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A DYNAMIC MICROECONOMIC MODEL WITH DURABLE GOODS AND ADAPTIVE EXPECTATIONS

by

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Abstract. This paper develops a new model of a dynamic economy with durable goods that is simple enough so that interesting examples can be solved, even with limited computational capacity. The model is of temporary equilibrium in continuous time, with money, bonds, and durable goods. Individuals form estimates of the marginal values of assets through a learning process of adaptive expectations. Properties of existence and uniqueness are shown for the equilibrium prices. A simple example is considered, to show how the model can simulate many of the dynamic adjustment problems of the business cycle.

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1. Introduction.

In a dynamic economy, the problem of estimating the values of durable assets (capital goods, inventories, and bonds) can be extremely complex; and the efforts of economic agents to solve this problem are a critical factor in the business cycle. Leijonhufvud (1968) has argued that the essential ideas of Keynes must be understood in the context of a dynamic economic model in which this problem is a central issue. Such a model should be able to show us how a shock in one sector of the economy can lead to systematic misvaluations of assets in other sectors, and how these misvaluations can endure for an extended period of time. The goal of this paper is to develop a new model of this type.

To be most useful, a theoretical model must be simple enough so that we can actually analyze or simulate interesting examples. To achieve such simplicity, it is common in the theoretical economic literature to assume either that there are no durable goods, or that there is only one good, or that the economy is in a steady state. Unfortunately, all of these assumptions eliminate the whole question of how individuals learn the values of durable intermediate goods. Thus, the main task of this paper is to develop a new set of simplifying assumptions, to create a <u>tractable</u> model of the dynamic economy with durable goods.

Our model will be a temporary equilibrium model in continuous time. The central assumption of temporary equilibrium (developed by Hicks (1965); see Grandmont (1977) for a survey of the literature) is that, at every point in time, the individuals should plan their current transactions so as to maximize their utility given the current market prices and given their expectations about the future, while the prices should always keep desired current sales equal to desired current purchases. It is assumed that expectations are some function of past experience, and may not prove to be always correct.

Since we are studying a temporary equilibrium model, with prices as the only market signals controlling individual behavior, it follows that we are ruling out involuntary unemployment by assumption, A more complicated model with quantity constraint signals as well as price signals could give us a concept of market equilibrium with involuntary unemployment; for example, see Barro and Grossman (1976), Drèze (197 5), and Myerson (1981). However, involuntary underemployment can arise in our model, depending on the state of expectations. That is, our model may describe situations in which engineers are driving taxicabs even though there is valuable engineering work to be done. The key idea may be found in the suggestion of Leijonhufvud (1968, Chapter VI:1) that we distinguish between the <u>control</u> and <u>communication</u> roles of prices. In our temporary equilibrium approach, we assume that prices are flexible enough to equalize desired sales with desired purchases at every point in time, so prices do succeed in their role of controlling the current balance of supply and demand. But we shall see that, in a dynamic environment, prices may fail to communicate to owners of durable intermediate goods how these goods can best be used and how highly they should be valued. This is the important fact which tends to be obscured in static models and in dynamic models without durable goods.

The plan of this paper is as follows. In Section 2 we define the basic parameters of our model, describing individuals, assets, and the technology of production and exchange. In Section 3, ideas from dynamic programming are used to show how an individual might derive, from his long-term objective function, a linear value formula to guide his production and transactions decisions at a given point in time. We then state our first main result: that the there exists a unique market-clearing price for each actively traded asset at any point in time. Section 4 develops a theory of how individuals might actually compute their personal values of assets, as they learn from past experiences; and Section 5 extends the theory from Section 4 to include the effect of inflation. In Section 6, we summarize our model, and we state assumptions sufficient to guarantee existence of a temporary equilibrium path for the economy, starting from given initial conditions. Section 7 introduces the government sector. In Section 8, we discuss the efficiency of stationary equilibria. In Section 9, we discuss the dynamic behavior of our model as a description of the business cycle. Section 10 contains a simple example, illustrating how the model may be used. All proofs are found in the Appendix.

2. Basic structures

We consider an economy with I individuals, numbered 1 to I. Throughout this paper, we let i represent a typical individual. To avoid stockholder-unanimity questions, we ignore firms in this model, and assume that productive activities will be carried out by the individuals themselves using the capital which they own.

There are J+1 assets in our model, including money, goods, and bonds. In our notation scheme, money is asset number 0. The other J assets are numbered from 1 to J. Throughout this paper, we let j represent a typical asset.

The first G assets are the (physical) goods, and the other J-G assets are government bonds. Throughout, we let g represent a typical good, and we let b represent a typical class of government bonds. Thus, g is always to be interpreted as a variable ranging over $g=1,2,\ldots,G$; and b is always to be interpreted as a variable ranging over $b=G+1,\ldots,J$.

All goods in our economy are assets, enduring through time, although some goods may have high rates of depreciation. For simplicity, we assume that all goods depreciate according to a simple exponential decay. We let c_g be the <u>rate of depreciation</u> for good g. We assume that $c_g \geq 0$ for every g.

The bonds are denominated so that one unit of any bond gives an income stream at the rate of one unit of money per unit of time. For simplicity, we assume that each class of bonds is issued with a constant redemption rate $c_b \geq 0$. Equivalently, we could think of c_b an exponential depreciation rate for bonds of class b. If b is a short-term bond then c_b should be a relatively high number; if b is a long-term bond then c_b should be close to zero (equal to zero for a perpetuity).

For any individual i and any asset j, we let $x_{ij}(t)$ denote the quantity of asset j owned by i at time t. We write $x_i(t)$ to denote the bundle of assets owned by i at time t, so

$$x_i(t) = (x_{i0}(t), x_{i1}(t), \ldots, x_{iJ}(t)) \in \mathbb{R}_+^{J+1}.$$

We shall use a superscript-G to denote the subvector listing only the components for the G physical goods, so

$$x_i^G(t) = (x_{i1}(t), \ldots, x_{iG}(t)) \in \mathbb{R}_+^G.$$

Individuals may have a variety of production activities available to them. We let $y_{ig}(t)$ denote the net rate of production of good g by individual i at time t. If i is consuming g (as a productive input) then y_{ig} will be negative. The vector of net production by i at time t is denoted

$$y_i(t) = (y_{i1}(t), \ldots, y_{iG}(t)).$$

The set of feasible production-rate vectors available to i at any time t may depend on his current stock of capital goods. Thus, to describe the production possibilities of any individual i, we assume that, for every bundle of goods x_i^G in $\mathbb{R}_+^G$, there exists a nonempty compact <u>production possibility set</u> $Y_i(x_i^G)$, a subset of $\mathbb{R}^G$. Then i's production-rate vector $y_i(t)$ must always satisfy the feasibility constraint:

(1) $y_i(t) \in Y_i(x_i^G(t))$.

Money and bonds are not produced by individuals in this model. In Section 7 we will introduce the government sector, which can introduce new quantities of money and bonds into the economy. (Until we introduce government monetary policy, it may be simplest to think of our economy as having a fixed supply of perpetuities, and no short-term bonds.)

Notice that we speak only of <u>rates</u> of production at a point in time. That is, we are assuming that in an infinitesimal period of time only infinitesimal quantities of assets can be produced, so it is the rate of production per unit time which we must discuss.

Similarly, we shall assume that in an infinitesimal period of time, an individual can only make infinitesimal sales or purchases of durable assets. So at any moment, we can only speak of the <u>rates</u> of sales and purchases per unit time. Foley (1975) has also suggested this assumption in continuous time models.

In most sectors of every modern economy, a seller must hold inventories in order to have a flow of sales. Similarly, in order to make purchases, an individual must hold positive money balances for some period of time. These basic facts are known to be important in macroeconomic modelling, and yet they are usually neglected in general equilibrium theory. The simplest way to introduce these facts into our dynamic model is to assume that there exist velocity constraints on the sales of assets and on the spending of money.

Thus, for every individual i and every asset j (including j=0) we assume that there exists a natural velocity constant, which we shall call n_{ij}. For any nonmoney asset j, the constant n_{ij} is the <u>velocity of j-inventories for i</u>. That is, the ratio of i's rate of sales of j over i's inventories of j can never be higher than n_{ij}, even if there were unlimited demand. So, in order to sell asset j at rate S, i would need to hold at least S/n_{ij} units of j in inventories.[1]

The constant n_{i0} is the <u>velocity of money for i</u>. That is, the ratio of i's rate of spending over i's money-balances can never be higher than n_{i0}. So, in order to spend money for purchases at rate D, i would need to hold at least D/n_{i0} units of money, for liquidity.

For any nonmoney asset j, we let $p_j(t)$ denote the price of asset j at time t. (We shall ignore taxes until Section 7, so buying prices and selling prices coincide.) We let $s_{ij}(t)$ denote the rate at which individual i offers to sell asset j at time t, and we let $d_{ij}(t)$ denote the rate at which i demands asset j at time t.

Given the velocity parameters, the quantities sold and demanded must satisfy the following constraints, for every individual i, at every time t:

$$(2) \quad s_{ij}(t) \leq n_{ij}x_{ij}(t) \qquad (j = 1,\ldots,J),$$

$$(3) \quad \sum_{j=1}^{J} p_j(t)d_{ij}(t) \leq n_{io}x_{io}(t),$$

$$(4) \quad s_{ij}(t) \geq 0, \; d_{ij}(t) \geq 0 \qquad (j = 1,\ldots,J).$$

Condition (2) asserts that an individual's rate of sale of an asset is limited by his current stock of the asset multiplied by the inventory velocity factor. Condition (3) asserts that an individual's total rate of spending for demand is limited by his current stock of money multiplied by the velocity of money. (Thus conditions (2) and (3) implicitly give us a transactions demand for money and inventories. Clower (1967) has proposed a constraint like (3) in his monetary model. Condition (4) states the obvious nonnegativity constraints.

We have now described four ways in which an individual's stocks may change: through production, depreciation, sales, and purchases. Thus, the stocks of nonmoney assets change according to the following differential equations:

$$(5) \quad \dot{x}_{ig}(t) = y_{ig}(t) - c_g x_{ig}(t) - s_{ig}(t) + d_{ig}(t),$$

$$(6) \quad \dot{x}_{ib}(t) = -c_b x_{ib}(t) - s_{ib}(t) + d_{ib}(t),$$

for all $i = 1,\ldots,I$; $g = 1,\ldots,G$; $b = G+1,\ldots,J$. (Recall that, for bonds, the "depreciation rate" c_b is a redemption rate.)

