate of time preference. It is defined as follows.

A stationary consumption plan is a function $x:A \to R_+^L$. A stationary allocation is of the form $(x_i) = (x_1,\ldots,x_I)$, where x_i is a stationary consumption plan. The bundle allocated to consumer i in period n is $x_i(s_n)$. The allocation (x_i) is feasible if $\sum_{i=1}^{I} (x_i(a) - \omega_i(a)) = 0$, for all $a\varepsilon A$.

To every stationary consumption plan x, there corresponds the infinite consumption program $\hat{x} = \hat{x}_n(a_1,\ldots,a_n)$, defined by $\hat{x}_n(a_1,\ldots,a_n) = x(a_n)$. A feasible stationary allocation (x_i) is said to be Pareto optimal if the corresponding allocation $(\hat{x}_i)$ is Pareto optimal.

Remark One can also conceive of stationary consumption plans and prices which would be functions of the infinite history $(\ldots,a_{n-1},a_n)$, and not just of the current state a_n. A stationary monetary equilibrium, if it existed, would be stationary in this sense. For the history would determine the current distribution of money balances.

A stationary price system with deflation rate r is of the form (p,r), where $r \geq 0$ is the deflation rate and $p:A \to R_+^L$ is such that $p(a) > 0$, for all a. The interpretation is that the price vector at time n is $(1+r)^{-n+1}p(s_n)$.

Given $p: A \to R_+^L$, the stationary budget set of consumer i is $\beta_i(p)$ $= \{x:A \to R_+^L \mid \sum_{a\varepsilon A} \pi_a\, p(a) \cdot (x(a) - \omega(a)) \leqq 0\}$, where π is the stationery distribution on A.

The <u>stationary expected utility</u> of consumer i is $U_i(x) = \sum_{a \in A} \pi_a u(x(a),a)$, where x is a stationary consumption program.

The <u>stationary demand</u> for consumer i, given a stationary price system (p,r), is the unique stationary consumption plan $\xi_i(p)$ which solves the problem $\max\{U_i(x) \mid x \in \beta_i(p)\}$.

If the deflation rate equals the consumer's rate of time preference, then $\xi_i(p)$ describes an infinite consumption program which is optimal given a long-run budget constraint. To be precise, suppose that the deflation r is positive. Then, the infinite consumption program $\hat{\xi}_i(p)$ corresponding to $\xi_i(p)$ solves the problem,

$$\max \{E[\sum_{n=1}^{\infty} (1+r)^{-n+1} u_i(x_n(s_1,\ldots,s_n),s_n)] \mid x \text{ is an infinite consumption program which satisfies } E[\sum_{n=1}^{\infty} (1+r)^{-n+1} p(s_n)\cdot(x_n(s_1,\ldots,s_n) - \omega_i(s_n))] \leqq 0\}.$$

Now suppose that there is no deflation. Then, $\xi_i(p)$ solves the problem

$$\max\{\liminf_{N\to\infty} N^{-1} E[\sum_{n=1}^{N} u_i(x_n(s_1,\ldots,s_n),s_n)] \mid x \text{ is an infinite consumption program which satisfies } \liminf_{N\to\infty} N^{-1} E[\sum_{n=1}^{N} p(s_n)\cdot(x_n(s_1,\ldots,s_n) - \omega_i(s_n))] \leqq 0\}.$$

I may now define stationary equilibrium. A <u>stationary equilibrium with deflation rate r</u> is a vector $(p,r,(x_i))$, where (p,r) is a stationary price system with deflation rate r, (x_i) is a feasible stationary allocation and $x_i = \xi_i(p)$, for all i. The rate of deflation r plays no role in the conditions defining a stationary equilibrium. It becomes important only when one interprets the equilibrium.

One may think of consumers in a stationary equilibrium as keeping the marginal utility of money constant. The marginal utility of money of consumer i is the Lagrange multiplier associated with the constraint $\sum_{a \in A} \pi_a p(a) \cdot x(a) \leqq \sum_{a \in A} \pi_a p(a) \cdot \omega(a)$. It is, of course, simply a positive number, λ_i. λ_i together with p determines consumer i's demand. That is, if $\xi_i(p) = (x_i(a))_{a \in A}$, then for each a, $x_i(a)$ is determined by the following set of inequalities.

$$\frac{\partial u_i(x_i(a),a)}{\partial z_k} \leqq \lambda_i p_k(a) \quad \text{, for all } k \text{, with equality if } x_{ik}(a) > 0.$$

3) Assumptions

Here, I collect the assumptions I use. Many have already been mentioned in the previous section.

3.1) $\{s_n\}$ is a stationary Markov chain.

The realization of the random variables s_n belong to the finite set A.

3.2) $\{s_n\}$ is ergodic and has no transient states.

$\pi = (\pi_a)_{a\varepsilon A}$ denotes the unique stationary distribution of $\{s_n\}$. $\pi_a > 0$, for all $a \varepsilon A$.

$\omega_i : A \to R_+^L$ describes the initial endowment of consumer i. I make use of the following conditions on the ω_i.

3.3) For every i, $\omega_i(a) \neq 0$, for some $a \varepsilon A$.

3.4) For every $a \varepsilon A$, $\sum_{i=1}^{I} \omega_i(a) \geqq I(1,\ldots,1)$.

The validity of this assumption depends on the choice of the units of commodities. In more general terms, I have simply assumed that $\sum_{i=1}^{I} \omega_i(a) >> 0$, for all a.

$u_i: R_+^L \times A \to (-\infty,\infty)$ is the utility function of consumer i. I make the following regularity assumptions about the u_i.

3.5) For all i and a, $u_i(\cdot,a)$ is everywhere twice continuously differentiable.

3.6) For every i and a and for every $x \varepsilon R_+^L$, $D^2u_i(x,a)$ is negative definite and $Du_i(x,a) >> 0$, The final assumption has to do with initial money balances and the tax system.

3.7) $\sum_{i=1}^{I} M_{io} = 1$, and for all i, $M_{io} > 0$ and $\tau_i = r\, M_{io}$, where r is the interest rate on money.

The next two assumptions guarantee that a monetary equilibrium exists. They are very strong. The γ appearing in these assumptions is some small positive constant.

3.8) For every $a \in A$, Prob $[\omega_{ik}(s_2) \leqq \gamma$, for $k = 1,\ldots,L \mid s_1 = a] > 0$, where $0 < \gamma < 1$.

3.9) There exists $Q \in R^L$ such that $Q >> 0$ and the following are true. For all i and a, $Du_i(x,a) >> Q$, whenever x is such that $x_k \leqq \gamma\, \underline{q}_k^{-1} \sum_{j=1}^{k} \bar{q}_j$, for all k. Also, for every k, a and i, $\frac{\partial u_i(x,a)}{\partial x_k} << Q_k$, if $x \in R_+^L$ is such that $x \leqq (1,\ldots,1)$ and $x_k = 1$.

The validity of this assumption depends, of course, on the choice of scale for the utility functions.

The next assumption expresses the idea that the Markov process $\{s_n\}$ is sufficiently random.

3.10) There exists a state $a \in A$ for which there are at least three distinct histories which begin and end with a. Each of these histories contains a state which is distinct from a and does not appear in either of the other two histories.

4) Theorems

In all the following theorems, I assume that assumptions (3.1) - (3.7) apply. Assumptions (3.8) and (3.9) are used only in theorems (4.1) and (4.4). Assumption (3.10) is used only in theorems (4.3) and (4.4).

4.1) Theorem Assume that assumptions (3.8) and (3.9) apply. If $\delta_i < (1+r)^{-1}$, for all i, then there exists a monetary equilibrium provided that $\min_i \delta_i$ is sufficiently large.

4.2) Theorem Suppose that $\delta_i < (1+r)^{-1}$, for all i. Let $(p,(x_i))$ be a monetary equilibrium such that $x_{in}(a_1,\ldots,a_n) \neq 0$, for all i, n and $a_1,\ldots,a_n$. Then, the allocation (x_i) is not Pareto optimal.

In the following theorem, Ω denotes the space $\{(\omega_1,\ldots,\omega_I)|\omega_i: A \to \text{int } R^L_+,$ for all $i\}$. If $\omega = (\omega_i) \varepsilon \Omega$, then $\mathcal{E}(\omega)$ denotes the economy with utility functions $u_1,\ldots,u_I$ and with initial endowment functions $\omega_1,\ldots,\omega_I$. Notice that Ω may be viewed as a subset of $R^{L|A|}$, where $|A|$ is the number of points in A. The statement "for almost every $\omega \varepsilon \Omega$" means "for all ω except for ω belonging to a subset of Ω of Lebesque measure zero."

Recall that P_{ab} denotes the transition probability from a to b in A.

4.3) Theorem Assume that $I \geq 2$ and that assumption (3.10) applies. If $\delta_i = (1+r)^{-1}$, for all i, then $\mathcal{E}(\omega)$ has no monetary equilibrium, for almost every $\omega \varepsilon \Omega$.

4.4) Theorem Assume that $I \geq 2$ and that assumptions (3.8) - (3.10) apply. Assume also that $\delta_i = (1+r)^{-1}$, for all i, where $r > 0$. Then for almost every $\omega \varepsilon \Omega$, the following is true. Let r_k be such that $0 \leqq r_k < r$, where $k = 1,2,\ldots$. For each k, let $(p^k,(x_i^k))$ be a monetary equilibrium for $\mathcal{E}(\omega)$ with interest rate r_k. If $\lim_{k\to\infty} r_k = r$, then $\lim_{k\to\infty} p_n^k(a_1,\ldots,a_n) = 0$, uniformly with respect to n and $a_1,\ldots,a_n$.

4.5) <u>Theorem</u> If $\delta_i = \delta \leqq 1$, for all i, then there exists a stationary equilibrium with deflation rate $\delta^{-1}-1$. Such an equilibrium is Pareto optimal.

An example given in section 13 illustrates the need for the special assumption in theorem (4.2) and for assumption (3.10) in theorems (4.3) and (4.4).

5. Review of the Literature

I review briefly the literature on the optimal quantity of money and on the relation of monetary theory to equilibrium theory. Not all of this literature is directly related to my own work. However, my own work falls in this general area. Since the literature is large and confusingly diverse, it seems worthwhile to review it. I first deal with the literature on the optimum quantity of money.

It seems to be impossible to attribute the idea of the optimum quantity of money to any one author. It must have been in the air for some time. In a paper of 1953 [27], Friedman discusses the fact that inflation leads consumers to economize unnecessarily on cash balances. This idea was formalized by Bailey [4] in a paper appearing in 1956. The idea was empirically tested by Cagan [14] in a paper of the same year. In a paper of 1963 ([62], p. 535), Samuelson mentions the idea that the real rate of interest on money should be positive, at least in idealized models. In a paper of 1963 ([43], p. 113), Harry Johnson remarked that money should earn the same real rate of interest as other assets. Samuelson developed his ideas somewhat in two papers of 1968 [63] and 1969 [64]. Tobin ([76], p. 846 discussed the same idea in a paper published in 1968. Both Samuelson and Tobin argued that from the point of view of efficiency, economic agents should be saturated with money balances. Hence, money should bear a real rate of return high enough to remove all incentive to economize on it. This idea was discussed at length by Milton Friedman in "The Optimum Quantity of Money" [28], which appeared in 1969.

There was a long debate about whether money should bear interest in reality. Friedman [28] advanced this idea. Harry Johnson [45], Tsiang [77], Clower

[18,19] and Phelps [58, pp. 201-220] and [59] made important contributions to the debate. Johnson was mainly concerned about substitution between money and interest bearing assets. Since money does not bear interest, consumers economize on it in order to buy other assets. Tsiang expresses the view that if money bore interest at a rate equal to the general rate of return on capital, then it would tend to displace all other assets. Clower's main point is that one cannot make practical recommendations about monetary policy in terms of models which do not capture those aspects of reality which make money useful. Phelps related the question of the optimal level of inflation (or deflation) to the theory of optimal taxation. He pointed out that inflation is a form of tax, so that there is a trade-off between dead weight losses caused by inflation and those caused by other taxes.

The issues raised by Johnson and Tsiang cannot be discussed in terms of my model, since money is the only asset in my model. Nor can I discuss the theory of optimal taxation, for I permit lump-sum taxes. My model is, of course, open to Clower's criticism. There are no transactions cost, no information problems and so on which could explain why money exists. But I do not make practical recommendations either.

I note in passing that in Inflation Policy and Unemployment Theory, Phelps mentions the idea that consumers would have an insatiable demand for liquidity if the real rate of interest equaled the rate of time preference (see [58], pp. 181-2). This is, of course, one of the main ideas of this paper.

The theory of the optimal quantity of money is related to the literature on the optimal rate of growth of the money supply from the point of view of growth theory. This is a vast literature. See, for example, Johnson [44], Levhari and Patinkin [48], Marty [50], Sidrauski [68] and Tobin [74,75]. This literature is surveyed in Stein [73]. One of the main preoccupations of the literature is the effect of the real rate of interest on saving and investment. Most of the discussion is in terms of Keynesian and Solow growth models.

A revival of this literature was initiated recently by Brock [11, 12]. He formulated the problem in terms of a mathematically rigorous, infinite horizon growth model. All consumers are identical and live forever. Utility is additively separable with respect to time. There is no uncertainty. Consumers have perfect foresight and maximize the discounted infinite sum of present and future utilities. The utility of each period depends on consumption, leisure and real balances. Brock's model of a single consumer is similar to my own, except that Brock puts money directly in the utility function. (In my model, uncertainty and the heterogeneity of consumers are what give money value.) The questions Brock asks are different from my own and also from those posed in the earlier literature on growth and money. His primary concern is with uniqueness of the perfect foresight equilibrium. He also studies the response of the model to anticipated future changes in the nominal supply of money. He discusses the optimal quantity of money and proves that it is infinite if the marginal utility of money is not eventually zero. (The marginal utility of money in Brock's money is measured directly by the utility function.) Calvo [15] studies the uniqueness of equilibrium in models similar to that of Brock. Calvo allows money to appear in the production function.

I now turn to the enormous literature on models which describe in detail how and why people use money and why it is socially useful to do so.

The early papers of Baumol [5] and Tobin [74] use an inventory theoretic model to explain why people hold money rather than interest bearing assets. Money is the sole means of payment and each purchase or sale of an interest bearing asset involves a fixed transaction cost.

Clower and Howitt [20] analyze an inventory theoretic model of consumer behavior in a model with both transaction costs and inventories of goods. They find that because of delicate questions of timing, average cash balances can depend in a discontinuous way on the parameters of the consumer's problem.

Feige and Parkin [24], Niehans [53], and Perlman [56] also introduce commodity inventories into the story told by Baumol and Tobin. They discuss the optimal quantity of money in a semi-formal general equilibrium framework. That is, they give general equilibrium interpretations of the first order conditions of consumer equilibrium, but they do not prove that equilibria exist. The work of Feige and Parkin and of Perlman has led to some controversy. See Feige, Parkin, Avery and Stones [25], Perlman [57] and Russel [60].

The model of consumer behavior most closely related to my own is that of Foley and Hellwig [26]. In their model, as in mine, money is needed only for self-insurance. Consumers live forever and maximize the expected value of a discounted infinite stream of utilities. Utility in each period depends on consumption and leisure. Consumers fluctuate between being employed and involuntarily unemployed. They use money to compensate for the resulting fluctuations in income. The model is of partial equilibrium in that it is a model of a single consumer. Foley and Hellwig demonstrate that the probability distribution of money holdings converges to a long run stationary distribution.

There is a large literature which analyzes in detail the role of money in transactions. Authors in this area try to show why exchange involving money is simpler and cheaper than barter. They also look for the essential difference between money and other goods. Works in this area include Brunner and Meltzer [13], Niehans [51,52], Ostroy [54], Ostroy and Starr [55], Saving [65] and Starr [70].

The papers just referred to explain why individuals would find money convenient if others were willing to accept it. They also explain why money is socially useful. But they do not describe a rigorous model in which it would be completely rational for every individual to accept and use money. The problem is that if one thinks in terms of a finite horizon model, money would have no value in the last period. By backward induction, it would have no value in any period. In order to bypass this problem, one must think of equilibrium as an ongoing process, as I do in this paper. Shubik does so as well in his game-theoretic approach to monetary theory. (See, for instance, [67].) Jones [46] treates equilibrium as an ongoing process in a model which includes costs of finding a trading partner. His equilibria may be interpreted as Nash equilibria. He also describes a process which leads in an evolutionary way to the adoption of a medium of exchange.

Samuelson's consumption-loan model [61] is another example of a model of an ongoing process in which money has value. This model has been much studied. See, for instance, Gale [29], Grandmont and Laroque [32], Cass, Okuno and Zilcha [16] and Wallace [78].

There have been many rigorous, finite horizon general equilibrium models in which money is given value by imposing somewhat artificial terminal conditions. These works include Hahn [39], Heller [40], Heller and Starr [41], Kurz [47], Sontheimer [69] and Starr [71]. All of these papers, except that of Starr, include transaction costs. Kurz's model allows barter and monetary trade to occur simultaneously with distinct transaction costs.

Another approach to giving money value is simply to assume that consumers believe it will have value in the terminal period. That is, the value of money is a consequence of consumer expectations. This is the approach taken by Grandmont [30]. Drandrakis [23] seems to have had the same approach in mind in his early work on temporary equilibrium theory. Grandmont proves the existence of a temporary equilibrium with a positive price for money in a two period model in which consumers believe that the real value of money in the second period is bounded away from zero. These beliefs are not necessarily rational. In my model, money also has value only because consumers believe it will be valuable in the future. Because I use an infinite horizon, I am able to prove that these beliefs are rational.