Each individual i gets money flowing in from bond dividends and from sales, while money flows out for purchases; so the differential equation for x_{i0} is

$$(7)\quad \dot{x}_{i0}(t) = \sum_{b=G+1}^{J} x_{ib}(t) + \sum_{j=1}^{J} p_j(t)s_{ij}(t) - \sum_{j=1}^{J} p_j(t)d_{ij}(t).$$

In a temporary equilibrium at any time t, two further conditions must be satisfied. First, all individuals must choose their current supply, demand, and production optimally with respect to their preferences and expectations. Second, the individual transaction flows must be in balance, so that total sales rates equal total purchase rates:

$$(8)\quad \sum_{i=1}^{I} s_{ij}(t) = \sum_{i=1}^{I} d_{ij}(t) \qquad (j=1,\ldots,J).$$

The market-clearing condition (8) gives us the J equations which will determine the J prices at every point in time. But first, the individuals' preferences and expectations must be described, so that we can formalize the individual-optimality condition. This task will occupy our next three sections.

3. Individual utility and values.

Let us suppose that, at any time t, each individual i wants his long-term strategy of production and transactions to maximize an objective function of the following form:

$$(9)\qquad \int_0^\infty e^{-r_i\tau}\, U_i(x_i^G(t+\tau))\,d\tau$$

for some <u>utility production function</u> $U_i : \mathbb{R}_+^G \to \mathbb{R}$ and some <u>personal discount rate</u> $r_i \geq 0$. ($e=2.718\ldots$). Notice that we assume that only ~~[illegible]~~ goods can generate utility directly; endowments of money and bonds are not included in the vector $x_i^G = (x_{i1},\ldots,x_{iG})$, so they are desired only for their power to purchase ~~[illegible]~~ goods.

(With some reinterpretation, our notation can accomodate the case where utility is derived from consumption <u>activities</u>, as well as from possession of goods. This can be done by introducing into the model some artificial goods to represent recent recent consumption activity. For example, if utility is derived from eating cake, rather than from having it, then we could introduce an artificial good to represent "recently eaten cake": utility could be derived from this artificial good, which in turn could only be produced by destroying some cake. These artificial goods should have zero transaction velocities ($n_{ig} = 0$) and high depreciation rates.)

For any $t \geq T$, and any $\bar{x}_i$ in $\mathbb{R}_+^{J+1}$, we let $V_i(\bar{x}_i,t,T)$ be the expected value of (9) if i uses an optimal (closed loop) strategy of production and transactions from time t onwards, subject to the constraints (1) - (7) and $x_i(t) = \bar{x}_i$, given i's expectations about future conditions as he estimates them <u>at time T</u>.

Now let h be a small positive number, representing a short time interval. Assuming V_i to be sufficiently smooth, the usual dynamic-programming approach gives us the following:

$$(10)\quad V_i(x_i(t),t,t) =$$

$$= U_i(x_i^G(t))h + \underset{\dot{x}_i(t)}{\text{maximum}}\; e^{-r_i h}\cdot V_i(x_i(t)+\dot{x}_i(t)h,\ t+h,t)+o(h)$$

$$= U_i(x_i^G(t))h+(1-r_i h)V_i(x_i(t),t+h,t)$$

$$+ \underset{\dot{x}_i(t)}{\text{maximum}}\; \Big(\sum_{j=o}^{J} \frac{\partial V_i}{\partial x_{ij}}(x_i(t),t+h,t)\dot{x}_{ij}(t)h\Big) + o(h).$$

The maximum in (10) is taken over all $\dot{x}_i(t) = (\dot{x}_{i0}(t),\ldots,\dot{x}_{ij}(t))$ satisfying (1)- (7); and o(h) denotes terms going to zero faster than h, so that $o(h)/h \to 0$ as $h \to 0$.

Equation (10) assures us that i can achieve an expected utility payoff within o(h) of the optimum if he uses the following strategy: between time t and t+h, choose production and transactions so as to maximize

$$\sum_{j=0}^{J} \frac{\partial V_i}{\partial x_{ij}}(x_i(t),t+h,t)\ \dot{x}_{ij}(t);$$

and after time t+h continue computing the optimal strategy (whatever it may be) and use it. Thus, taking the limit as $h \to 0$, the optimal strategy for i must be to choose production and transactions at every time t so as to maximize

$$\sum_{j=0}^{J} \frac{\partial V_i}{\partial x_{ij}}(x_i(t),t,t)\ \dot{x}_{ij}(t)$$

subject to (1)-(7).

We now define quantities $v_{ij}(t)$ so that

$$(11) \qquad v_{ij}(t) = \frac{\partial V_i}{\partial x_{ij}}(x_i(t),t,t) \qquad (j = 0,1,\ldots,J).$$

We may call $v_{ij}(t)$ the <u>marginal value</u> of asset j for individual i at time t, since it is the per-unit increase in the long run objective function (9) which i could expect if he had slightly more of asset j at time t. Using equations (5)-(7), we get:

$$(12) \qquad \sum_{j=0}^{J} v_{ij}(t)\dot{x}_{ij}(t) =$$

$$= \sum_{g=1}^{G} v_{ig}(t)y_{ig}(t) + \sum_{j=1}^{J} (v_{i0}(t)p_j(t) - v_{ij}(t))(s_{ij}(t) - d_{ij}(t))$$

$$+ v_{i0}(t)\Big(\sum_{b=G+1}^{J} x_{ib}(t)\Big) - \sum_{j=1}^{J} v_{ij}(t)c_j x_{ij}(t).$$

The last two terms in (12) are constants which i cannot affect at time t, since he only controls the y_{ig}, s_{ij}, and d_{ij} variables. So the optimal policy for i at any time t is to choose the y_{ig}, s_{ij}, and d_{ij} quantities so as to maximize

$$(13) \qquad \sum_{g=1}^{G} v_{ig}(t)y_{ig}(t) + \sum_{j=1}^{J}(v_{i0}(t)p_j(t) - v_{ij}(t))s_{ij}(t) + \sum_{j=1}^{J}(v_{ij}(t) - v_{i0}(t)p_j(t))d_{ij}(t)$$

subject to conditions (1)-(4). Thus, to compute the optimal policy for i, it is only necessary to know the J+1 marginal values $v_{ij}(t)$.

It might be suggested that an individual at time t should compute his marginal values from his subjective estimates of future prices. That is, for every asset j and every $\tau>0$, individual i at time t is supposed to assess some number or random variable $\hat{p}_j^t(t+\tau)$, representing his estimate of what the price of j will be at time $t+\tau$. Then i must solve the infinite-horizon optimization problem of maximizing (9) subject to the production and transactions constraints described in the preceding section, given his initial endowment $x_i(t)$ and given his current estimates of future prices. Using the methods of optimal control theory or dynamic programming, this problem can (in principle) be solved, giving i his marginal values and an optimal plan for production and transactions. However, only the current recommendations of this plan should actually be carried out, because at any later time t+h individual i will generally have a new series of future price estimates $\hat{p}_j^{t+h}(t+h+\tau)$. Thus he may have to solve a different infinite-horizon optimal control problem at time t+h, and at every other point in time!

This kind of analysis generally cannot give us a model of the dynamic economy which is practically computable, and it is doubtful whether the real individuals whom we want to describe can ever do these heroic computations either. But these computations are all based on subjective estimates of future prices, so it is just as reasonable to assume that individuals form their subjective estimates of the marginal values directly.[2] That is, instead of assuming that individuals first estimate future prices and then use these estimates to compute the marginal values, we may simply assume that they estimate the marginal values $v_{ij}(t)$ directly, as some function of their past experience up to time t. As situations change and as individuals learn from new experience, individual i may update his estimated marginal values according to some differential equation of the form

$$(14)\quad \dot{v}_i(t) = F_i(x_i(t),\ v_i(t),\ p(t)),$$

where

$$v_i(t) = (v_{i0}(t),\ v_{i1}(t),\dots,v_{iJ}(t)),$$
$$p(t) = (p_1(t),\dots,p_J(t)),$$

and $F_i : \mathbb{R}_+^{3J+2} \to \mathbb{R}^{J+1}$ is a function which is derived in some way from i's objective function (9).

Once the F_i functions are specified, we will have a complete model of the dynamic economy. At any time t, individual i chooses all $s_{ij}(t), d_{ij}(t)$, and $y_{ij}(t)$ quantities to maximize (13) subject to (1)-(4), given $x_i(t), v_i(t)$ and $p(t)$. Equations (5)-(7) and (14) describe how the endowments $x_i(t)$ and the values $v_i(t)$ evolve through time. The prices $p(t)$ are determined by the market-clearing condition (8).

In fact, the market-clearing conditions uniquely determine the prices, except possibly in inactive markets.

<u>Theorem 1</u>. Let t be fixed, and suppose that nonnegative endowment vectors $x_i(t)$ in $\mathbb{R}_+^{J+1}$ and strictly positive value

vectors $v_i(t)$ in $\mathbb{R}^{J+1}_{++}$ are given, for every individual i. Then there exists a strictly positive price vector p(t) in $\mathbb{R}^{J}_{++}$ such that the market clearing conditions (8) hold when all individuals maximize (13) subject to (1)-(4). Furthermore, for any nonmoney asset j such that $\sum_i s_{ij}(t) \neq 0$ (so that j is actually being traded), the market-clearing price $p_j(t)$ is uniquely determined. The price of an untraded asset may range over some closed interval.

The proof of Theorem 1 can be found in the Appendix. However, one key idea in the proof should be remarked here: it is the linearity of the individuals' maximands (13) which makes our uniqueness result possible. We have derived this linearity from the assumption that only infinitesimal transactions are possible in an infinitesimal time interval, which allowed us to consider only the linear first-order local approximation to the value function V_i. Thus, we have used the continuous time assumption in an essential way here, even though Theorem 1 refers to market equilibria at only a single point in time.

The problem of computing our market-clearing prices turns out to be a linear complementarity problem. Myerson (1981) presented an algorithm that computes these prices in finitely many iterations, by solving a series of linear programming problems.

4. Updating the values.

Equation (14) embodies the most important part of the dynamic model, since it describes how individuals learn to evaluate durable assets in terms of their contributions to utility maximization. If we only stated the general functional relationship as in (14) then we would be evading the central problem of the dynamic economy. On the other hand, to be more specific requires some specific assumptions about learning behavior. In this section, we examine one such set of assumptions, to get a specific form for the F_i functions, based on the utility production functions U_i and the personal discount rates r_i. Hopefully, other alternative assumptions about learning behavior might be similarly formalized in future research.

To develop our value-update equations, it will be helpful to begin with a discrete-time approximation to our model. So suppose individual i makes his production and transaction decisions in short time intervals, which we may call "days," each day being of length $h > 0$. On a typical day, at time t, individual i begins the day with an endowment vector $x_i(t)$ in $\mathbb{R}_+^{J+1}$ and a recently estimated marginal value vector in $\mathbb{R}_+^{J+1}$. This vector of estimated marginal values was calculated on the previous day, so we shall call it $v_i(t-h)$ to emphasize this fact. Individual i is then confronted with the current price vector $p(t)$, and in response he must perform two tasks: he must plan his production and transactions, to determine the next day's endowment

$$x_i(t + h) = x_i(t) + \dot{x}_i(t)h,$$

as discussed above in Section 3; and he must somehow compute his new vector of marginal values $v_i(t)$.

In this discrete-time approximation of the model, a natural simplifying assumption for us to make is that $v_i(t)$ should be computed as a function of the previous day's value vector $v_i(t-h)$ and the current price and endowment vectors $p(t)$ and $x_i(t)$. This is a kind of Markovian assumption. It asserts that all of i's information about events up to time $t-h$ is summarized by the sufficient statistic $v_i(t-h)$, for the purposes of estimating the marginal values of assets after time t. Equation (14) also implicitly makes this assumption.

Now let us consider again the basic value equation

$$(15)\quad V_i(x_i(t),t,t) = \\ = U_i(x_i^G(t))h + \underset{x_i(t+h)}{\text{maximum}}(1 - r_i h)V_i(x_i(t+h),t+h,t) + o(h).$$

To get an equation for the marginal value $v_{ij}(t)$, we would like to differentiate both sides of (15) (or (10)) with respect to $x_{ij}(t)$. But this begs the question, how should i estimate the expected future marginal value $\frac{\partial V_i}{\partial x_{ij}}(x_i(t+h),t+h,t)$? In view of our Markovian assumption, we should look for some way to write the expected future marginal values as functions of the most recent marginal values $v_{ij}(t-h)$, and of the current price and endowment vectors, $p(t)$ and $x_i(t)$. In this context, the simplest assumption we could make is to suppose that

$$(16)\qquad \frac{\partial V_i}{\partial x_{ij}}(x_i(t+h),t+h,t) = v_{ij}(t-h)$$

for all $x_i(t+h)$ in some open neighborhood of $x_i(t)$.

This assumption (16) is essentially an assumption of inelastic expectations, since it asserts that today's expectations about tomorrow's marginal values are independent of the new information received today ($p(t)$ and $x_i(t)$). We can also think of (16) as a <u>no trend</u> assumption. That is, if individual i believes that his marginal values are fluctuating randomly around some long-run stationary level, then any increase in the marginal value of an asset from yesterday to today would lead i to expect a similar decrease in the future. Thus, he might expect tomorrow's marginal values to be close to yesterday's marginal values, whatever today's marginal values might be. So let us investigate the implications of assuming (16). We will discuss some more sophisticated alternative assumptions later in this section and the next, but it is easiest to start with (16).

To simplify our notation, we shall use the following abbreviation:

$$z^+ = \max\{0,z\},$$

for any expression z. Also, we define a new function

$W_i : \mathbb{R}_+^G \times \mathbb{R}_+^{J+1} \to \mathbb{R}$ such that

$$(17) \qquad W_i(x_i^G, v_i) = \underset{y_i \in Y_i(x_i^G)}{\text{maximum}} \left(\sum_{g=1}^{G} v_{ig} y_{ig} \right).$$

That is, if the goods are evaluated using the v_{ig} values, then $W_i(x_i^G, v_i)$ is the highest value of net production feasible for i when x_i^G is his stock of goods.

When assumption (16) is substituted into equation (10) and the indicated maximization is carried out (using (1) - (7)), it is straightforward to derive the following equation:

$$\begin{aligned}
(18) \qquad & V_i(x_i(t),t,t) = \\
& = U_i(x_i^G(t))h + (1-r_i h) V_i(x_i(t),t+h,t) \\
& + W_i(x_i^G(t), v_i(t-h))h \\
& + \sum_{j=1}^{J} n_{ij} x_{ij}(t) h (v_{i0}(t-h) p_j(t) - v_{ij}(t-h))^+ \\
& + \underset{j=1,\dots,J}{\text{maximum}} \left(n_{i0} x_{i0}(t) h (v_{ij}(t-h)/p_j(t) - v_{ij}(t-h))^+ \right) \\
& + v_{i0}(t-h) \left(\sum_{b=G+1}^{J} x_{ib}(t) \right) h - \sum_{j=1}^{J} v_{ij}(t-h) c_j x_{ij}(t) h + o(h).
\end{aligned}$$

Now, assuming that the functions V_i, U_i, and W_i are sufficiently smooth, we can differentiate both sides of (18) with respect to each quantity variable $x_{ij}(t)$. The results are as follows:

$$(19) \qquad v_{ig}(t) = \frac{\partial U_i}{\partial x_{ig}}(x_i^G(t))h + \frac{\partial W_i}{\partial x_{ig}}(x_i^G(t), v_i(t-h))h$$

$$+ n_{ig}h(v_{iO}(t-h)p_g(t) - v_{ig}(t-h))^+ + (1-(c_g+r_i)h)v_{ig}(t-h) + o(h),$$

$$(20) \qquad v_{ib}(t) = v_{iO}(t-h)h + n_{ib}h(v_{iO}(t-h)p_b(t) - v_{ib}(t-h))^+$$

$$+ (1-(c_b+r_i)h)v_{ib}(t-h) + o(h),$$

$$(21) \qquad v_{iO}(t) = \underset{j}{\text{maximum}} \quad (n_{iO}h(\frac{v_{ij}(t-h)}{p_j(t)} - v_{iO}(t-h))^+)$$

$$+ (1-r_ih)v_{iO}(t-h) + o(h),$$

for every good g and bond b.

Now we can return from our discrete-time story to our original continuous-time model, by letting h go to zero. The (19)-(21) equations become

$$(22) \qquad \dot{v}_{ig}(t) = \frac{\partial U_i}{\partial x_{ig}}(x_i^G(t)) + \frac{\partial W_i}{\partial x_{ig}}(x_i^G(t), v_i(t))$$

$$+ n_{ig}(v_{iO}(t)p_g(t) - v_{ig}(t))^+ - (c_g+r_i)v_{ig}(t),$$

$$(23) \qquad \dot{v}_{ib}(t) = v_{iO}(t) + n_{ib}(v_{iO}(t)p_b(t) - v_{ib}(t))^+$$

$$- (c_b + r_i)v_{ib}(t),$$

$$(24) \qquad \dot{v}_{iO}(t) = \underset{j=1,\ldots,J}{\text{maximum}} \quad (n_{iO}(\frac{v_{ij}(t)}{p_j(t)} - v_{iO}(t))^+) - r_iv_{iO}(t),$$

for every good g and every bond b. These value equations give us a specific formulation of (14), as we had hoped for. Equation (22) shows how an individual's value for a good derives from its direct contribution to his personal utility, from its use in producing other goods, and from the sales income which it can provide. Equation (23) shows how an individual's value for a bond derives from its power to provide dividend income and capital gains. And equation (24) shows how an individual's value for money derives from its power to purchase goods and bonds.

More general versions of equation (16) are straightforward to analyze. For example, we might want to consider the assumption

$$(25) \quad \frac{\partial V_i}{\partial x_{ij}}(x_i(t+h),t+h,t) = (1-\alpha)v_{ij}(t-h) + \alpha v_{ij}(t),$$

for every $x_i(t+h)$ in an open neighborhood of $x_i(t)$. If we repeat our analysis with (25) instead of (16), then we get the same differential equations except that the left-hand sides of (22)-(24) must be multiplied by $(1-\alpha)$.

The case of $\alpha = 2$ might seem particularly interesting, since this is the <u>projected-trend</u> case, with

$$v_{ij}(t+h) = v_{ij}(t) + (v_{ij}(t) - v_{ij}(t-h)).$$

When $\alpha > 1$, however, the resulting differential equations tend to be highly unstable.

In differentiating (18) with respect to the $x_{ij}(t)$ variables, we were able to treat the $v_{ij}(t-h)$ marginal values as constants because (16) was assumed to hold in an open neighborhood around $x_i(t)$, so that

$$(26) \qquad \frac{\partial^2 V_i}{\partial x_{ij}\partial x_{ik}}(x_i(t),t+h,h) = 0,$$

for any assets j and k. That is, we assumed that i's estimated future marginal values are insensitive to small changes in his endowment. This insensitivity assumption is technically necessary as long as we make our simple first-order Markovian assumption: that the recently estimated marginal values $v_{ij}(t-h)$ summarizes for current purposes everything which i has learned about marginal values up to time t-h. In a more sophisticated learning model, we might suppose that each individual at every point in time computes estimates of all first <u>and</u> second derivatives of his value function; then the natural analogue of (16) is to suppose, at time t, that the expected value function at time t+h should be quadratic, with first and second derivatives equal to those recently estimated at time t-h. We leave further analysis of such second-order learning models to future research.

5. Effects of inflation.

In deriving the value-update equations (22)-(24), the essential assumption was embodied in equation (16). We assumed there that, on any "day", the individual uses the marginal values calculated "yesterday" to estimate the value of the assets which he may hold "tomorrow". In effect, this amounts to assuming that individuals project their experiences from the recent past into the near future, without considering any general trends which might make the future different from the past. In this section, we revise these formulas to account for the effect of perceived inflationary trends.

Let us suppose that at any point in time, there is some number a(t) which is the <u>expected inflation rate,</u> based on the rate of price changes throughout the economy in the recent past. This expected inflation rate may affect individuals' value estimates, especially for money and bonds. Thus, our value-update equations should be revised to include a dependence on a(t), so that our general equation (14) becomes:

$$(27)\quad \dot{v}_i(t) = F_i(x_i(t), v_i(t), p(t), a(t))$$

In the presence of a perceived inflationary trend, it is still reasonable for an individual to use the marginal value of a physical good yesterday to estimate the value of that good tomorrow. To see why this is so, observe that these values are in terms of contribution to long-term utility maximization, so that the marginal values are in units of personal utility per unit of asset. When the asset is a physical good, there is no reason to believe that the marginal value should vary systematically with the price level.

However, the individuals must recognize that money and bonds (which are denominated in money-units) ultimately contribute to utility only through the power of money to purchase goods. Thus, a general increase in the prices of goods should cause a proportionate decline in the utility-values of money and bonds. When the current inflation rate is $a(t)$, then the price level should be expected to increase by a factor of $(1+2a(t)h)$ during the interval between time $t-h$ ("yesterday") and time $t+h$ ("tomorrow"), and the marginal values of money and bonds should be expected to decline by the same factor. To account for inflation, therefore, each individual i may use the following formulas to estimate "tomorrow's" marginal values:

$$\frac{\partial V_i}{\partial x_{ig}}(x_i(t+h),\ t+h,t) = v_{ig}(t-h), \tag{28}$$

$$\frac{\partial V_i}{\partial x_{ib}}(x_i(t+h),\ t+h,t) = \frac{v_{ib}(t-h)}{1+2a(t)h}, \tag{29}$$

$$\frac{\partial V_i}{\partial x_{i0}}(x_i(t+h),\ t+h,t) = \frac{v_{i0}(t-h)}{1+2a(t)h}, \tag{30}$$

for any good g, any bond b, and any endowment vector $x_i(t+h)$ in some open neighborhood of $x_i(t)$. That is, (19) should be revised dividing the expected marginal values of money and bonds by $(1+2a(t)h)$.