Yet another way to obtain equilibrium with a positive price for money is to use the Clower constraint in an infinite horizon model with rational expectations. The Clower constraint is the requirement that goods can be exchanged only for money. It was proposed by Clower in 1967 [17]. The Clower constraint serves to make money useful. The infinite horizon does away with the problem of the value of money in the terminal period. Grandmont and Younes [33,34], Hool [42], Lucas [49] and Wilson [79] all take this approach. Grandmont and Younes prove the existence of a stationary monetary equilibrium and analyze the optimal quantity of money. Hool solves a difficulty met by Grandmont and Younes. Wilson analyzes in detail the nature of the equilibria in his model.

The Clower constraint has a curious interpretation. In monetary models which specify transaction costs, it is usually automatic that goods can be exchanged only for money (or for other goods). This is so in the papers of Hahn [39], Heller [40], Heller and Starr [41] and Kurz [47]. However in models which do not specify transactions, the Clower constraint must be interpreted as a payments lag. It takes one period for money to pass from buyer to seller.

My work is closely related to the literature on temporary equilibrium. Both monetary and stationary equilibrium, as I define them, are forms of temporary equilibrium. Unlike many models of temporary equilibrium, my models have rational expectations. The literature on temporary equilibrium has been surveyed by Grandmont [31].

6) Lemmas

The lemmas of this section express relations between the marginal utility of expenditure and equilibrium prices. I assume throughout that assumptions (3.1) - (3.10) apply.

Let $\mathcal{E}$ (a), for a ε A, be the pure trade economy corresponding to state a ε A. That is, $\mathcal{E}$ (a) has I consumers and L commodities. The utility function of the i^{th} consumer is $u_i(\cdot, a)$: $R^L_+ \to (-\infty,\infty)$. His initial endowment is $\omega_i(a) \; \varepsilon \; R^L_+$.

An equilibrium with transfer payments for $\mathcal{E}$ (a) is of the form $(q,(y_i))$, where $q \; \varepsilon \; R^L_+$ is the price vector and (y_i) is a feasible allocation for $\mathcal{E}$ (a). These must satisfy $q > 0$ and $u_i(y_i,a) = \max\{u_i(y,a) | y \; \varepsilon \; R^L_+ \text{ and } q\cdot y \leqq q\cdot y_i\}$, for all i. The transfer payment of consumer i is $q \cdot (\omega_i(a) - y_i)$. Clearly, if every consumer's transfer payment is zero, then the equilibrium is in fact a Walrasian equilibrium in the usual sense.

The marginal utility of expenditure of consumer i associated with $(q,(y_i))$ is defined to be the Lagrange multiplier, α_i, associated with the problem $\max \{u_i(y,a) | \; y \; \varepsilon \; R^L_+ \text{ and } q\cdot y \leqq q\cdot y_i\}$. That is

6.1) α_i is the smallest number t such that $\dfrac{\partial u_i(y_i,a)}{\partial x_k} \leqq t\, q_k$,

for all k with equality if $y_{ik} > 0$.

Throughout this section, $\underline{q}$ and $\bar{q}$ are as in (2.11).

6.2) Lemma $(\max_i \alpha_i^{-1})\, \underline{q} << q << (\max_i \alpha_i^{-1})\bar{q}$, whenever $(q,(y_i))$ is on equilibrium with transfer payments for $\mathcal{E}$ (a), for some $a\varepsilon A$, and where (α_i) is the vector of marginal utilities of expenditure associated with $(q,(y_i))$.

<u>Proof</u> If $(q_1(y_i))$ is an equilibrium with transfer payments for $\mathcal{E}(a)$, then $0 \leq y_i \leq \bar{\omega}$, so that $\underline{q} << Du_i(y_i,a) << \bar{q}$, for all i.

By the definition of α_j, $\alpha_j q \geqq Du_j(y_j,a)$, so that $q >> (\max_i \alpha_i^{-1})\underline{q}$. This proves the first inequality.

$\sum_{i=1}^{I} y_i = \sum_{i=1}^{I} \omega_i(a) >> 0$, so that for each $k = 1,\ldots,L$, $y_{ik} > 0$, for some i. (Here I have used assumption 3.4.) For this i, $\frac{\partial u_i(y_i,a)}{\partial x_k} = \alpha_i q_k$, so that $q_k < \alpha_i^{-1}\bar{q}_k$. This proves the second inequality.

Q.E.D.

6.3) <u>Lemma</u> Let $(q, (y_i))$ be an equilibrium with transfer payments for $\mathcal{E}(a)$, for some a, and let (α_i) be the vector of associated marginal utilities of expenditure. Then, $\max_i \alpha_i < b \min_i \alpha_i$, where $b = \max_k \underline{q}_k^{-1}\bar{q}_k$.

<u>Proof</u>. It follows from the definition of α_i that for each i, $\frac{\partial u_i(y_i,a)}{\partial x_k} = \alpha_i q_k$, for some k. But $\bar{q}_k > \frac{\partial u_i(y_i,a)}{\partial x_k}$. Also, by the previous lemma, $q_k > (\min_j \alpha_j)^{-1}\underline{q}_k$. Putting these inequalities together, I obtain $\bar{q}_k > \alpha_i(\min_j \alpha_j)^{-1}\underline{q}_k$. It follows that $\alpha_i < \underline{q}_k^{-1}\bar{q}_k \min_j \alpha_j \leqq b(\min_j \alpha_j)$.

Q.E.D.

An <u>equilibrium for $\mathscr{E}$(a) with transfer payments and marginal utilities of money</u> is defined to be $(q,(y_i), (\lambda_i))$, where $(q,(y_i))$ is an equilibrium with transfer payments for $\mathscr{E}$(a) and where $\lambda_i \geqq \alpha_i$, for all i, with equality if $y_i > 0$. Here (α_i) is the vector of marginal utilities of expenditure associated with $(q, (y_i))$.

6.4) <u>Lemma</u> Let $(q,(y_i), (\lambda_i))$ be an equilibrium for $\mathscr{E}$(a) with transfer payments and marginal utilities of money, where $a \in A$. Then,

$(\max_i \lambda_i^{-1}) \underline{q} << q << b(\max_i \lambda_i^{-1}) \bar{q}$, where b is as in lemma 6.3.

<u>Proof</u> The first inequality follows trivially from lemma 6.1, since $\lambda_i \geqq \alpha_i$, for all i.

In order to prove the second inequality, let i be such $\alpha_i = \min_j \alpha_j$. By lemma 6.2, $q << \alpha_i^{-1} \bar{q}$. By assumption 3.4, there exists j such that $y_j > 0$. Then, $\lambda_j = \alpha_j$, and by lemma 6.3, $\alpha_j \leqq b\alpha_i$. Putting these inequalities together, I obtain $q << b\lambda_j^{-1} \bar{q}$.

7) Proof of Theorem 4.1

The first step of the proof is to truncate the economy at the N^{th} period, artificially giving money utility in the N^{th} period. I use a standard fixed point argument to prove that the truncated economy has an equilibrium in which money has price one in every period. I then prove that the N-period equilibrium prices are uniformly bounded away from zero and infinity. This fact allows me to apply a Cantor diagonal argument in order to obtain a monetary equilibrium in the limit as N goes to infinity. The hard part of the proof is the demonstration that N-period equilibrium prices are bounded above and bounded away from zero. Prices are bounded above because money is needed for self-insurance and because high prices make the real stock of money low. Prices are bounded away from zero because there is a limit to the level of real balances that consumers will hold. This limit exists because the interest rate is less than consumers' rates of time preference.

The Finite Horizon Economy

I truncate the economy at period N. In the truncated economy, it is sufficient to deal with N-period price systems and programs. These specify prices and consumption bundles in the first N periods only. An N-period allocation (x_i) is feasible if $\sum_{i=1}^{I}(x_{in}(a_1, \ldots, a_n) - \omega_i(a_n)) = 0$, for all histories $a_1, \ldots, a_n$ and for all n such that $1 \leqq n \leqq N$.

Given an N-period price system p, $\xi_i^N(p)$ denotes the unique N-period program which solves the following maximization problem.

7.1) $\max\{E[\sum_{n=1}^{N} \delta_i^{n-1} u_i(x_n(s_1, \ldots, s_n), s_n) + \delta_i^{N-1} M_{iN}(p, x; s_1, \ldots, s_N)] | x$

is an N-period consumption program and $M_{in}(p, x; a_1, \ldots, a_n) \geqq 0$, for all histories $a_1, \ldots, a_n$ and for $1 \leqq n \leqq N\}$

Notice that money is given utility in the last period.

An N-period monetary equilibrium is of the form $(p, (x_i))$, where p is an N-period price system and each x_i is an N-period program. These must satisfy the following conditions.

7.2) $x_i = \xi_i^N(p)$, for all.

7.3) (x_i) is a feasible allocation.

7.4) $p_n(a_1, \ldots, a_n) > 0$, for all histories $a_1, \ldots, a_n$ and for all n such that $1 \leqq n \leqq N$.

7.5) <u>Lemma</u> For each $N \geqq 1$, there exists an N-period monetary equilibrium.

<u>Proof</u> For the purposes of this proof, I allow money to have a different price in every period. The L + 1st component of the vector of prices in any one period corresponds to the price of money. Price vectors vary over

$$\Delta = \prod_{n=1}^{N} \prod_{a_1, \ldots, a_n} \Delta^L, \text{ where } \Delta^L = \{q \in R_+^{L+1} | \sum_{i=1}^{L+1} q_i = 1\}.$$ If $q \in \Delta$, I write

$q = (q_n(a_1, \ldots, a_n))$.

I now add a vector $\vec{\epsilon} = (\epsilon, \ldots, \epsilon)$ to the initial endowment of each consumer in every state of the world, where $\epsilon > 0$. That is, I assume that the initial endowment of consumer i in state a is $\omega_i(a) + \vec{\epsilon}$, for all i and a. I also give each consumer ϵ units of money in each period. Later, I will let ϵ go to zero.

The plan of consumer i is denoted by (x_i, M_i), where $x_i = (x_{in}(a_1, \ldots, a_n))$ and $M_i = (M_{in}(a_1, \ldots, a_n))$.

I truncate the consumption sets as follows. Let $\bar{\omega} \epsilon R_+^L$ be such that $\sum_{i=1}^{L} \omega_i(a) + \vec{\epsilon} << \bar{\omega}$, for all $a \epsilon A$. I forbid each consumer to demand more than $\bar{\omega}_j$ units of good j, for all j, and to hold more than two units of money. In precise terms, I truncate consumer i's budget set to be the following compact set, given $q \epsilon \Delta$.

$\beta_i^T(q, \epsilon) = \{(x_i, M_i) | 0 \leqq x_{in}(a_1, \ldots, a_n) \leqq \bar{\omega}, 0 \leqq M_{in}(a_1, \ldots, a_n) \leqq 2$ and $q_n(a_1, \ldots, a_n) \cdot (x_{in}(a_1, \ldots, a_n), M_{in}(a_1, \ldots, a_n)) \leqq q_n(a_1, \ldots, a_n) \cdot (\omega_i(a_n) + \vec{\epsilon}, (1+r)M_{i,n-1}(a_1, \ldots, a_{n-1}) + \epsilon - \tau_i)$, for all histories $a_1 \ldots, a_n$ and for $n = 1, \ldots, N\}$. It follows from assumption 3.7 that $\beta_i^T(q,\epsilon)$ is non-empty.

I let $\xi_i^T(q)$ be the set of solutions to the problem,

$$\max\{E[\sum_{n=1}^{N} \delta_i^{n-1} u_i(x_{in}(s_1, \ldots, s_n), s_n) + \delta_i^{N-1} M_{iN}(s_1, \ldots, s_N)] | (x_i, M_i) \epsilon \beta_i^T(q, \epsilon)\}.$$

Since a consumer begins every period with a positive amount of every good, including money, it follows that $\xi_i^T(q)$ is a continuous function of q.

The monotonicity of u_i implies that

7.6) if $\xi_i^T(q) = (x_i, M_i)$ then $q_n(a_1, \ldots, a_n) \cdot (x_{in}(a_1, \ldots, a_n), M_{in}(a_1, \ldots, a_n))$
$= q_n(a_1, \ldots, a_n) \cdot (\omega_i(a_n) + \vec{\epsilon}, (1+r)M_{i, n-1}(a_1, \ldots, a_{n-1}) + \epsilon - \tau_i)$, for all i, n, and $a_1, \ldots, a_n$.

I define the aggregate excess demand function, Z(q), as follows. Let $q \epsilon \Delta$

and let $\xi_i^T(q) = (x_i, M_i)$, for each i. Then $Z(q) = (Z_n(q; a_1, \ldots, a_n))$, where $Z_n(q; a_1, \ldots, a_n) = (\sum_{i=1}^{I} (x_{in}(a_1, \ldots, a_n) - \omega_i(a_n) - \vec{\varepsilon}, M_{in}(a_1, \ldots, a_n) - (1+r)M_{i,n-1}(a_1, \ldots, a_{n-1}) - \varepsilon + \tau_i)$.

(7.6) implies that

$$q_n(a_1, \ldots, a_n) \cdot Z_n(q; a_1, \ldots, a_n) = 0 \text{, for } q \in \Delta \text{ and}$$
$$\text{for all } n \text{ and } a_1, \ldots, a_n.$$

This is the version of Walras' law appropriate for the price space Δ. Hence, by a slight extension of the standard fixed point argument, there is $q \in \Delta$ such that $Z(q) \leqq 0$. Let $\xi_i^T(q) = (x_i, M_i)$, for i=1, ..., I. (The standard fixed point argument may be found in Debreu [21], p. 82, or in Arrow and Hahn [3], p. 28. My proof is much like that of Hahn in [37].) I call $(q, (x_i), (M_i))$ an ε-modified equilibrium.

I now let ε_k, k=1, 2, be a sequence of positive numbers converging to zero. For each k, let $(q^k, (x_i^k), (M_i^k))$ be an ε_k - modified equilibrium. By passing to a subsequence, I may assume that $\lim_{k\to\infty}(q^k, (x_i^k), (M_i^k)) = (q, (x_i), (M_i))$. I will show that $q \gg 0$. Let $p = (p_n(a_1, \ldots, a_n))$ be defined by

$$p_n(a_1, \ldots, a_n) = q^{-1}_{n, L+1}(a_1, \ldots, a_n)(q_{n1}(a_1, \ldots, a_n), \ldots, q_{nL}(a_1, \ldots, a_n)). \tag{7.7}$$

It will be seen that $(p, (x_i))$ is an N-period monetary equilibrium.

Before proving that $q \gg 0$, I collect some facts.

First of all,

7.8) $$\sum_{i=1}^{I} (x_{in}(a_1, \ldots, a_n) - \omega_i (a_n)) \leqq 0 \quad \text{and}$$

$$\sum_{i=1}^{I} [M_{in}(a_1, \ldots, a_n) - (1+r)M_{i,n-1}(a_1, \ldots, a_{n-1}) + \tau_i] \leqq 0$$

for all n and $a_1, \ldots, a_n$.

It follows from (7.8) that $\sum_{i=1}^{I} M_{in}(a_1, \ldots, a_n) \leqq 1$, for all n and $a_1 \ldots, a_n$. It is easy to see that $\sum_{i=1}^{I} M^k_{in}(a_1, \ldots, a_n) \geqq 1$, for all k, n and $a_1, \ldots, a_n$. Hence, $\sum_{i=1}^{I} M_{in}(a_1, \ldots, a_n) \geqq 1$. In conclusion,

7.9) $$\sum_{i=1}^{I} M_{in}(a_1, \ldots, a_n) = 1, \text{ for all n and } a_1, \ldots, a_n.$$

Next, I observe that

7.10) $$q_n(a_1, \ldots, a_n) \cdot [\sum_{i=1}^{N} (x_{in}(a_1, \ldots, a_n) - \omega_i(a_n)),$$

$$\sum_{i=1} (M_{in}(a_1, \ldots, a_n) - (1+r)M_{i,n-1}(a_1, \ldots, a_{n-1}) + \tau_i)] \leqq 0,$$

for all n and $a_1, \ldots, a_n$.

I now prove that $q_n(a_1, \ldots, a_n) \gg 0$, for all n and $a_1, \ldots, a_n$. The proof is by backwards induction n.

Let n = N and fix $a_1, \ldots, a_N$. I first show that $q_{N,L+1}(a_1, \ldots, a_N) > 0$. Suppose that $q_{N,L+1}(a_1, \ldots, a_n) = 0$. Then, $q_{Nk}(a_1, \ldots, a_N) > 0$, for some $k \leqq L$. There is some i such that $\omega_{ik}(a_N) > 0$ (by assumption 3.4). Then,

$q_N(a_1, \ldots, a_N)\cdot(\omega_i(a_N), (1+r)M_{i,N-1}(a_1, \ldots, a_{N-1}) - \tau_i) > 0$. It follows easily that

7.11) $(x_{iN}(a_1, \ldots, a_N), M_{iN}(a_1, \ldots, a_N))$ solves the problem,

$$\max\{u_i(x, a_N) + M \mid q_N(a_1, \ldots, a_N)\cdot(x,M) \leqq q_N(a_1, \ldots, a_N)\cdot(\omega_i(a_N),$$
$$(1+r)M_{i,N-1}(a_1, \ldots, a_{N-1}) - \tau_i) \text{ and } 0 \leqq x \leqq \bar{\omega} \text{ and } 0 \leqq M \leqq 2\}.$$

Since $q_{N,L+1}(a_1, \ldots, a_N) = 0$, it follows that $M_{iN}(a_1, \ldots, a_N) = 2$. This contradicts (7.9). Hence, $q_{N,L+1}(a_1, \ldots, a_N) > 0$.

By (7.9), $M_{i,N}(a_1, \ldots, a_N) > 0$, for some i. For this i, (7.11) is true. It follows at once from the monotoncity of u_i that if $q_{N,k}(a_1, \ldots, a_N) = 0$, then $x_{iNk}(a_1, \ldots, a_N) = \bar{\omega}_k > \sum_{j=1}^{I} \omega_j(a_N)$. This contradicts (7.8). Hence, $q_N(a_1, \ldots, a_N) \gg 0$.

Now suppose by induction that $q_{n+k}(a_1, \ldots, a_{n+k}) \gg 0$, for all histories $a_1, \ldots, a_{n+k}$, and for $k = 1, \ldots, N-n$. It follows easily that for each i and $a_1, \ldots, a_n$, x_i solves the problem, $\max\{E[\sum_{k=1}^{N-n} \delta_i^{n+k-1} u_i(\hat{x}_{n+k}(s_1, \ldots, s_{n+k}), s_{n+k})$
$+ \delta_i^{N-1} \hat{M}_N(s_1, \ldots, s_N) | s_n = a_n] \mid (\hat{x}, \hat{M}) \in \beta_i^T(q, 0)$ and $\hat{M}_n(a_1, \ldots, a_n) = M_{in}(a_1, \ldots, a_n)\}$
That is, x_i solves the maximization problem for periods n+1 and beyond. It follows that money is useful in period n and hence I may repeat the argument just made in order to prove that $q_n(a_1, \ldots, a_n) \gg 0$, for all $a_1, \ldots, a_n$. This completes the proof that $q \gg 0$.

I must now show that $(p, (x_i))$ is an N-period monetary equilibrium; where p is defined by (7.7). It follows from what has been said that $x_i = \xi_i^N(p)$. The feasibility of (x_i) follows from (7.8), (7.10) and the fact that $q \gg 0$.

Clearly, $p \gg 0$, so that $(p, (x_i))$ satisfies conditions (7.2) - (7.4) of the definition of any N-period monetary equilibrium.

Q.E.D.

Remark The proof of lemma 7.5 made no use of assumptions (3.1) - (3.3), (3.8) or (3.9). The proof applies even if the utility functions are only continuous, strictly concave and strictly monotone.

Boundedness From Above

I next prove that prices in N-period monetary equilibria are uniformly bounded from above. It now becomes important to keep track of marginal utilities of money. If $(p, (x_i))$ is an N-period equilibrium, the marginal utility of money of consumer i associated with $(p, (x_i))$ is a vector $\lambda_i = (\lambda_{in}(a_1, \ldots, a_n))$. Similarly, let $\alpha_i = (\alpha_{in}(a_1, \ldots, a_n))$ be the vector of marginal utilities of expenditure of consumer i associated with $(p, (x_i))$. $\alpha_{in}(a_1, \ldots, a_n)$ is defined by (2.7). λ_i satisfies (7.12) - (7.14) below, for all histories $a_1, \ldots, a_n$ and for $n = 1, \ldots, N$.

7.12) $\lambda_{iN}(a_1, \ldots, a_N) = \max(\alpha_{iN}(a_1, \ldots, a_N), 1)$. If $n < N$, then

$$\lambda_{in}(a_1, \ldots, a_n) = \max [\alpha_{in}(a_1, \ldots, a_n), \delta_i(1+r)E(\lambda_{i,n+1}(a_1, \ldots, a_n, s_{n+1}) | s_n = a_n)]$$

7.13) $\lambda_{iN}(a_1, \ldots, a_N) > 1$ only if $M_{iN}(p, x_i; a_1, \ldots, a_N) = 0$.

If $n < N$, then $\lambda_{in}(a_1, \ldots, a_n) > \delta_i(1+r)E[\lambda_{i,n+1}(a_1, \ldots, a_n, s_{n+1}) | s_n = a_n]$ only if $M_{in}(p, x_i; a_1, \ldots, a_n) = 0$

7.14) For all n, $\lambda_{in}(a_1, \ldots, a_n) > \alpha_{in}(a_1, \ldots, a_n)$ only if $x_{in}(a_1, \ldots, a_n) = 0$.

This subsection is devoted to the proof of the following.

7.15) Lemma There exist $\bar{p} \epsilon R^L_+$, $\underline{\lambda} > 0$ and $\underline{\delta}$ such that $0 < \underline{\delta} < (1+r)^{-1}$ and the following are true. Let $(p, (x_i))$ be any N-period monetary equilibrium and let (λ_i) be the associated vector of marginal utilities of money. If $\delta_i \geqq \underline{\delta}$, for all i, then $p_n(a_1, \ldots, a_n) \leqq \bar{p}$ and $\lambda_{in}(a_1, \ldots, a_n) \geqq \underline{\lambda}$, for all i, n and all histories $a_1, \ldots, a_n$.

In order to prove this lemma, I need some preliminary lemmas, which exploit assumptions 3.4 and 3.9. The economies $\mathcal{E}(a)$ appearing in the next lemma were defined at the beginning of section 6. γ is as in assumptions 3.8 and 3.9.

7.16) Lemma Let $(q,(y_i))$ be a Walrasian equilibrium for $\mathcal{E}(a)$, for any $a \epsilon A$, and let (α_i) be the associated vector of marginal utilities of expenditure, defined by (6.1). If i and a are such that $\omega_{ik}(a) \leqq \gamma$, for all k, then $\alpha_i > \max \{\alpha_j | j \text{ is such that } q \cdot y_j \geqq \sum_{k=1}^{L} q_k\}$.

Remark: There exists j such that $q.y_j > \sum_{k=1}^{L} q_k$, for by assumption 3.4, $q \cdot \sum_{i=1}^{I} y_j \geqq q \cdot \sum_{i=1}^{I} \omega_i(a) \geqq I \sum_{k=1}^{L} q_k$.

In order to prove the above lemma, I make use of the following fact.

7.17) For each i and a, $u_i(x,a) \geqq u_i((1, \ldots, 1), a)$ implies that $\frac{\partial u_i(x,a)}{\partial x_k} < Q_k$, for some k such that $x_k > 0$. Here, $Q = (Q_1, \ldots, Q_k)$ is as in assumption 3.9.

This fact follows from assumption 3.9 and from the concavity of $u_i(x,a)$.

<u>Proof of Lemma 7.16</u>

By assumption, $\omega_{ik}(a) \leqq \gamma$, for all k. It follows $\underline{q}_k y_{ik} \leqq (\min_j \alpha_j) q_k y_{ik}$ $\leqq (\min_j \alpha_j) q \cdot y_i = (\min_j \alpha_j) q \cdot \omega_i(a) \leqq \bar{q} \cdot \omega_i(a) \leqq \gamma \sum_{m=1}^{L} \bar{q}_m$, for all k. Here, I have made use of lemma 6.2. In summary, $y_{ik} \leqq \gamma \underline{q}_k^{-1} \sum_{m=1}^{L} \bar{q}_m$, for all k. Therefore, by assumption 3.9,

$$7.18) \qquad \frac{\partial u_i(y_i,a)}{\partial x_k} > Q_k, \qquad \text{for all k.}$$

Let j be such that $q \cdot y_j \geqq \sum_{m=1}^{L} q_m = q \cdot (1,\ldots,1)$. Then, $u_j(y_j,a)$ $\geqq u_j((1,\ldots,1),a)$, so that by (7.17)

$$7.19) \qquad Q_k > \frac{\partial u_j(y_j,a)}{\partial x_k}, \qquad \text{for some k such that } y_{jk} > 0.$$

By the definition of α_i and α_j, (see (6.1)),

$$7.20) \qquad \alpha_i q_k \geqq \frac{\partial u_i(y_i,a)}{\partial x_k} \quad \text{and} \quad \frac{\partial u_j(y_j,a)}{\partial x_k} = \alpha_j q_k,$$

where k is as in (7.19).

Putting (7.18) - (7.20) together, it follows that $\alpha_i > \alpha_j$. This proves the lemma.

Q.E.D.

The next lemma says that lemma 7.16 holds uniformly.

7.21) Lemma There exists $\varepsilon > 0$ such that the following is true. Let $(q, (y_i))$ be any equilibrium with transfer payments for $\mathcal{E}(a)$, where $a \in A$. Let (α_i) be the vector of associated marginal utilities of expenditure. Suppose that $|q \cdot (y_i - \omega_i(a))| \leqq \varepsilon \max_j \alpha_j^{-1}$, for all i. Then, $\alpha_i \geqq (1+\varepsilon) \max\{\alpha_j | j$ is such that $q \cdot y_j \geqq \sum_{k=1}^{L} q_k\}$, for any i such that $\omega_{ik}(a) \leqq \gamma$, for all k.

Proof If ε did not exist, then for some $a \in A$, there would exist a sequence $(q^k, (y_i^k))$, k = 1, 2, ..., of equilibria with transfer payments for $\mathcal{E}(a)$, such that

$$7.22) \quad |q^k \cdot (\omega_i(a) - y_i^k)| \leqq k^{-1} \max_j (\alpha_j^k)^{-1}, \text{ for all i and}$$

$$7.23) \quad \alpha_i^k < (1+k^{-1})\alpha_j, \text{ where } \quad q^k \cdot y_j^k \geqq \sum_{n=1}^{L} q_n^k \text{ and } \omega_{ik}(a) \leqq \gamma,$$

for all k.

Here, (α_i^k) is the vector of marginal utilities of expenditure associated with $(q^k, (y_i^k))$.

I now apply a compactness argument. Without loss of generality, I may assume that $\min_j \alpha_j^k = 1$, for I may replace q^k by $(\min_j \alpha_j^k)q^k$. Since $\min_j \alpha_j^k = 1$, lemma 6.2 implies that $\underline{q} \ll q^k \ll \bar{q}$, for all k. The set of feasible allocations for $\mathcal{E}(a)$ is compact. Hence, I may choose a convergent subsequence of equilibria. The limit, $(q, (y_i))$ is an equilibrium for $\mathcal{E}(a)$ with transfer payments. The corresponding subsequence of (α_i^k) converges to (α_i), where (α_i) is the vector of marginal utilities of expenditure associated with $(q, (y_i))$. Passing to the limit in (7.22), I obtain $q \cdot (\omega_i(a) - y_i) = 0$, so that $(q,(y_i))$ is a Walrasian equilibrium for $\mathcal{E}(a)$. Passing to the limit in (7.23), I obtain $\alpha_i \leqq \alpha_j$, where $q \cdot y_j \geqq \sum_{n=1}^{L} q_n$ and $\omega_{ik} \leqq \gamma$, for all k. This contradicts lemma 7.16.

Q.E.D.

<u>Proof of lemma 7.15</u> It is sufficient to prove that there exist $\underline{\delta}$ and $\underline{\lambda}$ as in the lemma. For by lemma 6.4, I may let $\bar{p} = b\underline{\lambda}^{-1}\bar{q}$.

Let $\underline{\delta} = (1+\epsilon^2)^{-1}(1+r)^{-1}$ and let $\underline{\lambda} = \epsilon(1+\epsilon)^{-1}$, where $\epsilon > 0$ is so small that it satisfies the conditions of lemma 7.21 and

7.24) $\quad \epsilon \leqq \min\{P_{ab} \mid a,b\epsilon A \text{ and } P_{ab} > 0\}.$

(P_{ab} is the probability of transition from a to b.)

I now prove that $\underline{\delta}$ and $\underline{\lambda}$ satisfy the conditions of lemma 7.15. Assume that $(1+r) > \delta_i \geqq \underline{\lambda}$, for all i. I must prove the following.

7.25) $\quad \lambda_{in}(a_1, \ldots, a_n) \geqq \underline{\lambda}$, for all i, for all histories $a_1, \ldots, a_n$ and for $n = 1, \ldots, N$.

I prove (7.25) by backwards induction on n. Clearly, (7.25) is true for $n = N$, for $\lambda_{iN}(a_1, \ldots, a_N) \geqq 1 > \underline{\lambda}$.

Suppose that (7.25) is true for n+1. First of all, I claim that

7.26) $\quad$ for any history $a_1, \ldots., a_{n+1}$, $\lambda_{i,n+1}(a_1, \ldots, a_{n+1}) \geqq (1+\epsilon)\underline{\lambda}$, whenever $\omega_{ik}(a_{n+1}) \leqq \gamma$, for all k.

For suppose that $\omega_{ik}(a_{n+1}) \leqq \gamma$, for all k, and that $\lambda_{i,n+1}(a_1,\ldots,a_{n+1}) < (1+\epsilon)\underline{\lambda}$. loss of generality, I may assume that $i = 1$.

Observe that $[p_{n+1}(a_1, \ldots, a_{n+1}), (x_{i,n+1}(a_1, \ldots, a_{n+1}))]$ forms an equilibrium with transfer payments for $\mathcal{E}(a_n)$. These transfer payments are made with money. Since there is only one unit of money in the economy, $|p_{n+1}(a_1, \ldots, a_{n+1})\cdot[x_{i,n+1}(a_1, \ldots, a_{n+1}) - \omega_i(a_{n+1})]| \leqq 1 = \epsilon((1+\epsilon)\underline{\lambda})^{-1}$ $< \epsilon(\lambda_{1,n+1}(a_1, \ldots, a_{n+1}))^{-1}$. Now by (7.12), $\lambda_{i,n+1}(a_1, \ldots, a_{n+1})$ $\geqq \alpha_{i,n+1}(a_1, \ldots, a_{n+1})$, for all i, so that $|p_{n+1}(a_1, \ldots, a_{n+1})\cdot[x_{i,n+1}(a_1, \ldots, a_{n+1})$ $- \omega_i(a_{n+1})]| \leqq \epsilon \max_i(\alpha_{i,n+1}(a_1, \ldots, a_{n+1}))^{-1}$. Therefore, by lemma 7.21.

$\alpha_{1,n+1}(a_1, \ldots, a_{n+1}) \geqq (1+\epsilon)\alpha_{i,n+1}(a_1, \ldots, a_{n+1})$, where i is such that $p_{n+1}(a_1, \ldots, a_{n+1})\cdot x_{i,n+1}(a_1, \ldots, a_{n+1}) \geqq \sum_{k=1}^{L} p_{n+1,k}(a_1, \ldots, a_n)$ > 0.

Since $x_{i,n+1}(a_1 \ldots, a_{n+1}) > 0$, $\alpha_{i,n+1}(a_1, \ldots, a_{n+1}) = \lambda_{i,n+1}(a_1, \ldots, a_{n+1})$ Therefore, $\lambda_{1,n+1}(a_1, \ldots, a_{n+1}) \geqq \alpha_{1,n+1}(a_1, \ldots, a_{n+1}) \geqq (1+\epsilon)\lambda_{i,n+1}(a_1, \ldots, a_{n+1})$ $\geqq (1+\epsilon)\underline{\lambda}$, where the last inequality follows from the induction hypothesis. This contradicts the hypothesis about $\lambda_{1,n+1}(a_1, \ldots, a_{n+1})$ and so proves (7.26).

I now prove that $\lambda_{in}(a_1, \ldots, a_n) \geqq \underline{\lambda}$, for all i and $a_1, \ldots, a_n$. By assumption 3.8 and by condition 7.24 on ϵ, $\text{Prob}[\omega_{ik}(s_{n+1}) \leqq \gamma, \text{ for all } k | s_n = a_n] \geqq \epsilon$ Therefore by the induction hypothesis and by (7.12) and (7.26), $\lambda_{in}(a_1, \ldots, a_n)$ $\geqq \delta_i(1+r)E(\lambda_{i,n+1}(a_1, \ldots, a_n, s_{n+1})|s_n = a_n) > \underline{\delta}(1+r)[(1-\epsilon)\underline{\lambda} + \epsilon(1+\epsilon)\underline{\lambda}] = \underline{\lambda}$. This completes the induction step in the proof of (7.25) and hence proves the lemma.

Q.E.D.

Boundedness from Below

The next lemma asserts that prices in N-period monetary equilibrium are uniformly bounded away from zero.

7.27) Lemma If $\delta_i < (1+r)^{-1}$, for all i, then there exist $\underline{p} \in R_+^L$ and $\bar{\lambda} > 0$ such that $\underline{p} \gg 0$ and the following are true. If $(p, (x_i))$ is an N-period monetary equilibrium and (λ_i) is the associated vector of marginal utilities of money, then $p_n(a_1, \ldots, a_n) \geqq \underline{p}$ and $\lambda_{in}(a_1, \ldots, a_n) \leqq \bar{\lambda}$, for all histories $a_1, \ldots, a_n$ and for all n.

Proof It is sufficient to find $\bar{\lambda}$ as in the lemma, for by lemma 6.4, I may let $\underline{p} = \bar{\lambda}^{-1}\underline{q}$.

I prove the lemma only for the case $r > 0$, since the proof for the case $r = 0$ is similar and slightly easier. Let

7.28) $$\bar{\lambda} = b + b^2(\bar{q}\cdot\bar{\omega})\left(\sum_{k=1}^{K}(1+r)^{k-1}\right)r\left(\max_i \tau_i^{-1}\right)$$

where $b = \max_k \underline{q}_k^{-1}\bar{q}_k$ and $\bar{\omega}$, $\underline{q}$ and $\bar{q}$ are as in (2.11). Here, K is a positive integer such that

7.29) $$\min_i(\delta_i(1+r))^{-K}b^{-1} > 1.$$

Notice that by assumption 3.7 $\tau_i > 0$, for all i, so that $\bar{\lambda} < \infty$.

It is sufficient to prove the following.

7.30) $\quad \lambda_{in}(a_1, \ldots, a_n) \leqq \bar{\lambda}$, for all i, for all histories $a_1, \ldots, a_n$ and for $n=1, \ldots, N$.

I prove (7.30) by backwards induction on n. First of all, (7.30) is true if $n=N$. To see that this is so, fix $a_1, \ldots, a_N$ and let i be such that $M_{iN}(p, x_i; a_1, \ldots, a_N) > 0$. Then by (7.13), $\lambda_{iN}(a_1, \ldots, a_N) = 1$. Also, $\alpha_{iN}(a_1, \ldots, a_N) \leqq \lambda_{iN}(a_1, \ldots, a_N) = 1$. Hence by lemma 6.3 $\alpha_{jN}(a_1, \ldots, a_N) \leqq b$, for all j. But then by (7.12), $\lambda_{jN}(a_1, \ldots, a_N) \leqq \max(b,1) = b$, for all j. Finally by (7.28), $b \leqq \bar{\lambda}$. This proves (7.30) for $n=N$.