It is now straightforward to repeat the analysis of Section 4, replacing (16) by (28)-(30). The analysis is the same, except that the marginal values for money and bonds at t-h must be multiplied by a factor of

$$(1+2a(t)h)^{-1} = 1-2a(t)h + o(h)$$

throughout equations (18)-(21). Then the final result is to give us the following value-update equations, replacing (22)-(24):

$$(31)\quad \dot{v}_{ig}(t) = \frac{\partial U_i}{\partial x_{ig}}(x_i^G(t)) + \frac{\partial W_i}{\partial x_{ig}}(x_i^G(t), v_i(t))$$

$$+ n_{ig}(v_{i0}(t)p_g(t)-v_{ig}(t))^+ - (c_g+r_i)v_{ig}(t),$$

$$(32)\quad \dot{v}_{ib}(t) = v_{i0}(t)+n_{ib}(v_{i0}(t)p_b(t)-v_{ib}(t))^+ - (c_b+r_i+2a(t))v_{ib}(t),$$

$$(33)\quad \dot{v}_{i0}(t) = \underset{j=1,..,J}{\text{maximum}}\; (n_{i0}(\frac{v_{ij}(t)}{p_j(t)} - v_{i0}(t))^+) - (r_i + 2a(t))v_{i0}(t),$$

for all goods $g=1,...,G$ and all bonds $b=G+1,...,J$.

Suppose that the individuals derive their inflation expectations from some aggregate price level index published by some government statisticians. Specifically, suppose that the statisticians compute the cost of some <u>standard bundle</u> of goods $(\beta_1,...,\beta_G)$, and they report their findings with some <u>reporting lag</u> $\lambda \geq 0$. Then, at time t, the statisticians report an <u>aggregate price level index</u>,

$$Q(t) = \sum_{g=1}^{G} \beta_g p_g(t-\lambda).$$

The simplest way to derive inflation expectations from $Q(t)$ would be by an adaptive expectations formula:

$$(34)\quad \dot{a}(t) = r_0(\frac{\dot{Q}(t)}{Q(t)} - a(t)),$$

where $r_0 \geq 0$ is the rate of adaptation of inflation expectations. However, there is a mathematical difficulty here, because the aggregate price level might not be a differentiable function of time. Consider, therefore, the following pair of equations:

$$(35) \quad \dot{q}(t) = r_0 q(t)(\log(\sum_{g=1}^{G} \beta_g p_g(t-\lambda)) - \log(q(t))),$$

$$(36) \quad a(t) = \dot{q}(t)/q(t).$$

(Here log is the natural logarithm, base e.) It is easy to check that (35) and (36) do imply (34), when the aggregate price level is a differentiable function of time. The quantity $q(t)$ may be interpreted as a kind of geometric mean of the recent price levels.

For the theoretical analysis of our model we shall use (35) and (36) to determine the expected inflation rate. However, in practical simulations it might be easier to use some discrete-time approximation based on (34) instead.

We shall see that a positive lag λ is needed to guarantee that the differential equations in our model do not diverge to infinity in finite time. It might seem inconsistent to assume that individuals respond directly to the current prices when they plan their transactions, but that the individuals have a lagged response to the price level when they form their estimates of inflation. However, these assumptions can be justified without inconsistency if we recall that our model is intended only as an aggregated representation of a huge economy with great diversity of products and regions. In practice, any individual at any point in time will only need to check a few local prices to select his current transactions. On the other hand, he might also feel that the general price index can give the best information about overall inflationary trends, because this general index assimilates information from all sectors of the economy. And since the general index of prices depends on a broader data base, there might well be some reporting lag for the general index, even though each individual can check without lag the current prices most important to him.

Even with $\lambda > 0$, the economy may tend to get caught in accelerating inflationary or deflationary spirals, if r_0 is too large. To prevent such extreme instability in our models, the expectations adaptation rate r_0 should generally not be chosen much larger than the individuals' discount rates r_i. Choosing $r_0 \leq r_i$ guarantees that the process of learning to perceive inflation (or deflation) will not proceed faster than the other learning processes embodied in the value-update equations (31)-(33).

6. Existence of temporary equilibrium paths.

One basic requirement which our model must satisfy is that it should be logically consistent, in the sense that there do generally exist solutions to the equations we have written down. In this section, we summarize our mathematical model and we state assumptions which are sufficient to guarantee existence of the temporary equilibrium paths described by our equations.

The basic parameters which have defined our economy may be listed as follows:

$$(37)\quad E = (J, G, (c_j)_{j=1}^{J}, I, ((n_{ij})_{j=0}^{J})_{i=1}^{I}, (U_i, r_i, Y_i)_{i=1}^{I},$$

$$r_0, (\beta_g)_{g=1}^{G}, \lambda).$$

At any time t, the state of the market is described by the following quantities:

$$(38)\quad (((x_{ij}(t),v_{ij}(t))_{j=0}^{J},(s_{ij}(t),d_{ij}(t))_{j=1}^{J},\ (y_{ig}(t))_{g=1}^{G})_{i=1}^{I},$$

$$(p_j(t))_{j=1}^{J},\ q(t),a(t)),$$

For each i, the $y_{ig}(t)$, $s_{ij}(t)$, and $d_{ij}(t)$ quantities are chosen to maximize (13) subject to (1)-(4), given the $p_j(t)$, $v_{ij}(t)$, and $x_{ij}(t)$ quantities. The prices $p_j(t)$ are implicitly determined by the market-clearing equations (8) (uniquely, according to Theorem 1). The $x_{ij}(t)$,$v_{ij}(t)$, and $q(t)$ quantities are determined through time by the differential equations (5)-(7), (31)-(33), and (35). The expected inflation rate a(t) is defined by (36).

Mathematically, it may be too much to ask for the differential equations in our model to be satisfied at all t, because the right hand sides may have jumps at some points in time. Thus, we can only ask that the x_{ij}, v_{ij}, and q quantities should be absolutely continuous functions of time, and that these differential equations should hold at almost all times $t \geq 0$. With this one qualifications, we say that a temporary equilibrium path for the economy E is any solution to the equations and maximizations described above, over all $t \geq 0$.

A set of initial conditions for the economy must specify nonnegative values for all $x_{ij}(0)$ and $v_{ij}(0)$ (for every i=1,...,I and j=0,...,J), for q(0), and for the recent aggregate price levels $\sum_{g=1}^{G} \beta_g p_g(t)$ as a function of t in the interval $-\lambda \leq t < 0$.

We may now ask, when does there exist a temporary equilibrium path satisfying given initial conditions?

In order to prove existence, we need to make several assumptions about the economy and the initial conditions. Our first assumption guarantees that the U_i term in (31) will not cause v_{ig} to diverge to infinity or below zero within finite time.

<u>Assumption 1</u> For every i, $U_i : \mathbb{R}^G_+ \to \mathbb{R}$ is continuously differentiable, with bounded nonnegative derivatives.

Our second assumption guarantees that the y_i vectors are always well-defined, and that they never cause x_{ig} to go below zero.

<u>Assumption 2</u> For every i, and every x_i^G in $\mathbb{R}^G_+$, the set $Y_i(x_i^G)$ is nonempty, compact, and convex, and the point-to-set correspondence $Y_i(.)$ is continuous on $\mathbb{R}^G_+$. For every i and g, there exists a constant k such that, for all x_i^G in $\mathbb{R}^G_+$ and all y_i in $Y_i(x_i^G)$, $y_{ig} \geq -kx_{ig}$. (That is, rates of input consumption are bounded in proportion to the available stocks.)

Our third assumption also refers to production. It assures us that the W_i term in (31) is well-defined, and cannot cause v_{ig} to diverge to infinity or to zero. Furthermore, it also implies that the y_{ig} term in (5) cannot drive x_{ig} to infinity within finite time.

<u>Assumption 3</u> For every i and g, the derivative $\frac{\partial W_i}{\partial x_{ig}}$ is well-defined and continuous on $\mathbb{R}^{G+(J+1)}_+$; and there exists some bound K such that, for every (x_i^G, v_i) in $\mathbb{R}^{G+(J+1)}_+$,

$$0 \leq \frac{\partial W_i}{\partial x_{ig}}(x_i^G, v_i) \leq K \sum_{g=1}^{G} v_{ig}.$$

Our fourth assumption refers to the initial conditions, and guarantees that prices will be well-behaved.

<u>Assumption 4</u> For every i=1,...,I and every j = 0,1,...,J, $v_{ij}(0)>0$, and if $n_{ij}>0$ then $x_{ij}(0)>0$. Furthermore, the recent general price levels $\sum_g \beta_g p_g(t)$ are bounded away from 0 and ∞, and are continuous on the interval $-\lambda \leq t < 0$.

Finally, we require that the inflationary expectations feedback must have some lag.

Assumption 5 $\lambda > 0$.

Theorem 2 Given an economy and initial conditions satisfying the five assumptions above, there exists a temporary equilibrium path (over all $t \geq 0$) satisfying the given initial conditions.

The proof uses a general existence theorem for multivalued differential equations due to Lazota and Opial (1965) (see also Henry (1973)), and is presented in the Appendix. The proof relies heavily on the fact that, at any point in time, the $x_{ij}(t)$ and $v_{ij}(t)$ determine an (essentially) unique equilibrium price vector and a convex set of equilibrium supply and demand vectors, which depend upper-semicontinuously on the $x_{ij}(t)$ and $v_{ij}(t)$ state variables. This fact is necessary because the Lazota-Opial theorem requires that the right-hand side of the multivalued differential equation must be a convex-valued upper-semicontinuous correspondence. The essential problem is that, if the set of market equilibria were not convex, our economy might tend to chatter with infinite frequency between two temporary equilibria. Researchers who wish to investigate other ways to formulate models of temporary equilibrium in continuous time should be aware of this need for a convex set of market equilibria (given endowments and expectations), if the Lazota-Opial existence theorem is to guarantee that the model's differential equations can be meaningfully solved.

7. The government sector.

The government can be introduced as one more individual into our model, as individual i=0. However, the government is unlike other individuals in that it can create new money and government bonds. Furthermore, the government's spending and sale of bonds are not determined by any utility function, but rather must be thought of as exogenously specified policy instruments.

In an economy with government, we let $d_{0j}(t)$ denote the government's net rate of demand for asset j at time t (where j is any nonmoney asset). In our simple model we may assume that the government can buy or sell bonds, but can only buy goods. (We may ignore public goods, which are not marketed.) Thus, for any good g and bond b, we have

$$d_{0g}(t) \geq 0, \quad d_{0b}(t) \text{ unconstrained.}$$

When d_{0b} is negative, $-d_{0b}$ is the government's net rate of sales of new bonds of class b. With government transactions, the market clearing condition (8) becomes:

$$(8') \qquad \sum_{i=1}^{I} s_{ij}(t) = \sum_{i=1}^{I} d_{ij}(t) + d_{0j}(t) \qquad (j=1,\ldots,J).$$

Obviously, (8') cannot be satisfied if the government demands any asset faster than the maximum rate at which the private sector can supply the asset, so we must have

$$(39) \qquad d_{0j}(t) \leq \sum_{i=1}^{I} n_{ij}x_{ij}(t),$$

for every nonmoney asset j. If the government's demand rates are given as continuous functions of time satisfying (39), then the existence results in Theorems 1 and 2 can still be proven when (8) is replaced by (8').

The government could also introduce taxes or subsidies into the transactions process. A tax or subsidy on a nonmoney asset j can be described by a function $\theta_j:\mathbb{R}_+ \to \mathbb{R}_+$, such that $\theta_j(p_j)$ is the price received by sellers when buyers are paying price p_j for j. So if the government puts a tax θ_j on asset j, then we should replace p_j by $\theta_j(p_j)$ throughout the supply side of our model, that is, whenever p_j is multiplied by n_{ij} or s_{ij}. Again, with appropriate assumptions about the tax functions θ_j, our existence results can still be proven.

8. Efficiency of stationary equilibria

An important purpose of macroeconomic modelling is to help us understand what kinds of problems the unguided economy can get into. In this section, we consider the question of whether our economy can become caught in a permanent state of depression or Pareto inefficiency. We shall consider an economy with no government taxes or spending, fixed money supply, and no government bonds. Thus G=J in this section.

Of course, questions of efficiency depend on what standards of ideal efficiency are used for comparison. The simplest standard to use for our current purposes is efficiency among feasible stationary states. A <u>real feasible stationary state</u> for our economy is any collection of vectors

$$((x_i^G,y_i))_{i=1}^I = ((x_{i1},\ldots,x_{iG},y_{i1},\ldots,y_{iG}))_{i=1}^I$$

such that

(40) $\quad y_i \in Y_i(x_i^G)$ (for all i=1,...,I),

(41) $\quad \sum_i (y_{ig}-c_g x_{ig}) = 0$ (for all g=1,..,G),

(42) $\quad y_{ig}-c_g x_{ig} \le n_{ig} x_{ig}$ (for all i and g).

As usual, x_i^G is i's bundle of goods and y_i is i's vector of production rates. Constraint (40) requires that each individual's production must be feasible for him at his given endowment. Constraint (41) requires that total production of each good must equal total depreciation, so that the aggregate endowment can be maintained. Constraint (42) derives from the inventory velocity constraint, that deliveries of good g from i to other individuals cannot be made at an average rate of more than $n_{ig}x_{ig}$. Presumably these inventory velocities are based on the need for inventories in the delivery system, and do not depend on the form of the market organization, whether our monetarized price system is used or some other system of barter or centralized command. Then (42) says that, in a stationary state, an individual's net rate of production minus depreciation cannot exceed his maximal rate of deliveries to other individuals (otherwise his endowment would have to increase over time).

An <u>efficient stationary state</u> is a real feasible stationary state $(x_i^G,y_i)_{i=1}^I$ such that there exists no other real feasible stationary state $(\hat{x}_i^G,\hat{y}_i)_{i=1}^I$ such that $U_i(\hat{x}_i^G) \geq U_i(x_i^G)$ for all i, with at least one strict inequality.

A <u>stationary equilibrium</u> is any temporary equilibrium path, as defined in Section 6, such that the state of the market, as described in (38), is constant over time. We say that a stationary equilibrium is <u>stationary-efficient</u> if its x_{ig} and y_{ig} components form an efficient stationary state. We can now present our basic efficiency result.

<u>Theorem 3</u> Suppose that, for all i, $r_i=0$, U_i is a concave function, and the set

$$\{(x_i^G,y_i) \mid y_i \in Y_i(x_i^G)\}$$

is convex. Then any stationary equilibrium is stationary-efficient.

Thus, Theorem 3 tells us that our economy cannot get permanently trapped in an unchanging inefficient state, if all individuals use negligibly small discount rates.[3] So we cannot hope to account for major depressions as stationary equilibria in our dynamic model. But this is precisely the argument from Leijonhufvud (1968) which was cited at the beginning of this paper: that we must try to understand depressions and underemployment as part of a dynamic economic process or business cycle. That is, we should be less concerned with the study of stationary equilibria; instead we should ask: what can go wrong in the changing dynamic economy, and how long are the natural corrective adjustments likely to take? We discuss these questions in terms of our model in the next section.

9. Dynamic adjustments and the business cycle.

To see what can go wrong in our dynamic economy, the best place to start is by looking at the initial conditions. The initial conditions in our model consist of the initial endowments and marginal value estimates of goods, money, and bonds for all individuals at time zero, and some data about recent price levels before time t=0. It is not fruitful for economists to think of the initial endowments as "wrong," since these are part of the physical realities with which the economy must cope. However, we certainly can speak of misadjusted value estimates, which may be either too high (bullish) or too low (bearish); and the recent price levels may embody inflationary or deflationary momentum.

For example, suppose that an economy has a unique efficient stationary equilibrium path, for a given set of initial endowments. Then we may say that an asset j is <u>undervalued</u> (or <u>overvalued</u>) by individual i if his marginal value v_{ij} is below (or above) the corresponding marginal value in the stationary equilibrium path. With arbitrarily chosen initial values $v_{ij}(0)$, it is easy to construct examples with large initial undervaluations or overvaluations. The temporary equilibrium paths which follow from such misadjusted initial value estimates may be very inefficient, with too little production of goods which are relatively undervalued, etc.

However, it is certainly not very deep to observe that individuals may make serious economic mistakes if their expectations about the future are wrong; what we want from our model is to help us keep track of how such errors may develop and how long they are likely to last in a complex economic system. So we must consider the question of how quickly do individuals adjust their values, as they try to learn from their environment. This is where prices play their communicative role, as we discussed in Section 1. So let us now consider the value-update equations, which describe the learning processes in our model.

If $p_g v_{iO} > v_{ig}$, then equation (31) becomes

$$(43)\quad \dot{v}_{ig} = (c_g+r_i+n_{ig})\left(\frac{\frac{\partial U_i}{\partial x_{ig}} + \frac{\partial W_i}{\partial x_{ig}} + n_{ig}p_g v_i}{c_g + r_i + n_{ig}} - v_{ig}\right).$$

If $p_g v_{iO} \le v_{ig}$, then equation (31) becomes

$$(44)\quad \dot{v}_{ig} = (c_g+r_i)\left(\frac{\frac{\partial U_i}{\partial x_{ig}} + \frac{\partial W_i}{\partial x_{ig}}}{c_g+r_i} - v_{ig}\right).$$

Thus, if i is selling good g, and if the inventory velocity n_{ig} is high, then (43) tells us that v_{ig} should rapidly converge towards

$$(45)\quad \left(\frac{1}{c_g+r_i+n_{ig}}\right)\left(\frac{\partial U_i}{\partial x_{ig}}(x_i^G) + \frac{\partial W_i}{\partial x_{ig}}(x_i^G,v_i) + n_{ig}p_g v_{iO}\right).$$

If i is not selling good g, then (44) applies, and we see that v_{ig} must converge towards

$$(46) \quad \left(\frac{1}{c_g+r_i}\right)\left(\frac{\partial U_i}{\partial x_{ig}}(x_i^G) + \frac{\partial W_i}{\partial x_{ig}}(x_i^G,v_i)\right),$$

during any period in which this quantity is stable, and the rate of convergence is proportional to (c_g+r_i). Thus, the slowest values to adjust are those of durable goods which are not being offered for sale, and their adjustment rates are as slow as the sum of their depreciation rates plus the individual's discount rate. For major capital goods, these rates could easily permit serious misevaluations to last for years.

There are many cross-effects between values in our dynamic system. One individual's values can influence other individuals' values, through the effect of his demand and supply decisions on prices. And a particular individual's value for one good can influence his value for other goods, through the $\frac{\partial W_i}{\partial x_{ig}}(x_i^G,v_i)$ terms. Thus, one individual's undervaluation of a durable good can distort the stable values of (45) and (46) for other goods and other individuals, so that even relatively nondurable goods may also suffer prolonged undervaluation. These secondary undervaluations cannot be corrected until the durable good's value adjusts first. Value estimates for durable capital goods thus play a central role in our picture of the business cycle, just as they do in the Keynesian model.

There cannot be involuntary unemployment in our model, since we have assumed that prices always balance effective supply and demand. However, when durable intermediate goods are undervalued, then other resources may be underutilized because efficient roundabout productive processes are not used. For example, architects may be involuntarily underemployed as unskilled workers when the buildings they could design are undervalued. Also, there may be voluntary unemployment of resources if the suppliers of these resources consider the current prices too low. For example, suppose that individual i produces good g, but the good has no use for him in either utility or production; that is,

$\frac{\partial U_i}{\partial x_{ig}} = \frac{\partial W_i}{\partial x_{ig}} = 0$. If p_g and v_{iO} have been stable long enough, then we should find (approximately)

$$v_{ig} = (n_{ig}p_g v_{iO})/(n_{ig}+r_i+c_g).$$

Now if the price suddenly drops to $\hat{p}_g$, where

$$\hat{p}_g < n_{ig}p_g/(n_{ig}+r_i+c_g)$$

then i will refuse to sell g until $v_{ig}(t)$ falls below $\hat{p}_g v_{iO}$. During this period when i does not sell g,

$$\dot{v}_{ig}(t) = -(c_g+r_i)v_{ig}(t) \;.$$

If v_{iO} is constant, then i will refuse to sell g for a period of length

$$\frac{\log(p_g/\hat{p}_g)-\log((n_{ig}+r_i+c_g)/n_{ig})}{r_i+c_g}$$

The lower c_g is (or the more durable g is), the longer this period of voluntary unemployment will be.

People often search for weeks or months looking for a job, refusing to work at wages which they consider too low. This observation suggests that labor should be thought of as a moderately durable good, in applications of our model. That is, it is not daily labor which is sold on the labor market, but rather it is labor service contracts expiring over a period of months which are sold. Obviously, we cannot fit labor exactly into our ideal of a perfectly divisible good with exponential depreciation, but the best approximation for macroeconomic modelling purposes may be to assume that labor is sold in units which depreciate over the course of several months.

Voluntary unemployment of money can arise in the same way as voluntary unemployment of goods, and represents a form of liquidity trap in our model. That is, if an individual's marginal value of money v_{i0} is higher than v_{ij}/p_j for all j, then he begins to hoard his money (all $d_{ij}=0$), and (33) becomes

$$\dot{v}_{i0} = -(r_i+2a)v_{i0}.$$

If prices are stable, then the required decline of v_{i0} may be quite slow, if r_i is small. But matters can be even worse, because i's hoarding may tend to drive prices down, and the resulting deflation makes $a(t)<0$, which further delays the downward adjustment of v_{i0}. So an overvaluation of money may endure for an extended period of time.

The problems of inflationary and deflationary momentum can also be observed in our model. In equations (35) and (36) we explicitly assume that individuals develop their perceptions of inflation gradually, averaging the rates at which the price level has changed over a broad period of recent history. The lower r_0 is, the broader is the period considered. The result is that inflationary expectations take time to develop, but they also take time to be extinguished. In the initial conditions, if q(0) is chosen low enough then the economy will start with an inflationary momentum, because a low q(0) represents a memory of low prices in the past. Inflationary expectations interact with the real sector of the economy through the effect of a(t) in (33) on the estimated marginal value of money, which in turn influences decisions to buy and sell.