Suppose by induction that (7.30) is true for $n+1, \ldots, N$ and that for some i and $a_1, \ldots, a_n$, $\lambda_{in}(a_1, \ldots, a_n) > \bar{\lambda}$. Without loss of generality, I may assume that $i=1$, so that

7.31) $\quad \lambda_{1n}(a_1, \ldots, a_n) > \bar{\lambda}$.

I will prove that (7.31) implies the following.

7.32) $\quad$ There exist i and a history $a_{n+1}, \ldots, a_{n+T}$ following a_n such that $\lambda_{i,n+t}(a_1, \ldots, a_{n+t}) \geqq (\delta_i(1+r))^{-t} b^{-1}\bar{\lambda}$ and $M_{i,n+t}(p, x_i; a_1, \ldots, a_{n+t}) \geqq r^{-1}\tau_i - \bar{\lambda}^{-1} b^2 (\bar{q}\cdot\bar{\omega})\left(\sum_{k=1}^{t}(1+r)^{k-1}\right)$, for $t=0, \ldots, T$, where $T = \min(K, N-n)$.

(7.32) leads to a contradiction. First of all, suppose that $T = N-n$. Then, (7.28) and (7.32) imply that $M_{iN}(p, x_i; a_1, \ldots, a_N) > 0$. But then $\lambda_{iN}(a_1, \ldots, a_N) = 1$. However by (7.28) and (7.32), $\lambda_{iN}(a_1, \ldots, a_N) \geqq (\delta_i(1+r))^{n-N} b^{-1}\bar{\lambda} \geqq b^{-1}\bar{\lambda} > 1$, which is a contradiction.

Suppose that $T = K$. Then, (7.29) and (7.32) imply that $\lambda_{i,n+K}(a_1, \ldots, a_{n+K}) \geqq (\delta_i(1+r))^{-K} b^{-1}\bar{\lambda} > \bar{\lambda}$, which contradicts the induction hypothesis. This proves that (7.32) leads to a contradiction and hence that (7.31) is impossible. Hence, the induction step in the proof of (7.30) will be completed once (7.32) is proved.

I now prove (7.32). Let i be such that $rM_{i,n}(p, x_i; a_1, \ldots, a_n) \geqq \tau_i$, where $a_1, \ldots, a_n$ are as in (7.31). Such an i exists by assumption 3.7.

I first show that $\lambda_{in}(a_1, \ldots, a_n) \geqq b^{-1}\bar{\lambda}$. Observe that $\bar{\lambda} < \lambda_{1n}(a_1, \ldots, a_n) = \max(\alpha_{1n}(a_1, \ldots, a_n), \delta_1(1+r)E(\lambda_{1,n+1}(a_1, \ldots, a_n, s_{n+1})|s_n = a_n)) \leqq \max(\alpha_{1n}(a_1, \ldots, a_n), \delta_1(1+r)\bar{\lambda}) = \alpha_{1n}(a_1, \ldots, a_n)$. The second inequality follows from the induction hypothesis on n (regarding (7.30)). Hence by lemma 6.3, $\lambda_{in}(a_1, \ldots, a_n) \geqq \alpha_{in}(a_1, \ldots, a_n) \geqq b^{-1}\alpha_{1n}(a_1, \ldots, a_n) > b^{-1}\bar{\lambda}$. I have now proved that i exists such that the inequalities of (7.32) are satisfied for $t = 0$.

I now prove by induction on t that $a_{n+1}, \ldots, a_{n+T}$ exist as in (7.32). Suppose that the conditions of (7.32) are satisfied for t no larger than some non-negative integer, call it t again. I may suppose that $t < T$. Then, $M_{i,n+t}(p, x_i; a_1, \ldots, a_{n+t}) \geqq r^{-1}\tau_i - \bar{\lambda}^{-1}b^2(\bar{q}\cdot\bar{\omega})\left(\sum_{k=1}^{t}(1+r)^{k-1}\right) > 0$. The last inequality follows from (7.28). Hence by (7.13), $\lambda_{i,n+t}(a_1, \ldots, a_{n+t}) = \delta_i(1+r)E[\lambda_{i,n+t+1}(a_1, \ldots, a_{n+t}, s_{n+t+1}) \big| s_{n+t} = a_{n+t}]$, so that for some a_{n+t+1}, $\lambda_{i,n+t+1}(a_1, \ldots, a_{n+t+1}) \geqq (\delta_i(1+r))^{-1}\lambda_{i,n+t}(a_1, \ldots, a_{n+t}) \geqq (\delta_i(1+r))^{-(t+1)}b^{-1}\bar{\lambda}$. The last inequality follows from the induction hypothesis on t.

I now show that $M_{i,n+t+1}(p, x_i; a_1, \ldots, a_{n+t+1}) \geqq r^{-1}\tau_i - \bar{\lambda}^{-1}b^2(q\cdot\bar{\omega})\left(\sum_{k=1}^{t+1}(1+r)^{k-1}\right)$. If $\alpha_{i,n+t+1}(a_1, \ldots, a_{n+t+1}) < \lambda_{i,n+t+1}(a_1, \ldots, a_{n+t+1})$, then by (7.14) $x_{i,n+t+1}(a_1, \ldots, a_{n+t+1}) = 0$, so that $M_{i,n+t+1}(p,x_i; a_1 \ldots, a_{n+t+1})$

$\geqq (1+r)M_{i,n+t}(p, x_i; a_1, \ldots, a_{n+t}) - \tau_i \geqq (1+r)\left[r^{-1}\tau_i - \bar{\lambda}^{-1}b^2(q\cdot\bar{\omega})\left(\sum_{k=1}^{t}(1+r)^{k-1}\right)\right]$
$-\tau_i \geqq r^{-1}\tau_i - \bar{\lambda}^{-1}b^2(q\cdot\bar{\omega})\left(\sum_{k=1}^{t+1}(1+r)^{k-1}\right)$. The third inequality follows from the induction hypothesis on t.

Suppose now that $\alpha_{i,n+t+1}(a_1, \ldots, a_{n+t+1}) = \lambda_{i,n+t+1}(a_1, \ldots, a_{n+t+1})$. Then, by the choice of a_{n+t+1}, $\alpha_{i,n+t+1}(a_1, \ldots, a_{n+t+1}) > b^{-1}\bar{\lambda}$. It follows from lemma 6.3 that $\min_j \alpha_{j,n+t+1}(a_1,\ldots, a_{n+t+1}) > b^{-2}\bar{\lambda}$, so that by lemma 6.2, $p_{n+t+1}(a_1, \ldots, a_{n+t+1}) \leqq b^2\bar{\lambda}^{-1}\bar{q}$. Hence, $p_{n+t+1}(a_1, \ldots, a_{n+t}) \cdot x_{i,n+t+1}(a_1, \ldots, a_{n+t+1}) \leqq b^2\bar{\lambda}^{-1}(\bar{q}\cdot\bar{\omega})$. It follows that $M_{i,n+t+2}(p,x_i; a_1\ldots, a_{n+t+1}) \geqq (1+r)M_{i,n+t+1}(p, x_i; a_1, \ldots, a_{n+t+1}) - \tau_i - b^2\bar{\lambda}^{-1}(\bar{q}\cdot\bar{\omega})$
$\geqq (1+r)[r^{-1}\tau_i - \bar{\lambda}^{-1}b^2(\bar{q}\cdot\bar{\omega}) \sum_{k=1}^{t}(1+r)^{k-1}] - \tau_i - b^2\bar{\lambda}^{-1}(\bar{q}\cdot\bar{\omega}) = r^{-1}\tau_i$
$- \bar{\lambda}^{-1} b^2(\bar{q}\cdot\bar{\omega})\sum_{k=1}^{t+1}(1+r)^{k-1}$.

This completes the proof that the two inequalities of (7.32) are satisfied for t+1, and so completes the induction step in the proof of (7.32).

This completes the proof of lemma 7.27.

Q.E.D.

Passage to the Limit

I now apply a Cantor diagonal argument to the N-period equilibria in order to obtain a monetary equilibrium in the limit.

Let $\underline{\delta}$ be as in lemma 7.15 and suppose that $\underline{\delta} < \delta_i < (1+r)^{-1}$, for all i. For each positive integer N, let $(p^N, (x_i^N))$ be an N-period monetary equilibrium and let (λ_i^N) be the vector of associated marginal utilities of money. By lemmas 7.14 and 7.27, $\underline{p} \leqq p_n^N(a_1, \ldots, a_n) \leqq \bar{p}$ and $\underline{\lambda} \leqq \lambda_{in}^N(a_1, \ldots, a_n) \leqq \bar{\lambda}$, for all i, n, N and $a_1 \ldots, a_n$. Similarly, $0 \leqq x_{in}^N(a_1, \ldots, a_n) \leqq \bar{\omega}$, for all i, n, N and $a_1, \ldots, a_n$. Hence, the components of the vectors $(p^N, (x_i^N))$ and

(λ_i^N) are uniformly bounded. There are countably many of these components. There exists a subsequence of N such that one of those components converges. There exists a subsequence of this subsequence such that another component converges. Continuing in this way, I choose a sequence of subsequences, one for each component. Taking the k^{th} member of the k^{th} subsequence, I obtain a subsequence of N such that all components converge. Let $p = (p_n(a_1, \ldots, a_n))$, $x_i = (x_{in}(a_1, \ldots, a_n))$ and $\lambda_i = (\lambda_{in}(a_1, \ldots, a_n))$ be the limits of this Cantor subsequence. I claim that $(p, (x_i))$ is a monetary equilibrium with associated marginal utilities of money (λ_i).

Clearly, $\underline{p} \leqq p_n(a_1, \ldots, a_n) \leqq \bar{p}$, for all n and $a_1, \ldots, a_n$. Hence, condition 2.5 of the definition of a monetary equilibrium is satisfied.

Since $\sum_{i=1}^{I} (x_{in}^N(a_1, \ldots, a_n) - \omega_i(a_n)) = 0$, the same is true in the limit and so (x_i) is a feasible allocation. This is condition 2.4 of the definition of equilibrium.

I now show that $x_i \in \xi_i(p)$, for all i and so verify the last condition, (2.6).

First of all, $M_{i,n}(p^N, x_i^N; a_1, \ldots, a_n) = (1+r)M_{i,n-1}(p^N, x_i^N; a_1, \ldots, a_{n-1}) - \tau_i + p_n^N(a_1, \ldots, a_n)\cdot(\omega_i(a_n) - x_{in}^N(a_1, \ldots, a_n)) \geqq 0$, for all N. Passing to the limit in these expressions and using the fact that M_{i0} is given, I obtain that $M_{in}(p, x_i; a_1, \ldots, a_n) \geqq 0$, for all i, n and $a_1, \ldots, a_n$. Hence, x_i satisfies the constraints of the consumer maximization problem (2.3).

I now prove that $(p, (x_i))$ and (λ_i) satisfy conditions (2.8)-(2.10), (2.13), and (2.14). First of all, it should be clear that $M_{in}(p, x_i; a_1, \ldots, a_n) \leqq \sum_{j=1}^{I} M_{jn}(p, x_j; a_1, \ldots, a_n) = 1$, for all i,n and $a_1, \ldots, a_n$. This is condition (2.13). (2.14) follows by passage to the limit in the inequalities $\underline{\lambda} \leqq \lambda_{in}^N(a_1, \ldots, a_n) \leqq \bar{\lambda}$. It remains to verify conditions (2.8) - (2.10).

Let (α_i^N) be the vector of marginal utilities of expenditure associated with the N-period monetary equilibrium $(p^N,(x_i^N))$. The convergence of the subsequence of $(p^N,(x_i^N))$ implies that the corresponding subsequence of (α_i^N) converges to (α_i), where (α_i) is the vector of marginal utilities of expenditure associated with $(p,(x_i))$. (The (α_i) are defined by (2.7).) $(p^N,(x_i^N))$, (λ_i^N) and (α_i^N) together satisfy (7.12) - (7.14). Passage to the limit in these inequalities gives (2.8) - (2.10).

It now follows by definition that $x_i \in \xi_i(p)$, when $\delta_i = 1$. I must now show $x_i = \xi_i(p)$, when $\delta_i < 1$. If $x_i \neq \xi_i(p)$, then there exists $\bar{x} \in \beta_i(p)$ such that $E[\sum_{n=1}^{\infty} \delta_i^{n-1} u_i(\bar{x}_n(s_1,\ldots,s_n),s_n)] > E[\sum_{n=1}^{\infty} \delta_i^{n-1} u_i(x_{in}(s_1,\ldots,s_n),s_n)] + \varepsilon$, where $\varepsilon > 0$. Choose N such that $\delta_i^{N-1} < (2\bar{\lambda})^{-1}\varepsilon$ and $E[\sum_{n=N+1}^{\infty} \delta_i^{n-1} |u_i(\bar{x}_n(s_1,\ldots,s_n),s_n)| < \varepsilon/4$ and $E[\sum_{n=n+1}^{\infty} \delta_i^{n-1} |u_i(x_{in}(s_1,\ldots,s_n),s_n)| < \varepsilon/4$. (It is easy to see that these series converge.). x_i solves the problem.

$$7.33) \qquad \max \{E[\sum_{n=1}^{N} \delta_i^{n-1} u_i(x_n(s_1,\ldots,s_n),s_n) + \delta_i^{N-1}\lambda_{iN}(s_1,\ldots,s_n)M_{in}(p,x_i;s_1,\ldots,s_n)] \mid x \in \beta_i(p)\},$$

since p, x_i and λ_i satisfy (2.8) - (2.10). However, $E[\sum_{n=1}^{N} \delta_i^{n-1} u_i(\bar{x}_n(s_1,\ldots,s_n),s_n) + \delta_i^{N-1}\lambda_{iN}(s_1,\ldots,s_N)M_{iN}(p,\bar{x};s_1,\ldots,s_n)] \geq E[\sum_{n=1}^{\infty} \delta_i^{n-1} u_i(\bar{x}_n(s_1,\ldots,s_n),s_n)] - \varepsilon/4 > E[\sum_{n=1}^{\infty} \delta_i^{n-1} u_i(x_n(s_1,\ldots,s_n),s_n)] + 3\varepsilon/4 > E[\sum_{n=1}^{N} \delta_i^{n-1} u_i(x_n(s_1,\ldots,s_n),s_n) + \delta_i^{N-1}\lambda_{iN}(s_1,\ldots,s_n)M_{iN}(p,x_i;s_1,\ldots,s_N)]$. The last inequality follows from the fact that

$E[\delta_i^{N-1}\lambda_{iN}(s_1,\ldots,s_N)M_{iN}(p,x_i;s_1,\ldots,s_N)] < \delta_i^{N-1}\bar{\lambda} < \varepsilon/2$. Hence, I have contradicted the fact that x_i solves the problem 7.33. This proves that $x_i = \xi_i(p)$.

I have now completed the verification of condition (2.6) and so have proved theorem 4.1.

Q.E.D.

8) Proof of Theorem 4.2

First of all, I observe that

8.1) $\lambda_{in}(a_1,\ldots,a_n) = \alpha_{in}(a_1,\ldots,a_n)$ for all i, n and $a_1,\ldots,a_n$,

where (α_i) is the vector of marginal utilities of expenditure associated with the monetary equilibrium $(p,(x_i))$. (8.1) follows from (2.8), (2.10), and the assumption that $x_{in}(a_1,\ldots,a_n) \neq 0$, for all i, n and $a_1,\ldots,a_n$.

Next, I observe that if the allocation (x_i) is Pareto optimal, then

8.2) $\lambda_{in}(a_1,\ldots,a_n) = \delta_i(1+r)E[\lambda_{i,n+1}(a_1,\ldots,a_n,s_{n+1})|s_n = a_n]$, for all i, n and $a_1,\ldots,a_n$,

where (λ_i) is the vector of marginal utilities of money associated with the monetary equilibrium $(p,(x_i))$.

By (2.8), the left hand side of (8.2) is at least as great as the right hand side. Suppose that for some i,n and $a_1,\ldots,a_n$, $\lambda_{in}(a_1,\ldots,a_n) > \delta_i(1+r)E[\lambda_{i,n+1}(a_1,\ldots,a_n,s_{n+1})|s_n = a_n]$. Then by (2.9), $M_{in}(a_1,\ldots,a_n) = 0$, so that for some $j \neq i$, $M_{jn}(a_1,\ldots,a_n) > 0$. Again by (2.9), $\lambda_{jn}(a_1,\ldots,a_n) = \delta_j(1+r)E[\lambda_{j,n+1}(a_1,\ldots,a_n,s_{n+1})|s_n = a_n]$. Now, I use (8.1) and find that $\alpha_{in}(a_1,\ldots,a_n) > \delta_i(1+r)E[\alpha_{i,n+1}(a_1,\ldots,a_n,s_{n+1})|s_n = a_n]$ and $\alpha_{jn}(a_1,\ldots,a_n) = \delta_j(1+r)E[\alpha_{j,n+1}(a_1,\ldots,a_n,s_{n+1})|s_n = a_n]$. A standard argument now shows tha a Pareto improvement could be made. Roughly speaking, consumer i should spend ϵ units of money more in period n (and when history $a_1,\ldots,a_n$ occurs), where $\epsilon > 0$ is very small. He should spend $(1+r)\epsilon$ units less in the next period. Consumer j should spend ε less units of money in period n and $(1+r)\varepsilon$ more in the following period. Thus, I have contradicted the assumption that (x_i) is Pareto optimal. This proves (8.2).

It follows from (8.2) that for each n and each history $a_1,\ldots,a_n$, there exists a_{n+1} following a_n such that $\lambda_{1,n+1}(a_1,\ldots,a_{n+1}) \geqq (\delta_1(1+r))^{-1}\lambda_{1n}(a_1,\ldots,a_n)$. Hence, there exists an infinite sequence $a_1,a_2,\ldots.$ such that $\lambda_{1n}(a_1,\ldots,a_n) \geqq (\delta_1(1+r))^{-n+1}\lambda_{11}(a_1) > 0$. Since $\delta_1(1+r) < 1$, it follows that $\lim_{n\to\infty} \lambda_{1n}(a_1,\ldots,a_n) = \infty$. This contradicts (2.12), so that (x_i) cannot be Pareto optimal.

Q.E.D.

9) A Lemma

In this section I prove a lemma which is in turn used in the next section to prove theorem 4.3. The statement of this lemma involves the concept of stationary equilibrium with transfer payments. Such an equilibrium is of the form $(p,(x_i))$, where p is a stationary price system and (x_i) is a stationary allocation. Each x_i must solve the problem.

$$\text{(9.1)} \qquad \max \{ \sum_{a\varepsilon A} \pi_a u_i(y(a),a) \mid y: A \to R_+^L \text{ and } \sum_{a\varepsilon A} \pi_a p(a)\cdot(y(a) - x_i(a)) \leqq 0\},$$

where (π_a) is the stationary distribution on A. The transfer payment of consumer i is $\sum_{a\varepsilon A} \pi_a p(a)\cdot(\omega_i(a) - x_i(a))$.

Given a stationary equilibrium with transfer payments $(p,(x_i))$, money holdings are defined as before. That is, $M_{in}(p,x_i;a_1,\ldots,a_n) = (1+r)M_{i,n-1}(p,x_i;a_1,\ldots,a_{n-1}) + p(a_n)\cdot(\omega_i(a_n) - x_i(a_n)) - \tau_i$. I now allow the initial holdings, M_{i0}, to be arbitrary, though I continue to assume that $\sum_{i=1}^{I} M_{i0} = 1$ and $\sum_{I=1}^{I} \tau_i = r$.

9.2) Lemma For almost every $\omega \in \Omega$, the following is true. Let $(p,(x_i))$ be any stationary equilibrium for $\mathcal{E}(\omega)$ with transfer payments. Then for any distribution of initial money balances and for any $a_1 \in A$, $M_{in}(p,x_i;a_1,\ldots,a_n) < 0$, for some i and some history $a_2,\ldots,a_n$ following a_1.

The proof of this lemma involves the marginal utilities of money associated with a stationary equilibrium $(p,(x_i))$. These are the Lagrange multipliers associated with the constrained maximization problems (9.1). The marginal utility of money of consumer i is as number λ_i.

Stationary equilibrium with transfer payments may be thought of as a function of the associated marginal utilities of money. This fact is expressed by the following lemma, which I do not prove. Its proof is contained in [8].

9.3) Lemma To each $(\lambda_i) >> 0$, there corresponds a unique stationary equilibrium with transfer payments such that (λ_i) is the corresponding vector of marginal utilities of money.

The proof of lemma 9.2 depends on the fact that the relation between stationary equilibrium and marginal utilities of money is nearly differentiable.

In order to express this fact, I drop $a \in A$ from the notation, for the moment. Let $u_i : R_+^L \to (-\infty,\infty)$ satisfy (3.5) and (3.6), for $i = 1, \ldots, I$. Given $\lambda > 0$ and $p \in R_+^L$ such that $p >> 0$, $X_i(p,\lambda)$ denotes the unique vector in R_+^L which satisfies the following (if such a vector exists).

$$(9.4) \qquad \frac{\partial u_i(X_i(p,\lambda))}{\partial z_k} \leqq \lambda p_k \text{ , for } k = 1, \ldots, L \text{ with equality if } X_{ik}(p,\lambda) > 0 \text{ .}$$

$X_i(p,\lambda)$ is consumer i's demand as a function of prices and his marginal utility of money. $X_i(p,\lambda)$ may not be defined if some price is too low relative to λ. I let $G = \{(p,\Lambda) \in \text{int } R_+^L \times \text{int } R_+^I \mid X_i(p,\Lambda_i) \text{ is defined for all } i\}$. It is easy to see that G is an open set and that each of the functions X_i is continuous on G.

Now let $\omega \in R_+^L$ be such that $\omega >> 0$. Think of ω as the total initial endowment of the economy. Given $\Lambda = (\Lambda_1, \ldots, \Lambda_I) \in \text{int } R_+^I$, $P(\Lambda)$ denotes the unique vector $p \in \text{int } R_+^L$ such that $\sum_{i=1}^{I} X_i(p,\Lambda_i) = \omega$. $P(\Lambda)$ is a market clearing price vector, given the demand functions $X_i(p,\lambda)$. Clearly, $P(\Lambda) >> 0$. I prove in [8] that P is a continuous function.

Observe that $P(\Lambda)$ is homogeneous of degree minus one with respect to Λ. That is, $P(t\Lambda) = t^{-1}P(\Lambda)$, for all $t > 0$. Hence, I may restrict Λ to $\operatorname{int} \Delta^{I-1} = \{\Lambda \in \operatorname{int} R^I_+ \mid \sum_{i=1}^{I} \Lambda_i = 1\}$.

(9.5) <u>Lemma</u> $\operatorname{int} \Delta^{I-1}$ is the union of finitely many sets, closed in $\operatorname{int} \Delta^{I-1}$, on each of which the function $P(\Lambda)$ is continuously differentiable. Similarly, G is the union of finitely many sets, closed in G, on each of which all of the functions $X_i(p,\Lambda_i)$ are continuously differentiable.

<u>Proof</u> First of all, I deal with the functions $X_i(p,\Lambda_i)$.

Let $\mathscr{S}$ be the set of all subsets of $\{1,\ldots,L\}$. For each $S \in \mathscr{S}$, let

$$C_{Si} = \{(p,\lambda) \in G \mid \frac{\partial u_i(X_i(p,\lambda))}{\partial z_k} = \lambda p_k\ , \quad \text{for } k \in S \quad \text{and } X_{ik}(p,\lambda) = 0, \text{ for } k \notin S\}.$$

Clearly, C_{Si} is closed in G and $G = \cup\{C_{Si} \mid S \in \mathscr{S}\}$. I show that X_i is continuously differentiable on each set C_{Si}.

Let $X_{Si}(p,\lambda)$ be the function defined by

$$(9.6) \qquad \frac{\partial u_i(X_{Si}(p,\lambda))}{\partial z_k} = \lambda p_k\ , \text{ if } k \in S$$

$$X_{Sik}(p,\lambda) = 0\ , \text{ if } k \notin S\ .$$

Clearly, if $(p,\lambda) \in C_{Si}$, then $X_{Si}(p,\lambda)$ is well-defined and equals $X_i(p,\lambda)$.

Recall that a function defined on a closed set $C \subset R^n$ is said to be differentiable if it has a differentiable extension, $\hat{f}$, defined on an open neighborhood of C . Hence, I must show that X_{Si} has a continuously differentiable extension to an open neighborhood of C_{Si} .

Since $u_i : R^L_+ \to (-\infty,\infty)$ is continuously differentiable, it has a continuously differentiable extension $\hat{u}_i : V \to (-\infty,\infty)$, where V is an open neighborhood of R^L_+ .

I now apply the implicit function theorem to the equation (9.6) with $\hat{u}_i$ substituted for u_i . By the implicit function theorem, X_{Si} is defined and differentiable on an open neighborhood of C_{Si} if the matrix of partial derivatives of the left hand side of (9.6) with respect to the components of X_{Si} is invertible, these partial derivatives being evaluated at an arbitrary point $X_{Si}(p,\lambda)$ for $(p,\lambda) \in C_{Si}$. This matrix of partial derivatives is given below, where I have assumed that $S = \{1,\ldots,K\}$. u_i appears in the matrix rather than $\hat{u}_i$, for the derivatives are evaluated at a point in the domain of u_i .

$$
(9.7)\qquad \begin{pmatrix}
\frac{\partial^2 u_i}{\partial z_1 \partial z_1} & \cdots & \frac{\partial^2 u_i}{\partial z_K \partial z_1} & \frac{\partial^2 u_i}{\partial z_{K+1} \partial z_1} & \cdots & & \frac{\partial^2 u_i}{\partial z_L \partial z_1} \\
\vdots & & & & & & \\
\frac{\partial^2 u_i}{\partial z_K \partial z_1} & \cdots & \frac{\partial^2 u_i}{\partial z_K \partial z_K} & \frac{\partial^2 u_i}{\partial z_{K+1} \partial z_K} & \cdots & & \frac{\partial^2 u_i}{\partial z_L \partial z_K} \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
\vdots & & \vdots & 0 & 1 & & \vdots \\
\vdots & & \vdots & \vdots & & \ddots & \vdots \\
\vdots & & \vdots & \vdots & & 1 & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & 1
\end{pmatrix}
$$

This matrix is invertible. For by assumption (3.6) the matrix $D^2u_i(x)$ $= \left(\frac{\partial^2 u_i}{\partial z_k \partial z_m}\right)$ is negative definite. Hence, the submatrix in the upper left hand corner of (9.7) is negative definite and so is invertible. It follows at once that the whole matrix is invertible. This completes the proof that X_i is differentiable on C_{Si}. Since the matrix (9.7) depends continuously on the components of X_{Si}, it follows that X_i is continuously differentiable on C_{Si}.

I now turn to the function $P(\Lambda)$. Let $\mathscr{S}^I = \{(S_1,\ldots,S_I) \mid S_i \in \mathscr{S}$, for all $i\}$. If $S \in \mathscr{S}^I$, let $C_S = \{\Lambda \in \text{int}\,\Delta^{I-1} \mid (P(\Lambda),\Lambda_i) \in C_{S_i i}$, for all $i\}$. Since P is a continuous function, C_S is closed in $\text{int}\,\Delta^{I-1}$. Clearly, $\text{int}\,\Delta^{I-1}$ $= \cup\{C_S \mid S \in \mathscr{S}^I\}$.

Now let $S \in \mathscr{S}^I$ be such that C_S is not empty. Recall that $P(\Lambda)$ satisfies the equation $\sum_{i=1}^{I} X_i(P(\Lambda),\Lambda_i) = \omega \gg 0$. Essentially, what I do is to apply the implicit function theorem to this equation. In order to do so, I let $f_S(p,\Lambda) = \sum_{i=1}^{I} X_{S_i i}(p,\Lambda_i)$, where $X_{S_i i}$ is defined by (9.6) and S_i is the i^{th} component of S. Let E be the matrix of partial derivatives of $f(p,\Lambda)$ with respect to the components of p, these derivatives being evaluated at $(P(\Lambda),\Lambda)$. I must show that E is invertible.

Let $D_p X_{S_i i}(P(\Lambda),\Lambda_i)$ denote the matrix of partial derivatives of the function $X_{S_i i}(p,\lambda)$ with respect to the components of p, these derivatives being evaluated at $(p,\lambda) = (P(\Lambda),\Lambda_i)$. It is easy to see that $D_p X_{S_i i}(P(\Lambda),\Lambda_i)$ is of the form $\Lambda_i E_i$, where E_i is defined as follows. Let $D^2_{S_i} u_i(X_{S_i i}(P(\Lambda),\Lambda_i))$ be the $|S_i| \times |S_i|$ matrix of second order partial derivatives of u_i with respect to variables with indices in S_i . This matrix is negative definite. If k and m belong to S_i, then the $(k,m)^{th}$ entry of E_i is that entry of the inverse of $D^2_{S_i} u_i(X_{S_i i}(P(\Lambda),\Lambda_i))$ which corresponds to k^{th} and m^{th} commodities. The rest of the entries of E_i are zero.

Let $\Lambda \in C_S$. For each $k = 1,\ldots,L$, $X_{ik}(P(\Lambda),\Lambda_i) > 0$, for some i . This statement follows from the fact that $\sum_{i=1}^{I} X_i(P(\Lambda),\Lambda_i) = \omega \gg 0$. Therefore, for each k , the k^{th} row of E_i is non-zero, for some i . Hence, since the Λ_i are all positive, every row and column of $E = \sum_{i=1}^{I} \Lambda_i E_i$ is non-zero. It now follows from the nature of the matrices E_i that E is negative definite. Hence, E is non-singular, as was to be proved.

Q.E.D.

For each $a \in A$, let $X_i(p,\lambda,a)$ be defined from $u_i(\cdot,a)$ by (9.4) Similarly, if $\Lambda \in \text{int } R^I_+$ and $\omega \in \text{int } R^L_+$, let $P(\Lambda,\omega,a)$ be the unique vector $p \in \text{int } R^L_+$ such that $\sum_{i=1}^{I} X_i(p,\Lambda,a) = \omega$. For each a and ω , lemma 9.5 applies to the functions $X_i(p,\lambda,a)$ and $P(\Lambda,\omega,a)$. Also, it is easy to see that $P(\Lambda,\omega,a)$ is a continuous function of ω . (In fact, P is just as differentiable with respect to ω as it is with respect to Λ.)

I now turn to the proof of lemma 9.2 Throughout the proof, if $\omega \in \Omega$, then $\omega^T = (\omega^T(a))_{a \in A}$ is defined by $\omega^T(a) = \sum_{i=1}^{I} \omega_i(a)$. ω^T is the total initial endowment of the economy.

<u>Proof of lemma 9.2</u> I first prove the lemma for $r = 0$.

By assumption 3.10, I may choose $a_1 \in A$ for which there are two histories going from a_1 to itself. Also, each of these histories contains an element which is distinct from a_1 and does not appear in the other history. For notational simplicitly, I assume that these distinct element occur just after a_1 in the histories. Let $a_1, a_2^1, \ldots, a_N^1$ and $a_1, a_2^2, \ldots, a_{N_2}^2$ be the two histories, where $a_{N_1}^1 = a_{N_2}^2 = a_1$. Then, a_2^1 does not appear in $a_2^2, \ldots, a_{N_2}^2$ and a_2^2 does not appear in $a_2^1, \ldots, a_{N_1}^1$. Also, I may assume that a_2^j does not appear in $a_3^j, \ldots, a_{N_j}^j$, for $j = 1,2$ for I may eliminate closed loops beginning with a_2^j.

Let $(p,(x_i))$ be a stationary equilibrium with transfer payments for $\mathcal{E}(\omega)$, where $\omega \in \Omega$. Let $\Lambda = (\Lambda_i)$ be the vector of associated marginal utilities of money. Then, $p(a) = P(\Lambda, \omega^T(a), a)$ and $x_i(a) = X_i(P(\Lambda, \omega^T(a), a), \Lambda_i, a)$ for all i and a. I now simplify the notation by writing $P(\Lambda, \omega^T(a), a)$ as $P(\Lambda, \omega^T, a)$ and by writing $X_i(P(\Lambda, \omega^T(a), a), \Lambda_i, a)$ as $X_i(\Lambda, \omega^T, a)$.

The net expenditure of consumer i during the course of the cycle $a_2^j, \ldots, a_N^j$ is $\sum_{n=2}^{N_j} p(a_n^j) \cdot (x_i(a_n^j) - \omega_i(a_n^j))$, for $j = 1,2$. If this quantity is not zero, for $j = 1,2$, then the money balances of some consumers must be negative at some time and for some history. For with positive probability, $a_2^j, \ldots, a_{N_j}^j$ could be repeated an arbitrarily large number of times in succession. In this case, consumer i would accumulate or lose an arbitrarily large quantity of money. In either case, someone would eventually hold a negative quantity of money. Therefore, in order that lemma 9.2 be true, it is enough that

9.8) for almost every $\omega \in \Omega$, $\sum_{n=2}^{N_j} p(a_n^j) \cdot (x_i(a_n^j) - \omega_i(a_n^j)) \neq 0$,

for $j = 1$ or 2 and for some i.

The equation $\sum_{n=2}^{N_j} p(a_n^j)\cdot(x_i(a_n^j) - \omega_i(a_n^j)) = 0$ holds if and only if $\omega_{i1}(a_2^j) = (p_1(a_2^j))^{-1} \left| \sum_{n=2}^{N_j} p(a_n^j)\cdot(x_i(a_n^j)-\omega_i(a_n^j)) + p_1(a_2^j)\omega_{i1}(a_2^j) \right|$, where $\omega_{i1}(a)$ is the first component of $\omega_i(a)$ and $p_1(a)$ is the first component of $p(a)$. Substituting the appropriate functions of Λ and ω^T, I obtain

9.9)
$$\omega_{i1}(a_2^j) = (P_1(\Lambda,\omega^T,a_2^j))^{-1} \left| \sum_{n=2}^{N_j} P(\Lambda,\omega^T,a_n^j)\cdot(X(\Lambda,\omega,a_n^j)-\omega_i(a_n^j)) + P_1(\Lambda,\omega^T,a_2^j)\omega_{i1}(a_2^j) \right|.$$

Notice that the right hand side of (9.9) does not depend on either $\omega_{i1}(a_2^1)$ or $\omega_{i1}(a_2^2)$, provided that $\sum_{i=1}^{I} \omega_{i1}(a_2^j)$ is held constant. Here, I use the assumption that for each j, a_2^j does not appear in $a_3^j,\ldots,a_{N_j}^j$ and does not appear in $a_2^k,\ldots,a_{N_k}^k$, for $k \neq j$. I now parameterize ω by $(\omega_\alpha,\omega_\beta)$, where ω_α is the vector $(\omega_{11}(a_2^1),\ldots,\omega_{I-1,1}(a_2^1),\omega_{11}(a_2^2),\ldots,\omega_{I-1,1}(a_2^2))$. ω_β is the vector
$$\left(\left(\sum_{i=1}^{I} \omega_{i1}(a_2^j)\right)_{j=1,2}, (\omega_{ik}(a_2^j))_{i=1,\ldots,I;k=2,\ldots,L;j=1,2}, (\omega_i(a))_{i=1,\ldots,I;a\neq a_2^1,a_2^2}\right).$$
Clearly, $(\omega_\alpha,\omega_\beta)$ is simply a coordinate system for Ω.