In the real economy, there are technological changes and exogenous political events, etc., which can affect expectations and create new misevaluations at any time (not just in the "initial conditions"). To account for such phenomena, stochastic terms could be added into our production functions and value-update equations. With suitable random disturbances added in, our model could effectively simulate the endless process of economic adjustment which is the business cycle.

10. Example

To illustrate the way in which our dynamic model may be used, let us consider a simple example of an economy with two individuals (I=2), two goods, and no bonds (J=G=2). In this example, the individuals produce goods at the following fixed rates:

$$y_{11} = 20,\ y_{12} = 0,\ y_{21} = 0,\ y_{22} = 12.5;$$

so that only individual 1 produces good 1, and only individual 2 produces good 2. The individuals' utility functions are

$$U_1(x_1^G) = .35x_{12},\ U_2(x_2^G) = 1.1x_{21},$$

so that each individual derives all his utility from consuming the other individual's product. The individual discount rates, depreciation rates, and transaction velocities are as follows:

$$r_1 = r_2 = .1,\ c_1 = 1,\ c_2 = .25,\ \text{and all } n_{ij} = 1.$$

Notice that good 2 is more durable than good 1, since good 2 has a lower depreciation rate.

Suppose for now that the total supply of money in the economy is 20. Then the economy has one stationary equilibrium, which is as follows:

$$(47)\quad x_{10} = 10,\ x_{11} = 10,\ x_{12} = 40,\ v_{10} = .909,\ v_{11} = .433,\ v_{12} = 1,$$

$$x_{20} = 10,\ x_{21} = 10,\ x_{22} = 10,\ v_{20} = .909,\ v_{21} = 1,\ v_{22} = .673,$$

$$p_1 = 1,\ s_{11} = d_{21} = 10,\ p_2 = 1,\ s_{22} = d_{12} = 10,\ a = 0.$$

That is, individual 1 sells good 1 to 2, individual 2 sells good 2 to 1, and both individuals are selling and buying at the fastest transaction rates allowed by the velocity constraints (2) and (3). To check that this equilibrium is indeed stationary, observe that all derivatives in (5), (7), (31), (33) are zero. For example, (33) becomes

$$v_{10} = n_{10}\left(\frac{v_{12}}{p_2} - v_{10}\right) - (r_1 + 2a)\,v_{10} = 1\left(\frac{1}{1} - .909\right) - (.1+0).909 = 0$$

so that $v_{10} = .909$ is a stationary value.

Let us investigate how such an economy would respond to an increase in the money supply, caused (perhaps) by an unexpected one-time cash grant from the government to the individuals. To simulate such a shock, we let the initial conditions $x_{ij}(0)$ and $v_{ij}(0)$ be as in the stationary equilibrium above (47), ex-cept that the initial endowments of money are changed to $x_{10}(0) = x_{20}(0) = 20$. Initially, the expected inflation rate is set at $a(0) = 0$, and $a(t)$ is updated through time using equation (34), with $r_0 = .1$, $\lambda = 0$, $\beta_1 = \beta_2 = 1$.

Figures 1, 2, and 3 show the paths of adjustment to this doubled money supply.[4] The quantity variables $x_{ij}(t)$ are not shown because they remain constant at their initial values $x_{ij}(0)$, as do the s_{ij} and d_{ij} variables. Also, each individual's value of his consumption good remains fixed at $v_{12}(t) = v_{21}(t) = 1$.

insert Figures 1-3 here

The process of adjustment to the doubled money supply can best be described in terms of four stages. The first stage is the initial shock, at $t = 0$. The prices of both goods immediately jump from 1.0 in (47) to $p_1(0) = p_2(0) = 1.1$. These are the highest prices at which each individual is willing to buy his consumption good. That is, these prices are determined by the equations

$$p_1 v_{10} = v_{12},\; p_2 v_{20} = v_{21};$$

so that each individual is indifferent between buying his consumption good and saving his money. Thus, most of the new money is simply hoarded, and does not affect supply and demand.

During the second stage, from t = 0 to t = 3.7, each individual gradually learns to lower his estimated value of money $v_{10}(t)$. This decline in the value of money causes an increase in prices, which causes the estimated inflation rate a(t) to increase, which in turn accelerates the fall of the estimated value of money. Throughout this period, the equations $p_1 v_{10} = v_{12}$ and $p_2 v_{20} = v_{21}$ continue to hold, so that individuals are willing to hoard the excess money. These excess cash balances, $n_{10}x_{10} - p_2 d_{12} = 20 - 10p_2$ and $n_{20}x_{20} - p_2 d_{21} = 20 - 10p_1$, diminish to zero at t = 3.8, when prices reach $p_1 = p_2 = 2$. During this period of rising prices, each individual also becomes more optimistic about the value of the good which he sells, so that $v_{11}(t)$ and $v_{22}(t)$ both rise to a maximum of 10% above their stationary equilibrium values.

In the third stage, from t = 3.7 to t = 7.0, prices are constant at the new "correct" level of $p_1 = p_2 = 2$, but the estimated values of money, v_{10} and v_{20}, continue to fall. The stationary equilibrium value for money would be $v_{10} = v_{20} = .455$ with our doubled money supply, but the values of money overshoot this level and drop to a low of .421 at t = 7.0. This undervaluation of money occurs because of the memory of past inflation, embodied in a(t), which causes individuals to be more pessimistic about holding money than they would be in a stationary equilibrium. The decline in the estimated value of money also leads each individual to become more pessimistic about the value of the good he sells, so that $v_{11}(t)$ and $v_{22}(t)$ drop to a minimum value of about 7% <u>below</u> their "correct" stationary equilibrium values.

During the fourth stage, after t = 7.0, the expected inflation rate decreases slowly to zero (following an exponential decay curve with parameter $r_0 = .1$). This decay of a(t) represents the gradual fading of memories of inflation. As a(t) approaches zero, the estimated values of money and of goods sold gradually rise to their stationary equilibrium levels.

Throughout this process of adjustment, there has been no real welfare effect, because the $x_{ij}(t)$ quantities have been constant at their stationary equilibrium levels. However, this constancy is mainly due to the fact that individuals have fixed production vectors. Suppose, for example, that we revised the example and we allowed individual 1 to increase production of good 1 (making $y_{11} > 20$) but at a high cost in terms of good 2 (so that $y_{12} < 0$). Then, when $v_{11}(t)$ rises above its stationary value, individual 1 would tend to increase his rate of production y_{11}. Similarly, we could easily revise the example so that the later decline in $v_{11}(t)$ would cause y_{11} to fall below the stationary equilibrium rate. Thus, we should interpret the $v_{11}(t)$ and $v_{22}(t)$ paths as suggesting that there may be an initial period of overproduction followed by a later period of underproduction, during the process of adjustment to an increase in the money supply.

Consider now the results of <u>halving</u> the money supply in our simple example, as shown in Figures 4 - 7.

insert Figures 4-7 here

These results were computed using the same initial conditions and update formulas as in the preceding case, <u>except</u> that the initial money balances are now $x_{10}(0) = 5$ and $x_{20}(0) = 5$. The variables not shown in the graphs, x_{10}, x_{20}, x_{11}, x_{21}, v_{12}, v_{21} and $s_{11} = d_{21}$, all remain constant over time at their initial levels.

In many ways, the response of our economy to halving the money supply is roughly a symmetric reflection of the doubling case, passing through four similar stages. For example, $v_{11}(t)$ first falls below and then later rises above its stationary equilibrium level, simply reversing the sequence we saw in the doubling case. (It may be worth noting that the amplitude of these variations is significantly larger in the halving case, however.)

There is one important qualitative difference between the two cases. Unlike the doubling case, we find real changes in supply rate s_{22} after halving the money supply, and these changes cause individual 1 to suffer real welfare losses.

To understand the s_{22} depression, observe first that, in the initial conditions, individual 2 is unwilling to sell good 2 for less than $v_{22}(0)/v_{20}(0) = .740$; by contrast, individual 1 is initially willing to sell good 1 for any price down to $v_{11}(0)/v_{10}(0) = .476$. Thus, in the initial shock when the money supply is halved, the price of good 2 can only drop to $p_2(0) = .74$, while the price of good 1 can drop immediately to its stationary equilibrium level of $p_1(0) = .5$. At these prices, individual 1 is only liquid enough to buy good 2 at rate

$$d_{12}(0) = n_{10}x_{10}(0)/p_2(0) = (1)(5)/(.74) = 6.8 = s_{22}(0).$$

Thus, individual 2 has an unemployed excess supply capacity of equal to $n_{22}x_{22}(0) - s_{22}(0) = (1)(10) - 6.8 = 3.2$, which he is willing to keep off the market because the market price $p_2(0)$ equals his personal reservation price $v_{22}(0)/v_{20}(0)$. In effect, the supplier of the more durable good 2 is unwilling to lower his prices as fast as the supplier of the less durable good 1, because it is more attractive to store good 2 in hopes of better prices in the future. Notice that this distinction between the price-flexibility of durable and nondurable goods appears only in the deflationary case; in the inflation after doubling the money supply, both prices rose at the same rate.

The price equation $p_2 = v_{22}/v_{20}$ and the excess supply inequality $n_{22}x_{22} > s_{22}$ continue to hold until $t = .4$. Then, at $t = .4$, the rise in v_{20} and the fall in v_{22} bring p_2 down to $p_2 = .47$, at which price individual 1 can afford to demand all of 2's supply of good 2. This price p_2 is below the stationary equilibrium level of .5, because of the extra inventories of good 2 accumulated by individual 2 during the period of excess supply.

In the period after $t = .4$, the price p_2 converges quickly to the stationary equilibrium level of $p_2 = .5$. The individuals' marginal values converge more slowly to their stationary equilibrium levels (with v_{10} and v_{20} ending up double their initial values), after first overshooting the limits around $t = 2.5$.

Reviewing these results, we may find some new insights into the dynamic effects of inflationary and deflationary monetary shocks. We can see that the two adjustment processes, although similar, are not completely symmetric phenomena. For example, excess supply unemployment may occur during deflation but not inflation; and downward price movements may be slower for durable goods than for non-durable goods, while this distinction may not occur in upward price movements. Notice, however, that we could not derive any such insights from an asymptotic analysis, or from a rational expectation analysis. In such models, we could only observe that prices must ultimately be proportional to the money supply.

Appendix: Proofs

Proof of Theorem 1. First we prove existence, using a fixed point argument. We shall use the variable D_{ij} to denote the rate at which i spends money for buying j. (So $D_{ij}=p_j d_{ij}$ in our usual notation.) We use the matrix notation
$D = ((D_{ij})_{j=1}^{J})_{i=1}^{I}$, $s = ((s_{ij})_{j=1}^{J})_{i=1}^{I}$.