The right hand side of (9.9) depends only on Λ and ω_β. Denote this right hand side by $f_{ji}(\Lambda,\omega_\beta)$, where $j = 1,2$ and $i = 1,\ldots,I$. (9.8) may now be rewritten as

9.10) for almost every $\omega \in \Omega$, $f_{ji}(\Lambda,\omega_\beta) \neq \omega_{i1}(a_2^j)$, for $j = 1$ or 2 and for some $i = 1,\ldots,I-1$.

In order that lemma 9.2 be true, it is sufficient that (9.10) be true for every $\Lambda \in \text{int } R_+^I$. More precisely, it is sufficient to prove the next statement.

9.11) For almost every $\omega \in \Omega$, the following is true. For every $\Lambda \in \text{int } R_+^I$, $f_{ji}(\Lambda,\omega_\beta) \neq \omega_{i1}(a_2^j)$, for $j = 1$ or 2 and for some $i = 1,\ldots,I-1$.

The functions f_{ji} are continuous, so that the set of ω in Ω for which (9.11) is true is measurable. It follows from the Fubini theorem that it is enough to prove (9.11) for ω with ω_β constant. More precisely, it is sufficient to prove the following.

9.12) For each fixed $\bar{\omega}_\beta$, the following is true for almost every $\omega_\alpha = (\omega_{11}(a_2^1),\ldots,\omega_{I-1,1}(a_2^2))$. For every $\Lambda \in \text{int } R_+^I$, $f_{ji}(\Lambda,\bar{\omega}_\beta) \neq \omega_{i1}(a_2^j)$, for $j = 1$ or 2 and for some i, $i = 1,\ldots,I-1$.

I now prove (9.12). Since $\bar{\omega}_\beta$ is constant, I may write $f_{ji}(\Lambda,\bar{\omega}_\beta)$ as $f_{ji}(\Lambda)$. Notice that f_{ji} is homogeneous of degree zero. That is, $f_{ji}(t\Lambda) = f_{ji}(\Lambda)$, for all $t > 0$. Hence, I may restrict Λ to int Δ^{I-1}. Consider the function $F\colon \text{int } \Delta^{I-1} \to R^{2I-2}$ defined by $F(\Lambda) = (f_{11}(\Lambda),\ldots,f_{1,I-1}(\Lambda), f_{21}(\Lambda),\ldots,f_{2,I-1}(\Lambda))$. (9.12) is simply the assertion that almost every ω_α does not belong to the range of F. Hence, it is sufficient to prove the following.

9.13) The range of F is of measure zero.

By lemma 9.5, F is continuously differentiable on each of a finite number of sets closed in int Δ^{I-1}, the union of which is int Δ^{I-1}. dim $\Delta^{I-1} = I - 1 < 2I - 2$, since $I \geqq 2$. Therefore, the range of F is of measure zero.

This completes the proof of lemma 9.2 for the case $r = 0$. I now turn to the case $r > 0$. The proof is quite similar.

I now use assumption 3.10 to obtain three histories, $a_1, a_2^j,\ldots,a_{N_j}^j$, where $j = 1,2,3$ and $a_{N_j}^j = a$, for all j. Also for each j, a_2^j does not appear in $a_3^j,\ldots,a_{N_j}^j$ and does not appear in either sequence $a_2^k,\ldots,a_{N_k}^k$, for $k \neq j$.

Let $(p,(x_i))$ be a stationary equilibrium. Suppose that consumer i is in state a_1 during the period one and has M_i units of money at the end of the period. Suppose that he then passes through the cycle $a_2^j,\ldots,a_{N_j}^j$. Then, at the end of period N_j, he has $M_i(1+r)^{N_j-1} + \sum_{n=2}^{N_j} (1+r)^{N_j-n}[p(a_n^j) \cdot(\omega_i(a_n^j)-x_i(a_n^j))-\tau_j]$ units of money. This sum must equal M_i, for all j, if it is to be true that no consumer ever holds negative money balances.

I now proceed as before. If $\omega \in \Omega$, then ω_α and ω_β are defined as follows. $\omega_\alpha = (\omega_{11}(a_2^1),\ldots,\omega_{I-1,1}(a_2^1),\omega_{11}(a_2^2),\ldots,\omega_{I-1,1}(a_2^2),\omega_{11}(a_2^3),\ldots,$ $\omega_{I-1,1}(a_2^3))$ and $\omega_\beta = \left(\left(\sum_{i=1}^{I} \omega_{i1}(a_2^j)\right)_{j=1,2,3};(\omega_{ik}(a_2^j))_{i=1,\ldots,i;k=2,\ldots,L;j=1,2,3;}\right.$ $\left.(\omega_i(a))_{i=1,\ldots,I;a\neq a_2^1,a_2^2,a_2^3}\right)$. $M = (M_i)$ denotes the vector of initial holdings of money, held at the end of period one in state a_1. M varies over $\Delta^{I-1} = \{M \in R_+^I \mid \sum_{i=1}^{I} M_i = 1\}$. It is sufficient to prove the next statement.

9.14) For each fixed $\overline{\omega}_\beta$, the following is true, for almost every ω_α. For every $(M,\Lambda) \in \Delta^{I-1} \times \text{int } R_+^{I-1}$, $f_{ji}(M,\Lambda,\overline{\omega}_\beta) \neq \omega_{i1}(a_2^j)$, for $j = 1,2,$ or 3 and for some $i = 1,\ldots,I-1$,

where $f_{ji}(M,\Lambda,\overline{\omega}_\beta) = -(1+r)^{2-N_j}(P_1(\Lambda,\omega^T,a_2^j))^{-1}\{M_i((1+r)^{N_j-1} - 1) + \sum_{n=2}^{N_j} (1+r)^{N_j-n}$ $[P(\Lambda,\omega^T,a_2^j)\cdot(\omega_i(a_n^j)-X_i(\Lambda,\omega^T,a_2^j))-\tau_j-(1+r)^{N_j-2}P_1(\Lambda,\omega^T,a_2^j)\omega_{i1}(a_2^j)]\}$. In this formula, ω^T is the total initial endowment determined by $\overline{\omega}_\beta$.

Now let $F\colon \Delta^{I-1} \times \text{int } R_+^{I-1} \to R^{3I-3}$ be defined by $F(M,\Lambda) = (f_{11}(M,\Lambda,\overline{\omega}_\beta),$ $\ldots,f_{1,I-1}(M,\Lambda,\overline{\omega}_\beta),f_{21}(M,\Lambda,\overline{\omega}_\beta),\ldots,f_{3,I-1}(M,\Lambda,\overline{\omega}_\beta))$. (9.14) is implied by the next statement.

9.15) For each $\overline{\omega}_\beta$ the range of F has measure zero.

The dimension of the domain of F is $2I-2$. The dimension of its range is $3I-3$, which exceeds $2I-2$ since $I \geq 2$. By lemma 9.5, F is continuously differentiable on each of finitely many sets closed in $\Delta^{I-1} \times \text{int } R_+^{I-1}$. These facts imply (9.15).

This completes the proof of lemma 9.2 and hence of theorem 4.6.

Q.E.D.

10) <u>Proof of Theorem 4.3</u>

Let $\omega \in \Omega$ be such that

10.1) if $(p,(x_i))$ is a stationary equilibrium for $\mathcal{E}(\omega)$ with transfer payments, then for any distribution of initial money balances and for any $a_1 \in A, M_{in}(p,x_i;a_1,\ldots,a_n) < 0$, for some i and some history $a_2,\ldots,a_n$ following a_1.

By lemma 9.2, it is enough to prove that if $\delta_i = (1+r)^{-1}$, then $\mathcal{E}(\omega)$ has no monetary equilibrium.

The outline of the proof that $\mathcal{E}(\omega)$ has no monetary equilibrium is as follows. If $\mathcal{E}(\omega)$ had a monetary equilibrium, then the associated marginal utilities of money for each consumer would form a supermartingale. Hence by the supermartingale convergence theorem, they would converge, so that they would eventually be nearly constant. If they were nearly constant, then some consumer would eventually exhaust his holdings of money. This contradiction establishes that $\mathcal{E}(\omega)$ has no monetary equilibrium. The idea that a consumer would exhaust his money holdings is used to prove the following lemma.

10.2 <u>Lemma</u> Let $\underline{\lambda}$ and $\overline{\lambda}$ be positive numbers and that $\underline{\lambda} < \overline{\lambda}$. Then, there exists a positive integer K, depending on $\underline{\lambda}$ and $\overline{\lambda}$, such that the following is true. Let $(p,(x_i))$ be any monetary equilibrium and let (λ_i) be the vector of associated marginal utilities of money. Suppose that $\underline{\lambda} \leq \lambda_{in}(a_1,\ldots,a_n) \leq \overline{\lambda}$, for all n and $a_1,\ldots,a_n$. Then for any history $a_1,\ldots,a_n$, the following must hold for some i and for some history

$a_{n+1},\ldots,a_{n+k}$ following a_n, where $1 \leq k \leq K$. $|\lambda_{i,n+k}(a_i,\ldots,a_{n+k}) - \lambda_{in}(a_i,\ldots,a_n)| > K^{-1}$.

<u>Proof</u> Suppose that the lemma were false. Then, there would exist a sequence of monetary equilibria $(p^K,(x_i^K))$ such that for some history $a_1^K,\ldots,a_{n_K}^K$, $|\lambda^K_{i,n_K+k}(a_1^K,\ldots,a_{n_K}^K,a_{n_K+1},\ldots,a_{n_K+k}) - \lambda^K_{in_K}(a_1^K,\ldots,a_{n_K}^K)| \leq K^{-1}$, for all histories $a_{n_K+1},\ldots,a_{n_K+k}$ following $a_{n_K}^K$, where $1 \leq k \leq K$.

Since there are only finitely many points in A, I may assume that $a_{n_K}^K = \bar{a}_1 \varepsilon A$, for all K. Also I may assume that $n_K = 1$, for all K. For I may restrict $(p^K,(x_i^K))$ to histories following $a_1^K,\ldots,a_{n_K}^K$. That is, I may define $(\hat{p},(\hat{x}_i))$ by $\hat{p}_n(\bar{a}_1,a_2,\ldots,a_n) = p^K_{n_K+n}(a_1^K,\ldots,a_{n_K}^K,a_2,\ldots,a_n)$ and so on. $(\hat{p},(\hat{x}_i))$ is defined only for histories starting with $\bar{a}_1$, but this is sufficient for my purposes. I use $(p^K,(x_i^K))$ again to denote $(\hat{p},(\hat{x}_i))$. In summary, I may assume that

10.3) $\quad |\lambda^K_{ik}(\bar{a}_1,a_2,\ldots,a_k) - \lambda^K_{i1}(\bar{a}_1)| \leq K^{-1}$, for all histories $\bar{a}_1,a_2,\ldots,a_k$ beginning with $\bar{a}_1$, where $1 \leq k \leq K$.

By a Cantor diagonal argument, I may obtain a subsequence of monetary equilibria, call it $(p^K,(x_i^K))$ again, such that $(p^K,(x_i^K))$ and (λ_i^K) all converge. The limit, $(p,(x_i))$, forms a monetary equilibrium (restricted to histories beginning with $\bar{a}_1$). In proving this fact, one proceeds as in the last section of the proof of theorem 4.1. The limit marginal utilities of money (λ_i) are the marginal utilities of money corresponding to $(p,(x_i))$. By passage to the limit in (10.3), I obtain that $\lambda_{in}(\bar{a}_1,a_2,\ldots,a_n) = \lambda_i(\bar{a}_1)$, for all i and for all histories $\bar{a}_1,a_2,\ldots$ beginning with $\bar{a}_1$.

By lemma 9.3 there is a unique stationary equilibrium $(\hat{p},(\hat{x}_i))$ with marginal utilities of money $(\lambda_i(\bar{a}_1))$. Hence, $x_{in}(\bar{a}_1,a_2,\ldots,a_n) = \hat{x}_i(a_n)$ and $p_n(\bar{a}_1,a_2,\ldots,a_n) = \hat{p}(a_n)$, for all histories $\bar{a}_1,a_2,\ldots,a_n$. It now follows from (10.1) that $M_{in}(p,x_i;\bar{a}_1,a_2,\ldots,a_n) < 0$, for some i and some $a_2,\ldots,a_n$. This contradicts the fact that $(p,(x_i))$ is a monetary equilibrium. This completes the proof of lemma 10.2

Q.E.D.

I may now prove theorem 4.3.

<u>Proof of theorem 4.3</u> Let $(p,(x_i))$ be a monetary equilibrium and let (λ_i) be the associated marginal utilities of money. Since $\delta_i(1+r) = 1$, (2.8) implies that

$$\lambda_{in}(a_1,\ldots,a_n) \geqq E\,[\lambda_{i,n+1}(a_1,\ldots,a_n,s_{n+1}) \mid s_n = a_n], \text{ for all } i,n \text{ and } a_1,\ldots,a_n. \tag{10.4}$$

(10.4) says that the random variables $\lambda_{in}(s_1,\ldots,s_n)$ form a supermartingale. Since the $\lambda_{in}(s_1,\ldots,s_n)$ are non-negative, I may apply the supermartingale convergence theorem (Doob [22], p. 324). This theorem implies that the $\lambda_{in}(s_1,\ldots,s_n)$ converge almost surely. Let $\lambda_{i\infty}(s_1,s_2,\ldots)$ be the limit random variable, for $i = 1,\ldots,I$.

By (2.12) or (2.14), the components of the λ_i are bounded away from zero and infinity, so that I may apply lemma 10.2. Let K be as in the lemma.

Since $\lim_{n \to \infty} \lambda_{in}(s_1,\ldots,s_n) = \lambda_{i\infty}(s_1,s_2,\ldots)$ almost surely, there exists N such that Prob $[\ |\lambda_{in}(s_1,\ldots,s_n) - \lambda_{i\infty}(s_1,s_2,\ldots)| \geqq (2K)^{-1}$, for some i and some $n \geqq N] < \epsilon^K$, where $\epsilon = \min\{P_{ab} \mid a,b \epsilon A, P_{ab} > 0\}$ and the P_{ab} are the transition probabilities of the Markov chain $\{s_n\}$. It follows that there exists a history $a_1,\ldots,a_N$ such that Prob $[\ |\lambda_{in}(a_1,\ldots,a_N,s_{N+1},\ldots,s_n) - \lambda_{iN}(a_1,\ldots,a_N)| \geqq K^{-1}$, for some $n > N \mid s_N = a_N] < \epsilon^K$. But for any history $a_{N+1},\ldots,a_n$ following a_N with $n \leqq N + K$, Prob $[(s_{N+1},\ldots,s_n) = (a_{N+1},\ldots,a_n) \mid s_N = a_N] \geqq \epsilon^K$. Therefore $|\lambda_{in}(a_1,\ldots,a_N,a_{N+1},\ldots,a_n) - \lambda_{iN}(a_1,\ldots,a_N)| < K^{-1}$, for all n such that $N < n \leqq N + K$ and for all $a_{N+1},\ldots,a_n$ following a_N. This statement contradicts lemma 10.2 This completes the proof of theorem 4.3.

Q.E.D.

11) Proof of Theorem 4.4

The rough idea of the argument is as follows. If the price systems p^k did not converge to zero, then the sequence $(p^k,(x_i^k))$ would have a limit point, $(p,(x_i))$, which would be much like a monetary equilibrium with interest rate equal to $\delta^{-1}-1$. An argument similar to the proof of theorem 4.3 shows that no such limit equilibrium exists, almost surely.

There is a snag in this argument. Prices in a monetary equilibria are uniformly bounded away from zero and they need not be so in $(p,(x_i))$. However, prices in $(p,(x_i))$ are bounded above and this fact makes it possible to imitate the proof of theorem 4.3.

The limit equilibrium $(p,(x_i))$ is what I call a pseudo monetary equilibrium.

A <u>pseudo</u> <u>monetary</u> <u>equilibrium</u> is a vector $(p,(x_i),(\lambda_i))$. $p=(p_n(a_1,\ldots,a_n))$ is a price system. (x_i) is an allocation in the usual sense. λ_i is a vector of marginal utilities of money, as before, except that I allow some or all of the numbers $\lambda_{in}(a_1, \ldots, a_n)$ to be infinite. $(p,(x_i),(\lambda_i))$ must satisfy the following conditions.

11.1) (x_i) is a feasible allocation.

11.2) For some $\underline{\lambda}>0$, $\lambda_{in}(a_1, \ldots, a_n) \geq \underline{\lambda}$, for all i, n, and $a_1, \ldots, a_n$.

For all i,n and $a_1, \ldots, a_n$,

11.3) $\lambda_{in}(a_1,\ldots,a_n) = \max\ [\alpha_{in}(a_1,\ldots,a_n), E(\lambda_{i,n+1}(a_1,\ldots,a_n,s_{n+1}) | s_n = a_n)]$,

11.4) $\lambda_{in}(a_1,\ldots,a_n) > E[\lambda_{i,n+1}(a_1,\ldots,a_n,s_{n+1}) | s_n = a_n]$,

only if $M_{i,n+1}(a_1, \ldots, a_n) = 0$,

11.5) $\lambda_{in}(a_1,\ldots,a_n) > \alpha_{in}(a_1,\ldots,a_n)$, only if $x_{in}(a_1,\ldots,a_n) = 0$, and

11.6) $M_{in}(p,x_i;a_1,\ldots,a_n) \geqq 0$ and $\sum_{i=1}^{I} M_{in}(p,x_i;a_1,\ldots,a_n) = 1$.

Notice that (11.2) and lemma 6.4 imply that $p_n(a_1,\ldots,a_n) \leqq b\,\underline{\lambda}^{-1}\,\bar{q}$, for all n and $a_1,\ldots,a_n$. Also, if $\lambda_{in}(a_1,\ldots,a_n) < \infty$ for some i , then by lemma 6.4 $p_n(a_1,\ldots,a_n) >> 0$. If $p_n(a_1,\ldots,a_n) > 0$ and $\lambda_{in}(a_1,\ldots,a_n) = \infty$, then $x_{in}(a_1,\ldots,a_n) = 0$.

I say that a pseudo monetary equilibrium is non-trivial if $\lambda_{in}(a_1,\ldots,a_n) < \infty$ for some i, n and $a_1,\ldots,a_n$.

In order to show that non-trivial pseudo-monetary equilibria do not exist almost surely, I introduce the concept of pseudo stationary equilibrium with transfer payments.

A <u>pseudo stationary equilibrium</u> with transfer payments is a vector $(p,(x_i),(\lambda_i))$, where p and the x_i are functions from A to R_+^L and each λ_i belongs to $(0,\infty]$. Notice that the λ_i may be infinite. $(p,(x_i),(\lambda_i))$ must satisfy the following conditions.

11.7) (x_i) is a feasible stationary allocation.

11.8) for all i and a, $\frac{\partial u_i(x_i(a),a)}{\partial z_k} \leqq \lambda_i p_k(a)$, for $k = 1,\ldots,L$, with equality if $x_{ik}(a) > 0$.

I say that a pseudo stationary equilibrium with transfer payments, $(p,(x_i),(\lambda_i))$, is non-trivial if $\lambda_i < \infty$, for some i. Then, $p(a) >> 0$, for all i. If $(p,(x_i),(\lambda_i))$ is non-trivial and $\lambda_i = \infty$, then $x_i(a) = 0$, for all a.

The lemma before simply says that lemma 9.2 applies to pseudo-stationary equilibrium.

11.9) Lemma. For almost every $\omega \in \Omega$, the following is true. Let $(p,(x_i)),(\lambda_i))$ be any non-trivial pseudo stationary equilibrium for $\mathcal{E}(\omega)$ with transfer payments. Then for any distribution of initial money balances and for any $a_1 \in A, M_{in}(p,x_i;a_1,\ldots,a_n) < 0$, for some i and some history $a_2,\ldots,a_n$ following a_1.

The proof of this lemma differs from that of (9.2) only in detail. Since there are only finitely many subsets of $\{1,\ldots,I\}$, one may fix the subset of consumers for whom $\lambda_i = \infty$. Those consumers consume nothing. The rest of the argument is as in section 9.

11.12) <u>Lemma</u> Let $\delta_i = \delta < 1$, for all i. There exist $\bar{p} \in R_+^L > 0$ and $\underline{r} > 0$ with $0 < \underline{r} < \delta^{-1} - 1$ such that the following are true. Let $(p,(x_i))$ be any monetary equilibrium with interest rate r and let (λ_i) be the vector of associated marginal utilities of money. If $\underline{r} \leq r < \delta^{-1} - 1$, then $p_n(a_1,\ldots,a_n) \leqq \bar{p}$ and $\lambda_{in}(a_1,\ldots,a_n) \geqq \underline{\lambda}$, for all i, n and all histories $a_1,\ldots,a_n$.

<u>Proof</u> It is sufficient to prove that $\underline{\lambda}$ and $\underline{r}$ exist as in the lemma. For by lemma 6.4, I may let $\bar{p} = b\underline{\lambda}^{-1}\bar{q}$.

Let $\varepsilon > 0$ be as in the proof of lemma 7.15 and let $\underline{\lambda} = \varepsilon(1+\varepsilon)^{-1}$ and $\underline{r} = \delta^{-1}(1+\varepsilon^2)^{-1}(1+2^{-1}\varepsilon^2) - 1$. By choosing ε sufficiently small, I may assure that $\underline{r} > 0$. Clearly, $\underline{r} < \delta^{-1} - 1$. Note that $\underline{\lambda}$ is as in lemma 7.15.

Suppose that $\underline{r} \leqq r < \delta^{-1} - 1$ and let $(p,(x_i))$ be a monetary equilibrium with interest rate r. Also, let (λ_i) be the vector of marginal utilities of money associated with $(p,(x_i))$. By (2.12) or (2.14) there exists $\lambda > 0$ such that $\lambda_{in}(a_1,\ldots,a_n) \geqq \lambda$, for all i,n and $a_1,\ldots,a_n$. Suppose that $\lambda < \underline{\lambda}$. I will show that

11.13) $\lambda_{in}(a_1,\ldots,a_n) \geqq \lambda(1+2^{-1}\varepsilon)^k$, for all i, n and $a_1,\ldots,a_n$ and for all positive integers k such that $\lambda(1+2^{-1}\varepsilon^2)^{k-1} < \underline{\lambda}$.

Clearly, (11.13) implies the lemma. For let k be such that $(1+2^{-1}\varepsilon^2)^{k-1}\lambda < \underline{\lambda}$ and $(1+2^{-1}\varepsilon^2)^k \lambda \geqq \underline{\lambda}$. By (11.13),

$\lambda_{in}(a_1,\ldots,a_n) \geq (1+2^{-1}\varepsilon^2)^k \lambda \geq \underline{\lambda}$, for all i,n and $a_1,\ldots,a_n$, as is to be proved.

I prove (11.13) by induction on k. (11.13) is true for $k = 0$, by the definition of λ. Suppose that k is such that $\lambda(1+2^{-1}\varepsilon^2)^k < \underline{\lambda}$ and assume by induction that $\lambda_{in}(a_1,\ldots,a_n) \geq \lambda(1+2^{-1}\varepsilon^2)^k$, for all i,n and $a_1,\ldots,a_n$. The proof of (7.26) proves that $\lambda_{in}(a_1,\ldots,a_n) \geq (1+\varepsilon)(1+2^{-1}\varepsilon^2)^k\lambda$, if $\omega_{ik}(a_n) \leq \delta$, for all k. But then, as in the proof of lemma 7.15, $\lambda_{in}(a_1,\ldots,a_n) \geq \delta(1+\underline{r})[1-\varepsilon) + \varepsilon(1+\varepsilon)](1+2^{-1}\varepsilon^2)^k\lambda = \lambda(1+2^{-1}\varepsilon^2)^{k+1}$. This completes the induction step in the proof of (11.13) and hence proves the lemma.

Q.E.D.

I now turn to the proof of theorem 4.4.

<u>Proof of Theorem 4.4</u> By lemma 11.9, I may assume that (11.10) applies to $\mathcal{E}(\omega)$. Let $(p^k,(x_i^k))$ be a sequence of monetary equilibria for $\mathcal{E}(\omega)$, the k^{th} having interest rate r_k. By lemma 6.4, it is sufficient to show that

11.14) if $(p^k,(x_i^k))$ is as in the theorem, then $\lim\limits_{k\to\infty} \lambda_{in}^k(a_1,\ldots,a_n) = \infty$ uniformly,

where (λ_i^k) is the vector of marginal utilities of money associated with $(p^k,(x_i^k))$.

Suppose that (11.14) were false. Then, there would exist a subsequence of $(p^k,(x_i^k))$, call it $(p^k,(x_i^k))$ again, with the following property. There is $\bar{\lambda} < \infty$ such that for each k, $\lambda^k_{i_k n_k}(a_1^k,\ldots a_{n_k}^k) \leqq \bar{\lambda}$, for some i_k and $a_1^k,\ldots a_{n_k}^k$. Since there are finitely many indices i and a, I may assume that $i_k = i$ and $a_{n_k}^k = \bar{a}_1$, for all k. Also, $(p^k,(x_i^k))$ forms an equilibrium when restricted to histories following $a_1^k,\ldots,a_{n_k}^k$. Therefore, I may assume that $n_k = 1$, for all k. In summary, I may assume that for some i and $\bar{a}_1$, $\lambda^k_{i1}(\bar{a}_1) \leqq \bar{\lambda}$, for all k. Without loss of generality, I may assume that $i = 1$, so that $\lambda^k_{11}(\bar{a}_1) \leqq \bar{\lambda}$, for all k.

By a Cantor diagonal argument, I can prove that there exists a subsequence of $(p^k,(x_i^k))$ such that $p_n^k(\bar{a}_1,a_2,\ldots,a_n)$, $x^k_{in}(\bar{a}_1,a_2,\ldots,a_n)$ and $\lambda^k_{in}(\bar{a}_1,a_2,\ldots,a_n)$ all converge for all i,n and for all histories $a_2,\ldots,a_n$ following $\bar{a}_1$. Let $(p,(x_i))$ and (λ_i) denote the vectors of limits. It is easy to see that $(p(x_i),(\lambda_i))$ is a pseudo monetary equilibrium, except that it is defined only for histories beginning with $\bar{a}_1$. (11.2) follows from lemma 11.12. (11.3) and (11.4) follow from the fact that $\lim_{k\to\infty}(1+r_k)\delta = 1$.

I now show that I may assume the following.

11.15) For every i, either $\lambda_{in}(\bar{a}_1,a_2,\ldots,a_n) = \infty$, for all $n \geqq 1$ and for all histories $\bar{a}_1,a_2,\ldots,a_n$, or $\lambda_{in}(\bar{a}_1,a_2,\ldots,a_n) < \infty$, for all $n \geqq 1$ and for all histories $\bar{a}_1,a_2,\ldots,a_n$.

Clearly, (11.3) implies that the following is true.

11.16) If $\lambda_{i1}(\bar{a}_1) < \infty$, then $\lambda_{in}(\bar{a}_1, a_2, \ldots a_n) < \infty$, for all i,n and $a_2, \ldots, a_n$.

I now proceed by induction on i. Since $\lambda_{11}(\bar{a}_1) \leq \bar{\lambda} < \infty$, (11.16) implies that (11.15) is true for $i = 1$. Suppose by induction that (11.15) is true for $i = 1, \ldots, j-1 < I$. If $\lambda_{j1}(\bar{a}_1) < \infty$, then (11.16) implies that (11.15) is true for j. If $\lambda_{jn}(\bar{a}_1, a_2, \ldots, a_n) = \infty$, for all n and $\bar{a}_1, a_2, \ldots, a_n$, then (11.15) is true for j. Suppose that $\lambda_{j1}(\bar{a}_1) = \infty$ and that $\lambda_{jn}(\bar{a}_1, a_2, \ldots, a_n) < \infty$, for some n and $a_2, \ldots a_n$. Then $(p, (x_i), (\lambda_i))$ forms a pseudo monetary equilibrium when restricted to histories beginning with $(\bar{a}_1, a_2, \ldots, a_n)$. Hence, I may relabel a_n as $\bar{a}_1$ and I have that (11.15) is true for $i = 1, \ldots, j$. This completes the induction step, and so I may assume that (11.15) is satisfied.

Now I proceed more or less as in the proof of theorem 4.3. Let $J = \{i=1, \ldots, I \mid \lambda_{i1}(\bar{a}_1) < \infty\}$. By assumption, $1 \in J$. For all $i \in J$, the random variables $\lambda_{in}(\bar{a}_1, s_2, \ldots, s_n)$ form a supermartingale. Hence, they converge almost surely.

Now let K be as in lemma 11.11 where the $\underline{\lambda}$ in lemma 11.11 is the same as the $\underline{\lambda}$ in lemma 11.12. Let the $\bar{\lambda}$ of lemma 11.11 be such that $\bar{\lambda} > 2\lambda_{11}(\bar{a}_1)$. Finally, let $\eta = \min\{P_{ab} \mid P_{ab} > 0,\ a, b, \in A\}$.

Since the random variables $\lambda_{in}(\bar{a}_1, s_2, \ldots s_n)$ converge almost surely, there exists N such that Prob $[|\lambda_{in}(\bar{a}_1, s_2, \ldots, s_N, s_{N+1}, \ldots, s_n) - \lambda_{iN}(\bar{a}_1, s_2, \ldots, s_N)| > K^{-1}$, for some $i \in J$ and for some $n > N \mid s_1 = \bar{a}_1] < 2^{-1}\eta^K$. Next, observe that $\text{Prob}[\lambda_{1N}(\bar{a}_1, s_2, \ldots, s_N) \leq \bar{\lambda} \mid s_1 = \bar{a}_1] \geq 1/2$, since $\bar{\lambda} > 2\lambda_{11}(\bar{a}_1)$ and the random variables $\lambda_{1n}(s_1, s_2, \ldots, s_n)$ form a non-negative supermartingale. These two inequalities imply that there exist $a_2, \ldots, a_N$ with the property that $\lambda_{1N}(\bar{a}_1, a_2, \ldots, a_N) \leq \bar{\lambda}$ and Prob $[|\lambda_{in}(\bar{a}_1, a_2, \ldots, a_N, s_{N+1}, \ldots, s_n) - \lambda_{iN}(\bar{a}_1, a_2, \ldots, a_N)| > K^{-1}$, for some $i \in J$ and some $n > N \mid s_N = a_N] < \eta^K$. Hence by the choice of η, $|\lambda_{in}(\bar{a}_1, a_2, \ldots, a_N, a_{N+1}, \ldots, a_n) - \lambda_{in}(\bar{a}_1, a_2, \ldots, a_N)| \leq K^{-1}$, for all $i \in J$ and for all $a_{N+1}, \ldots, a_n$ following a_N such that $N \leq n \leq N+K$. This contradicts lemma 11.11. This completes the proof of theorem 4.4.

Q.E.D.

12) Proof of Theorem 4.5

What follows is largely a reinterpretation of the argument given by Arrow in his pinoeering paper [1].

I first prove that a stationary equilibrium exists. In order to do so, I define a pure trade economy, $\mathcal{E}$, which represents, roughly speaking, a cross section of the economy at one moment in time. The commodity space of $\mathcal{E}$ is $R^{L|A|}$, where $|A|$ is the number of points in A. I write $x \in R^{L|A|}$ as $x = (x(a))_{a \in A}$, where $x(a) \in R^L$. The initial endowment of the i^{th} consumer is $\omega_i = (\omega_i(a))_{a \in A}$, $i = 1,\ldots,I$. The utility function of the i^{th} consumer is $\sum_{a \in A} \pi_a u_i(x_i(a),a)$, where (π_a) is the stationary distribution on A.

By Debreu [21], p. 83, $\mathcal{E}$ has an equilibrium, $(p,(x_i))$. By the strict monotonicity of the function $u_i(\cdot,a)$, $p(a) \gg 0$, for all a. Let $\bar{p} = (\bar{p}(a))$, where $\bar{p}(a) = \pi_a^{-1} p(a)$. $\bar{p}(a)$ is well-defined, since $\pi_a > 0$, for all a (see assumption 3.2). I claim that $(\bar{p}, \delta^{-1} - 1, (x_i))$ is a stationary equilibrium with deflation rate $\delta^{-1} - 1$. First of all, $\sum_{a \in A} \pi_a \bar{p}(a) \cdot x(a) = \sum_{a \in A} p(a) \cdot x(a) = p \cdot x$, for all $x \in R^{L|A|}$. Hence, the fact that $(p,(x_i))$ is an Arrow-Debreu equilibrium implies that for each i, x_i solves the problem,

$$12.1) \qquad \max \{ \sum_{a \in A} \pi_a u_i(x,a) \mid x \in R_+^{L|A|} \text{ and } \sum_{a \in A} \pi_a \bar{p}(a) \cdot (x(a) - \omega_i(a)) \leq 0 \}.$$

Clearly, $\bar{p}$ is a stationary price system and (x_i) is a feasible stationary allocation. Hence, $(\bar{p}, \delta^{-1} - 1, (x_i))$ is a stationary equilibrium with deflation rate $\delta^{-1} - 1$.

Suppose that $\delta = 1$ and that $(p, 0, (x_i))$ is a stationary equilibrium with deflation rate zero. I prove that (x_i) is Pareto optimal. Let $\lambda_i > 0$ be the Lagrange multiplier associated with the maximization problem 12.1. Recall that the consumption program $\hat{x}_i$ is defined by $\hat{x}_{in}(a_1, \ldots, a_n) = x_i(a_n)$, for all i, n and $a_1, \ldots, a_n$. I must show that $(\hat{x}_i)$ is Pareto optimal in the sense of (2.1). $(\hat{x}_i)$ solves the first order conditions of the social maximization problem,

$$12.2) \qquad \max \left\{ \sum_{i=1}^{I} \lambda_i^{-1} E \left[\sum_{n=1}^{N} u_i(y_{in}(s_1, \ldots, s_n), s_n) \right] \;\middle|\; (y_i) \text{ is a feasible allocation} \right\},$$

for all values of N. For by the definition of λ_i,

$$12.3) \qquad \frac{\partial u_i(x_i(a_n), a_n)}{\partial z_k} \leqq \lambda_i p_k(a_n), \quad \text{for all } k, i, n \text{ and } a_n,$$

$$\text{with equality if the } x_{ik}(a_n) > 0.$$

(12.3) gives the first order conditions for a solution to (12.2). Since the objective function of (12.2) is concave, it is sufficient to satisfy the first order conditions. Hence, $(\hat{x}_i)$ solves (12.2). This proves that (x_i) is Pareto optimal.

Suppose now that $\delta < 1$ and that $(p, \delta^{-1} - 1, (x_i))$ is a stationary equilibrium (with deflation rate $\delta^{-1} - 1$). The proof that (x_i) is Pareto optimal is exactly as in the previous paragraph, except that (12.2) and (12.3) become the two formulas below.

$$\max \{ \sum_{i=1}^{I} \lambda_i^{-1} E [\sum_{n=1}^{\infty} \delta^{n-1} u_i(y_{in}(s_1,\ldots,s_n),s_n)] \mid (y_i) \text{ is}$$

$$\text{a feasible allocation}\}.$$

$$\delta^{n-1} \frac{\partial u_i(x_i(a_n),a_n)}{\partial z_k} \leqq \lambda_i \delta^{n-1} p_k(a_n), \quad \text{for all } k,i,n \text{ and } a_n,$$

with equality if $x_{ik}(a_n) > 0$.