Let A_1 be the set of all matrix-pairs (s,D) such that, for all i and j:

$$0 \leq s_{ij} \leq n_{ij} x_{ij}(t),$$

$$0 \leq D_{ij} \quad \text{and} \quad \sum_{j'=1}^{J} D_{ij'} \leq n_{i0} x_{i0}(t).$$

Then for every j, let

$$\Pi_j^0 = \min_i \, (v_{ij}(t)/v_{i0}(t)), \tag{48}$$

$$\Pi_j^1 = \max_i \, (v_{ij}(t)/v_{i0}(t)). \tag{49}$$

Then let:

$$A_2 = \{p \in \mathbb{R}^J \,|\, \tfrac{1}{2}\Pi_j^0 \leq p_j \leq \Pi_j^1 + 1\}.$$

Let $A = A_1 \times A_2$, and observe that A is a nonempty compact convex subset of a finite dimensional vector space.

Given any p in A_2, let $Z_1(p)$ be the set of all matrix-pairs (s,D) in A_1 which maximize

$$(50)\quad \sum_{i=1}^{I}\sum_{j=1}^{J}(p_j v_{i0}(t)-v_{ij}(t))s_{ij}$$

$$+\sum_{i=1}^{I}\sum_{j=1}^{J}(v_{ij}(t)/p_j-v_{i0}(t))D_{ij}$$

subject to the constraint $(s,D)\in A_1$.
Given any (s,D,p) in A, let $Z_2(s,D,p)$ be the one-point set such that $\hat{p}\in Z_2(s,D,p)$ if and only if $\hat{p}\in A_2$ and, for every j

$$\hat{p}_j = \max\{\tfrac{1}{2}\ \Pi_j^0,\ \min\{\Pi_j^1+1,\ p_j+\sum_{i=1}^{I}(D_{ij}/p_j - s_{ij})\}\}.$$

Let $Z(s,D,p) = Z_1(p)\times Z_2(s,D,p)$. It is straightforward to check Z is an uppersemicontinuous correspondence from A to the non-empty compact convex subsets of A. Thus, by the Kakutani fixed-point theorem, there exists some $(\bar{s},\bar{D},\bar{p})$ in A such that $(\bar{s},\bar{D},\bar{p})$ $\in Z(\bar{s},\bar{D},\bar{p})$.

For every j, $\bar{p}\in Z_2(\bar{s},\bar{D},\bar{p})$ implies that one of the following must hold:

$$(51)\quad \sum_i(\bar{D}_{ij}/\bar{p}_j - \bar{s}_{ij}) = 0;\ \text{or}$$

$$(52)\quad \sum_i(\bar{D}_{ij}/\bar{p}_j - \bar{s}_{ij}) > 0\ \text{and}\ \bar{p}_j = \Pi_j^1 + 1;\ \text{or}$$

$$(53)\quad \sum_i(\bar{D}_{ij}/\bar{p}_j - \bar{s}_{ij}) < 0\ \text{and}\ \bar{p}_j = \tfrac{1}{2}\Pi_j^0\ .$$

If (52) were true, then we would have $\bar{p}_j>v_{ij}(t)/v_{i0}(t)$ for all i, so all $\bar{D}_{ij}=0$. (Recall (49) and (50).) Thus (52) is impossible. Similarly (53) would imply $\bar{p}_j<v_{ij}(t)/v_{i0}(t)$ and thus $\bar{s}_{ij}=0$, for all i. (Recall (48) and (50), and observe that $\Pi_j^0>\frac{1}{2}\Pi_j^0$, because the positivity of all $v_{ij}(t)$ implies that Π_j^0 is positive.)

Thus (53) is also impossible, and (51) must be true.

Now let $p_j(t)=\bar{p}_j$, $s_{ij}(t)=\bar{s}_{ij}$, and $d_{ij}(t)=\bar{D}_{ij}/\bar{p}_j$, for all i and j. Then (51) gives us the market clearing conditions (8), and $(\bar{s},\bar{D})\in L_1(p)$ implies that the $s_{ij}(t)$ and $d_{ij}(t)$ do maximize (13) subject to (2)-(4) (with the $y_{ij}(t)$ chosen accordingly). Thus we have proven the existence of market-clearing prices.

To prove uniqueness, suppose that p and $\hat{p}$ are two market-clearing price vectors. Let d_{ij} and s_{ij} denote the individuals' demand and supply quantities for the p prices, and let $\hat{d}_{ij}$ and $\hat{s}_{ij}$ denote the corresponding quantities for $\hat{p}$. Let H_o denote the set of all nonmoney assets j such that $p_j<\hat{p}_j$

Observe that, for every j in H_o and for every i,

(54) $s_{ij}\le\hat{s}_{ij}$,

because raising the price of j increases the coefficient of s_{ij} in (13) and the only constraint on s_{ij} is $0\le s_{ij}\le x_{ij}(t)$.

A comparison of (13) with constraint (3) shows that i will budget his money for assets j such that

$$v_{ij}(t)/p_j(t) = \max_{j'}(v_{ij'}(t)/p_{j'}(t)) \ge v_{i0}(t),$$

and he will budget all his money if the last inequality is strict. So if i buys any asset in H_o at the $\hat{p}$ prices, then i must budget <u>all</u> his money for assets in H_o at the p prices, since the assets in H_o become strictly better bargains relative to the other assets as we go from $\hat{p}$ to p. Thus, for every individual i,

(55) $$\sum_{j\in H_o} p_j d_{ij} \ge \sum_{j\in H_o} \hat{p}_j\hat{d}_{ij}.$$

Now suppose that, contrary to the theorem, there exists some j in H_o such that $\sum_i s_{ij}>0$ or $\sum_i \hat{s}_{ij}>0$. Then we can apply (54), (55), and the market-clearing conditions to get:

$$\sum_{i=1}^{I} \sum_{j\in H_o} p_j s_{ij} < \sum_i \sum_{j\in H_o} \hat{p}_j \hat{s}_{ij} =$$

$$= \sum_i \sum_{j\in H} \hat{p}_j \hat{d}_{ij} \leq \sum_i \sum_{j\in H_o} p_j d_{ij}$$

$$= \sum_i \sum_{j\in H_o} p_j s_{ij},$$

which is impossible. So any asset which has a strictly higher price in $\hat{p}$ than in p cannot be traded in either equilibrium. Then, reversing the roles of p and $\hat{p}$, we see that if $p_j>\hat{p}_j$ then j also cannot be traded in either equilibrium. So the prices of traded goods are uniquely determined.

Now let H_1 be the set of nonmoney assets which have uniquely determined market-clearing prices at time t. Suppose f is an asset which is not in H_1. Then f cannot be traded in any equilibrium at time t. Since f is not supplied, we must have

(56) $\quad p_f(t)\leq v_{if}(t)/v_{iO}(t),$

for all i such that $n_{ij}x_{ij}(t)>0$.

Since f is not demanded, we must have

(57) $\quad v_{if}(t)/p_f(t)\leq \max\{v_{iO}(t),\ \max_{j\in H_1} (v_{ij}(t)/p_j(t))\},$

for all i such that $n_{iO}x_{iO}(t)>0$.

Inequalities (56) and (57) together determine a closed interval for $p_f(t)$. As long as $p_f(t)$ is chosen anywhere in this interval, it will be neither supplied nor demanded, and it will not disturb the markets for assets in H_1. This proves the last part of the theorem.

Proof of Theorem 2. We prove first that a temporary equilibrium path exists on the interval $0 \leq t \leq \lambda$.

By Assumption 4, the recent general price levels are bounded and continuous on the interval $-\lambda \leq t < 0$, so the $\dot{q}$ equation (35) can be integrated to give us $q(t)$ and $a(t)$ as continuous and bounded functions of t in the interval between 0 and λ.

We shall use matrix notation here, letting x (or $x(t)$) denote the matrix of x_{ij} (or $x_{ij}(t)$) quantities, etc. Thus, we may say that $(p(t),y(t),s(t),d(t))$ is a temporary equilibrium for $(x(t), v(t))$ if:

$$(p(t),y(t),s(t),d(t)) \in \mathbb{R}_+^{J+IG+2IJ}, \quad (x(t),v(t)) \in \mathbb{R}_+^{2I(j+1)},$$

the market clearing equation (8) holds for every nonmoney asset j, and the linear functional (13) is maximized over (y,s,d) subject to (1)-(4) for every individual i.

For any asset j, let L_j be the set of individuals for whom j is liquid, that is,

$$L_j = \{i \mid n_{ij} > 0\}.$$

Given (x,v) in $\mathbb{R}_+^{2I(J+1)}$, we say that (x,v) is regular if, for every individual i and asset j, $v_{ij} > 0$ and if $i \in L_j$ then $x_{ij} > 0$.

Theorem 1 assured us that market-clearing prices are essential unique, except for the untraded assets. With regularity, we can continuously select a unique standard price, even for untraded assets.

For any regular (x,v), we say that a temporary equilibrium (p,s,d,y) is <u>standard</u> if, for every nonmoney asset j

(58) $p_j \geq \underset{i \in L_j}{\text{minimum}} \; (v_{ij}/v_{i0})$, and

(59) if $L_j=\emptyset$ then $p_j = \underset{i}{\text{maximum}} \; (v_{ij}/v_{i0})$.

Let $T(x,v)$ be the set of standard temporary equilibria for (x,v). Let $P(x,v)$ be the standard equilibrium price vector p, such that $(p,s,d,y) \in T(x,v)$ for some (s,d,y).

We must check that $P(x,v)$ is uniquely defined. If $L_j=\emptyset$, then j is not traded and (59) is the lowest price for j which prevents demand. If j is traded, then its equilibrium price is unique, and must satisfy (58) (or else no one would sell j). If $L_j \neq \emptyset$ and j is not traded, then (56) and (57) define the interval of possible equilibrium prices, and (with regularity) (58) tells us to pick the highest in this interval, that is $p_j = \underset{i \in L_j}{\text{minimum}} \; (v_{ij}/v_{i0})$. Thus $P(x,v)$ is a well-defined function as long as (x,v) is regular.

It is straightforward to check that $T(x,v)$ is a nonempty compact convext set for any regular (x,v), and that T is an uppersemicontinuous correspondence. Nonemptiness follows from the preceding paragraph. Compactness and convexity hold because the market-clearing and maximization conditions are linear, and constraints (1)-(4) define a compact and convex set given the positive prices. All of the conditions defining our standard temporary equilibria are continuous in all variables, so T is uppersemicontinuous. Because T is uppersemicontinuous and its price components P are unique at any regular (x,v), it follows that $P(x,v)$ is a continuous function on the set of regular (x,v).

For any regular $(x(t),v(t))$, let $R(x(t),v(t),a(t))$ be the set of all $(\dot{x}(t),\dot{v}(t))$ satisfying (5)-(7) and (31)-(33) for some $(p(t),s(t),d(t),y(t))$ in $T(x(t),v(t))$.