13) An Example

The following example illustrates why special assumptions are needed in theorems 4.2, 4.3 and 4.4. In the example, the Markov chain $\{s_n\}$ is cyclic and has two states. There are two consumers. One consumer has a relatively high preference for consumption in one state, and the other consumer prefers consumption in the other state. Therefore, there is a Pareto optimal allocation in which each consumer consumes the entire endowment of both consumers when his preferred state occurs. I call this allocation the alternating allocation. I assume that no interest is paid on money. At first, I assume that the consumers' rates of time preference are positive and show that the alternating Pareto optimal allocation is the allocation of a monetary equilibrium. Thus, theorem 4.2 does not hold, and the example justifies the assumption made in theorem 4.2 that each consumer always consumes something.

I next assume that each consumer's rate of time preference is equal to the interest rate, which is zero. In this case, the alternating Pareto optimal allocation is still that of a monetary equilibrium, so that the optimum quantity of money is finite. This is so, even if the endowment functions are allowed to vary over an open set. Hence, theorems 4.3 and 4.4 do not hold, and the example shows need for assumption 3.10 in these theorems.

I now describe the example. $A = \{a,b\}$. The transition probabilities are defined by $P_{aa} = P_{bb} = 0$ and $P_{ab} = P_{ba} = 1$. Thus, the stochastic process $\{s_n\}$ alternaties between a and b. The process starts at time 1 in state a with probability 1/2 and in state b with probability 1/2. There are two consumers and one good. The endowments are defined by $\omega_1(a) = \omega_1(b) = \omega_2(a) = \omega_2(b) = 1$. The utility functions are defined as follows. $u_1(x,a) = u_2(x,1) = 12 \log (1+x)$, where x is the quantity consumed of the good. $u_1(x,b) = u_2(x,a) = 3 \log (1+x)$. The initial holdings of money are

$M_{10} = M_{20} = 1/2$. Money earns no interest. That is, $r = 0$.

Suppose that $1/2 < \delta_i < 1$, for $i = 1,2$. I claim the following is a monetary equilibrium. The price system, p, is defined by $p_1(a) = p_1(b) = 1/2$ and $p_n(a_1,\ldots,a_n) = 1$, for all $n > 1$ and for all $a_1,\ldots,a_n$. The allocation, (x_1,x_2), is defined as follows. For all n and $a_1,\ldots,a_n$,

$$x_{1n}(a_1,\ldots,a_n) = \begin{cases} 2, & \text{if } a_n = a \\ 0, & \text{if } a_n = b, \text{ and} \end{cases}$$

$$x_{2n}(a_1,\ldots,a_n) = \begin{cases} 0, & \text{if } a_n = a \\ 2, & \text{if } a_n = b. \end{cases}$$

It should be clear for $n \geq 1$,

$$M_{1n}(a_1,\ldots,a_n) = \begin{cases} 0, & \text{if } a_n = a \\ 1, & \text{if } a_n = b, \text{ and} \end{cases}$$

$$M_{2n}(a_1,\ldots,a_n) = \begin{cases} 1, & \text{if } a_n = a \\ 0, & \text{if } a_n = b \end{cases} .$$

In order to verify that the above is an equilibrium, it is enough to verify that each consumer satisfies the first order conditions for his optimization problem. Since the consumers are symmetric, I need only deal with consumer 1. It is easy to see that his marginal utility of expenditure, $\alpha_1 = (\alpha_{1n}(a_1,\ldots,a_n))$, is as follows. $\alpha_{11}(a) = 8$. $\alpha_{11}(b) = 2$. If $n > 1$, then

$$\alpha_{1n}(a_1,\ldots,a_n) = \begin{cases} 4, \text{ if } a_n = a \\ 1, \text{ if } a_n = b \end{cases} .$$

It follows easily that his marginal utility of money is as follows.
$\lambda_{11}(a) = 8$. $\lambda_{11}(b) = 4\delta_1$. If $n > 1$, then

$$\lambda_{1n}(a_1,\ldots,a_n) = \begin{cases} 4, \text{ if } a_n = a \\ 4\delta_1, \text{ if } a_n = b. \end{cases}$$

Finally,

$$E(\lambda_{1,n+1}(a_1,\ldots,a_n,s_{n+1}) \mid a_n = a) = \begin{cases} 4\delta_1, \text{ if } a_n = a \\ 4, \text{ if } a_n = b \end{cases} .$$

It is easy to see that α_1 and λ_1 satisfy conditions (2.8) - (2.10). For instance, $\lambda_{11}(a) = 8 > 4\delta^2 = \delta_1 E[\lambda_{12}(a_1,s_2) \mid a_1 = a]$, and $M_{11}(a) = 0$. Also, $\alpha_{11}(b) = 2 < 4\delta_1 = \lambda_{11}(b)$ and $x_{11}(b) = 0$.

It should be clear that the allocation (x_1,x_2) is Pareto optimal, even though the rate of interest is less than the rate of time preference. Hence, some special condition is needed in theorem 4.2

Now suppose that $\delta_1 = \delta_2 = 1$. Let the initial endowments satisfy the condition $3/4 < \omega_i(c) < 5/4$, for $c = a,b$ and $i = 1,2$. Let the endowment of money and the utility functions be as before. I claim that $(p,(x_1,x_2))$ is a monetary equilibrium, where $(p,(x_1,x_2))$ is defined as follows.
$p_1(a) = (2\omega_2(a))^{-1}$, $p_1(b) = (2\omega_1(b))^{-1}$. If $n > 1$,

$$p_n(a_1,\ldots,a_n) = \begin{cases} (\omega_2(a))^{-1}, & \text{if } a_n = a \\ (\omega_1(b))^{-1}, & \text{if } a_n = b \end{cases} .$$

$$x_{in}(a_1,\ldots,a_n) = \begin{cases} \omega_1(a_n) + \omega_2(a_n), & \text{if } a_n = a \\ 0, & \text{if } a_n = b \end{cases} .$$

$$x_{2n}(a_1,\ldots,a_n) = \begin{cases} 0, & \text{if } a_n = a \\ \omega_1(a_n) + \omega_2(a_n), & \text{if } a_n = b \end{cases} .$$

It is easy to verify that this allocation is Pareto optimal in the sense of (2.1). Hence, theorems 4.3 and 4.4 do not apply.

REFERENCES

1. Arrow, Kenneth, "The Role of Securities in the Optimal Allocation of Risk-bearing," Review of Economic Studies, 31 (1963-4), 91-96.
2. Arrow, Kenneth, "Limited Knowledge and Economic Analysis," American Economic Review, 64 (1974), 1-10.
3. Arrow, Kenneth and Frank Hahn, General Competitive Analysis (Holden Day, San Francisco: 1971).
4. Bailey, Martin, "The Welfare Cost of Inflationary Finance," Journal of Political Economy, 64 (1956), 93-110.
5. Baumol, William, "The Transactions Demand for Cash: an Inventory Theoretic Approach," Quarterly Journal of Economics, 66 (1952), 545-556.
6. Bewley, Truman, "The Permanent Income Hypothesis and Welfare Economics," mimeographed, Univ. of Bonn, Germany, 1977.
7. Bewley, Truman, "The Permanent Income Hypothesis and Long-run Economic Stability," mimeographed, Univ. of Bonn, Bonn, Germany (1977).
8. Bewley, Truman, "The Permanent Income Hypothesis and Short-run Price Stability," mimeographed, Univ. of Bonn, Bonn, Germany (1977).
9. Bewley, Truman, "The Permanent Income Hypothesis: A Theoretical Formulation," Journal of Economic Theory, 16 (Dec. 1977), 252-292.
10. Breiman, Leo, Probability (Addison-Wesley, Reading, Mass: 1968).
11. Brock, William, "Money and Growth: the Case of Long Run Perfect Foresight," International Economic Review, 15 (1974), 750-777.
12. Brock, William, "A Simple Perfect Foresight Monetary Model," Journal of Monetary Economics, 1 (1975), 133-150.
13. Brunner, Karl and Allan Meltzer, "The Uses of Money: Money in the Theory of an Exchange Economy," American Economic Review, 61 (1971), 784-805.
14. Cagan, Phillip, "The Monetary Dynamics of Hyperinflation," in Milton Friedman (ed.) Studies in the Quantity Theory of Money (Univ. of Chicago Press, Chicago: 1956), pp. 25-117.
15. Calvo, Guillermo, "On Models of Money and Perfect Foresight," International Economic Review, 20 (1979), 83-103.
16. Cass, David and Masahiro Okuno and Itzhak Zilcha," The Role of Money in Supporting the Pareto Optimality of Competitive Equilibrium in Consumption-Loan Type Models," mimeographed, (Nov. 1978).

17. Clower, Robert, "A Reconsideration of the Microfoundations of Monetary Theory," Western Economic Journal, 6 (1967), 1-9.

18. Clower, Robert, "Comment: The Optimal Growth Rate of Money," Journal of Political Economy, 76 (1968), 876-880.

19. Clower, Robert, "Is There an Optimal Money Supply," Journal of Finance, 25 (1972), 425-433.

20. Clower, Robert and Peter Howitt, "The Transactions Theory of the Demand for Money: A Reconsideration," Journal of Political Economy, 86 (1976), 449-466.

21. Debreu, Gerard, Theory of Value (Wiley, New York: 1959).

22. Doob, J.L., Stochastic Processes, (Wiley, New York: 1953).

23. Drandrakis, Emmanuel, "On the Competitive Equilibrium in a Monetary Economy," International Economic Review, 7 (1966), 304-328.

24. Feige, Edgar and Michael Parkin, "The Optimal Quantity of Money, Bonds, Commodity Inventories, and Capital," American Economic Review, 61 (1971), 335-349.

25. Feige, Edgar and M. Parkin, R. Avery and C. Stones, "The Roles of Money in an Economy and the Optimum Quantity of Money," Economica n.s., 40 (1973), 416-431.

26. Foley, Duncan and Martin Hellwig, "Asset Management with Trading Uncertainty," Review of Economic Studies, 42 (July 1975), 327-396.

27. Friedman, Milton, "Discussion of the Inflation Gap," Essays in Positive Economics, (Chicago Univ. Press, Chicago: 1953), pp. 253-7.

28. Friedman, Milton, "The Optimum Quantity of Money," in The Optimum Quantity of Money and Other Essays (Aldine Publishing Co., Chicago: 1969), pp. 1-50.

29. Gale, David, "Pure Exchange Equilibrium of Dynamic Economic Models," Journal of Economic Theory, 6 (1973), 12-36.

30. Grandmont, Jean-Michel, "On the Short-run Equilibrim in a Monetary Economy," Allocations Under Uncertainty: Equilibrium and Optimality, Jacques Drèze (ed.), (Macmillan, Edinburgh: 1974).

31. Grandmont, Jean Michel, "Temporary General Equilibrium Theory," Econometrica, 45 (1977), 535-572.

32. Grandmont, Jean-Michel and Guy Laroque, "Money in the Pure Consumption Loan Model," Journal of Economic Theory, 6 (1973), 382-395.

33. Grandmont, Jean-Michel and Yves Younes, "On the Role of Money and the Existence of a Monetary Equilibrium," Review of Economic Studies, 39 (1972), 355-372.

34. Grandmont, Jean-Michel and Yves Younes, "On the Efficiency of a Monetary Equilibrium," Review of Economic Studies, 40 (April 1973), 149-165.

35. Hahn, Frank, "On Some Problems of Proving the Existence of an Equilibrium in a Monetary Economy," in The Theory of Interest Rates, F. Hahn and F. Brechling (eds.), (Macmillan, London: 1965).

36. Hahn, Frank, "Professor Friedman's Views on Money," Economica, 38 (Feb. 1971), 61-80.

37. Hahn, Frank, "Equilibrium with Transactions Costs," Econometrica, 39 (May 1971), 417-439.

38. Hahn, Frank, "On the Foundations of Monetary Theory," Essays in Modern Economics, Michael Parkin and A.R. Nobay (eds.) (Longman, London: 1973).

39. Hahn, Frank, "On Transaction Costs, Inessential Sequence Economies and Money," Review of Economic Studies, 40 (1973), 449-461.

40. Heller, Walter P., "The Holding of Money Balances in General Equilibrium," Journal of Economic Theory, 7 (1974), 93-108.

41. Heller, Walter P. and Ross Starr, "Equilibrium with Non-convex Transactions Costs: Monetary and Non-monetary Economies," Review of Economic Studies, 43 (June 1976), 195-215.

42. Hool, Bryce, "Money, Expectations and the Existence of a Temporary Equilibrium," Review of Economic Studies, 43 (1976), 439-445.

43. Johnson, Harry, "Equilibrium Under Fixed Exchanges," American Economic Review, 53 (May 1963), 112-119.

44. Johnson, Harry G., "Money in a Neo-classical One Sector Growth Model, in Essays in Monetary Economics (Allen and Unwin, London: 1967).

45. Johnson, Harry G., "Is there and Optimal Money Supply," Journal of Finance, 25 (1970), 425-442.

46. Jones, Robert, "The Origin and Development of Media of Exchange," Journal of Political Economy, 84 (1976), 757-775.

47. Kurz, Mordecai, "Equilibrium with Transaction Cost and Money in a Single Market Exchange Economy," Journal of Economic Theory, 7 (1974), 418-452.

48. Levhari, David and Don Patinkin, "The Role of Money in a Simple Growth Model," American Economic Review, 58 (1968), 713-753.

49. Lucas, Robert, "Equilibrium in a Pure Currency Economy," mimeo, Univ. of Chicago, Sept. 1968.

50. Marty, Alvin, "The Optimal Rate of Growth of Money," Journal of Political Economy, 76 (1968), 860-873.

51. Niehans, Jürg, "Money in a Static Theory of Optimal Payment Arrangements," Journal of Money, Credit and Banking, 1 (1969), 706-725.

52. Niehans, Jürg, "Money and Barter in General Equilibrium with Transactions Costs," American Economic Review, 61 (1971), 773-783.

53. Niehans, Jürg, "Interest and Credit in General Equilibrium with Transactions Costs," American Economic Review, 65 (1975), 548-566.

54. Ostroy, Joseph, "The Informational Efficiency of Monetary Exchange," American Economic Review, 63 (1973), 597-610.

55. Ostroy, Joseph and Ross Starr, "Money and the Decentralization of Exchange," Econometrica, 42 (1974), 1093-1113.

56. Perlman, Morris, "The Roles of Money in an Economy and the Optimum Quantity of Money," Economica n.s., 38 (1971), 233-252.

57. Perlman, Morris, "The Roles of Money in an Economy and the Optimum Quantity of Money: A Reply," Economica n.s., 40 (1973), 432-441.

58. Phelps, Edmund, Inflation Policy and Unemployment Theory (Norton, New York: 1972).

59. Phelps, Edmund, "Inflation in the Theory of Public Finance," Swedish Journal of Economics 75 (1973), 67-82.

60. Russel, Thomas,"Feige and Parkin on the Optimal Quantity of Money," American Economic Review, 64 (1974), 1074-6.

61. Samuelson, Paul, "An Exact Consumption-Loan Model of Interest With or Without the Social Contrivance of Money, Journal of Political Economy, 66 (1958), 467-82.

62. Samuelson, Paul, "D.A. Robertson (1890-1963)", The Quarterly Journal of Economics, 77 (1963), 517-536.

63. Samuelson, Paul, "What Classical and Neoclassical Monetary Theory Really Was," Canadian Journal of Economics, 1 (1968), 1-15.

64. Samuelson, Paul, "Nonoptimality of Money Holding Under Laissez Faire," Canadian Journal of Economics, 2 (1969), 303-308.

65. Saving, Thomas R., "Transactions Costs and the Demand for Money", American Economic Review, 61 (1971), 407-420.

66. Schechtman, Jack, "An Income Fluctuation Problem," Journal of Economic Theory, 12 (1976), 218-241.

67. Shubik, Martin, "The Capital Stock Modified Competitive Equilibrium," Cowles Foundation Discussion Paper No. 507, (Oct. 1978).

68. Sidrauski, M., "Inflation and Economic Growth," Journal of Political Economy, 75 (1967), 796-810.

69. Sontheimer, Kevin, "On the Determination of Money Prices," Journal of Money, Credit and Banking, 4 (1972), 489-508.

70. Starr, Ross, "The Structure of Exchange in Barter and Monetary Economies," Quarterly Journal of Economics, 86 (1972), 290-302.

71. Starr, Ross, "The Price of Money in a Pure Exchange Monetary Economy with Taxation," Econometrica, 42 (1974), 45-54.

72. Starrett, David, "Inefficiency and the Demand for 'Money' in a Sequence Economy," Review of Economic Studies, 40 (1973), 437-448.

73. Stein, Jerome, "Monetary Growth Theory in Perspective," American Economic Review, 60 (1970) 85-106.

74. Tobin, James, "The Interest-Elasticity of Transactions Demand for Money," Review of Economics and Statistics, 38 (1956), 241-247.

75. Tobin, James, "Money and Economic Growth," Econometrica, 33 (1965), 671-684.

76. Tobin, James, "Notes on Optimal Monetary Growth," Journal of Political Economy, 76 (1968), 833-859.

77. Tsiang, S.C., "A Critical Note on the Optimum Supply of Money," Journal of Money, Credit and Banking, 1 (1969), 266-280.

78. Wallace, Neil, "The Overlapping-Generations Model of Fiat Money," Staff Report #37, Univ. of Minnesota and Federal Reserve Bank of Minneapolis.

79. Wilson, Charles, "An Infinite Horizon Model with Money," Social Systems Research Institute paper No. 7814, Univ. of Wisconsin-Madison.