Since standard equilibrium prices are unique and since (5)-(7) are linear in (s,d,y), $R(x,v,a)$ is a nonempty compact convex set, for any regular (x,v). Also R is an uppersemicontinuous function at every (x,v,a) such that (x,v) is regular. (We use Assumption 1 and 3 here to guarantee that U_i and W_i terms in (31) are well-defined and continuous.)

Let k and K be as in Assumption 2 and 3, and let

$$\bar{y}_{ig} = \max \{y_{ig} | y_i \in Y_i(0,\ldots,0)\}.$$

Assumption 2 and 3 guarantee that, if $y_i \in Y_i(x_i^G)$, then for all g

$$-kx_{ig} \leq y_{ig} \leq \bar{y}_{ig} + K(\sum_{f=1}^{G} x_{if}).$$

(To derive the second inequality above, consider $\bar{v}_i=(0,..,0,1,0,..,0)$, with a single 1 in the g-component; and observe that $y_{ig} \leq W_i(x_i^G,\bar{v}_i)$ and $\frac{\partial W_i}{\partial x_{if}}(\tilde{x}_i^G,\bar{v}_i) \leq K$, for every good f and bundle $\tilde{x}_i^G$.)

The market clearing conditions (8), together with (2)-(4), imply that

$$(60) \qquad d_{ij} \leq \sum_{h=1}^{I} n_{hj}x_{hj} \text{ (for all } j=1, \ldots,J), \text{ and}$$

$$\sum_{j=1}^{J} p_j s_{ij} \leq \sum_{h=1}^{I} n_{h0}x_{h0},$$

for every individual i; otherwise the other individuals would be unable to balance i's demand and supply. Substituting these inequalities together with (2)-(4) into (5)-(7), we get

$$(61)\quad -(k+c_g+n_{ig})x_{ig} \le \dot{x}_{ig} \le \bar{y}_{ig} + K(\sum_{f=1}^{G} x_{if}) + \sum_{h=1}^{I} n_{hg}x_{hg}$$

$$(62)\quad -(c_b+n_{ib})x_{ib} \le \dot{x}_{ib} \le \sum_{h=1}^{I} n_{hb}x_{hb}$$

$$(63)\quad -n_{i0}x_{i0} \le \dot{x}_{i0} \le \sum_{b=G+1}^{J} x_{ib} + \sum_{h=1}^{I} n_{h0}x_{h0}$$

whenever $(\dot{x},\dot{v}) \in R(x,v,a)$.

For any nonmoney asset j and any individual i in L_j, if $n_{ij}(v_{i0}p_j - v_{ij}) > 0$ then i should offer all of his endowment of j for sale, so (8) would imply

$$p_j \le (\sum_{h=1}^{I} n_{h0}x_{h0})/(n_{ij}x_{ij})$$

(otherwise there would not be enough money to demand the j supplied by i). This implies that

$$(64)\quad n_{ij}(v_{i0}p_j - v_{ij})^+ \le \sum_{h=1}^{I} n_{h0}x_{h0}v_{i0}/x_{ij}.$$

Also, i should always budget all his money for assets which achieve the maximization in (33) (if it has a positive value) so

$$(65)\quad \max_j (n_{i0}(v_{ij}/p_j - v_{i0})^+) =$$

$$= \sum_{j=1}^{J} n_{i0}(v_{ij}/p_j - v_{i0})p_jd_{ij}/(n_{i0}x_{i0})$$

$$\le \sum_{j=1}^{J} v_{ij}d_{ij}/x_{i0} \le \sum_{h=1}^{I}\sum_{j=1}^{J} n_{hj}x_{hj}v_{ij}/x_{i0}.$$

(We used (60) at the last step.) Let K' be the upper bound on the partial derivatives of U_i, mentioned in Assumption 1. When (64) and (65) are substituted into (31)-(33) and Assumption 1 and 3 are applied, then $(\dot{x},\dot{v}) \in R(x,v,a)$ implies the following:

$$(66)\quad -(c_g + r_i)v_{ig} \le \dot{v}_{ig} \le \begin{cases} \le K' + K(\sum_{f=1}^{G} v_{if}), & \text{if } i \notin L_j; \\ \le K' + K(\sum_{f=1}^{G} v_{if}) + \sum_{h=1}^{I} n_{hO}x_{hO}v_{iO}/x_{ib}, & \text{if } i \in L_j; \end{cases}$$

$$(67)\quad -(c_b + r_i + 2|a|)v_{ib} \le \dot{v}_{ib} \le \begin{cases} \le v_{iO} + 2|a|v_{ib}, & \text{if } i \notin L_j; \\ \le v_{iO} + 2|a|v_{ib} + \sum_{h=1}^{I} n_{hO}x_{hO}v_{iO}/x_{ib}, & \text{if } i \in L_j; \end{cases}$$

$$(68)\quad -(r_i + 2|a|)v_{iO} \le \dot{v}_{iO} \le \begin{cases} \le 2|a|v_{iO}, & \text{if } i \notin L_O; \\ \le 2|a|v_{iO} + \sum_{h=1}^{I}\sum_{j=1}^{J} n_{hj}x_{hj}v_{ij}/x_{iO}, & \text{if } i \in L_O. \end{cases}$$

For any number $m>0$, let $M(x,v,a,m) = (\hat{x},\hat{v},\hat{a})$, where

$\hat{x}_{ij} = \max\{0, \min\{m, x_{ij}\}\}$, if $i \notin L_j$;

$\hat{x}_{ij} = \max\{\frac{1}{m}, \min\{m, x_{ij}\}\}$, if $i \in L_j$;

$\hat{v}_{ij} = \max\{\frac{1}{m}, \min\{m, v_{ij}\}\}$;

$\hat{a} = \max\{-m, \min\{m, a\}\}$.

By the theorem of Lazota and Opial (1965) (see Henry (1973)), for any $m > 0$, there exists an absolutely continuous solution on the interval $0 \le t \le \lambda$ to the multivalued differential equation

$$(69) \quad (\dot{x}(t), \dot{v}(t)) \in R(M(x(t), v(t), a(t), m))$$

satisfying the given initial conditions $(x(0), v(0))$. To see this, observe that the range of $M(.,.,.,.,m)$ is compact, and M gives us $(\hat{x}, \hat{v})$ pairs which are always regular. The left-hand and right-hand sides of (61)-(63) and (66)-(68) are continuous functions on the range of $M(.,.,.,.,m)$, and are therefore bounded on this range. So the range of $R(M(.,.,.m))$ is also bounded. The set $R(M(x,v,a(t),m))$ is always nonempty, compact and convex; and it depends uppersemicontinuously on (x,v) and measurably on t. (In fact, it is continuous in t.) Thus, all of the conditions for the Lazota-Opial existence theorem are satisfied.

It now only remains to show that, if m is chosen large enough, then our solution to (69) will also be a solution to $(\dot{x}, \dot{v}) \in R(x,v,a)$.

We define matrices $x^*(t)$ and $x_*(t)$ so that

$x_{*ij}(0) = \frac{1}{2} x_{ij}(0), x^*_{ij}(0) = x_{ij}(0) + 1.$

and the following differential equations are satisfied:

$$\dot{x}_{*ig}(t) = -(k + c_g + n_{ig}) x_{*ig}(t),$$

$$\dot{x}_{*ib}(t) = -(c_b + n_{ib}) x_{*ib}(t),$$

$$\dot{x}_{*i0}(t) = -n_{i0} x_{*i0}(t),$$

$$\dot{x}^*_{ig}(t) = \bar{y}_{ig} + K(\sum_{f=1}^{G} x^*_{if}(t)) + \sum_{h=1}^{I} n_{hg}\, x^*_{hg}(t),$$

$$\dot{x}^*_{ib}(t) = \sum_{h=1}^{I} n_{hb}\, x^*_{hb}(t),$$

$$\dot{x}^*_{io}(t) = \sum_{b=G+1}^{J} x^*_{ib}(t) + \sum_{h=1}^{I} n_{hO}\, x^*_{hO}(t).$$

for all i,j,g,b on the usual ranges. These equations are all linear, and they have solutions which are continuous and non-negative throughout the interval. Furthermore, for any j and any i in L_j, $x_{*ij}(t)$ is an exponential decay function with $x_{*ij}(O)>O$, so $x_{*ij}(t)$ is bounded away from O on the interval $O\le t\le\lambda$. So we can choose some number $\bar{m}_1$ large enough so that, for every i,j, and every t in $[O,\lambda]$,

(70) $x^*_{ij}(t)\le\bar{m}_1$, and if $i\in L_j$ then $x_{*ij}(t)\ge 1/\bar{m}_1$.

Next we define matrices $v^*(t)$ and $v_*(t)$ so that (for all i,j,g,b)

$$v_{*ij}(O) = v_{ij}(O)/2,\ v^*_{ij}(O)=v_{ij}(O)+1,$$

and the following differential equations are satisfied:

$$\dot{v}_{*ig}(t) = -(c_g + r_i)v_{*ig}(t),$$

$$\dot{v}_{*ib}(t) = -(c_b+r_i+2|a(t)|)v_{*ib}(t),$$

$$\dot{v}_{*io}(t) = -(r_i+2|a(t)|)v_{*io}(t),$$

$$\dot{v}^*_{ig}(t) = K' + K(\sum_{f=1}^{G} v^*_{if}(t)) + \sum_{h=1}^{I} n_{hO}(\bar{m}_1)^2 v^*_{iO}(t)$$

$$\dot{v}^*_{ib}(t) = v^*_{iO}(t) + 2|a(t)|v^*_{ib}(t)+ \sum_{h=1}^{I} n_{hO}(\bar{m}_1)^2 v^*_{iO}(t),$$

$$\dot{v}^*_{iO}(t) = 2|a(t)|v^*_{iO}(t) + \sum_{h=1}^{I}\sum_{j=1}^{J} n_{hj}(\bar{m}_1)^2 v^*_{ij}(t).$$

Again, these differential equations are linear and have continuous positive solutions on the interval $0 \le t \le \lambda$. So we can choose some $\bar{m}_2$ such that, for every i,j, and every t in $[0,\lambda]$,

$$1/\bar{m}_2 \le v_{*ij}(t), \text{ and } v^*_{ij}(t) \le \bar{m}_2.$$

Since a(t) is bounded on $0 \le t \le \lambda$, we can choose some $\bar{m}_3$ so that $-\bar{m}_3 \le a(t) \le \bar{m}_3$ for every t in $[0,\lambda]$. Now let

$$m = \text{maximum } \{\bar{m}_1, \bar{m}_2, \bar{m}_3\}.$$

Let $(x(t),v(t))$ solve (69) for this m, on the interval $0 \le t \le \lambda$.

Suppose that, at some t between 0 and λ, we have

$$(71) \quad x_{*ij}(t) \le x_{ij}(t) \le x^*_{ij}(t) \text{ and } v_{*ij}(t) \le v_{ij}(t) \le v^*_{ij}(t)$$

for every i and j. Then our choice of m guarantees that

$$M(x(t),v(t),a(t),m) = (x(t),v(t),a(t)),$$

and so

$$(\dot{x}(t),\dot{v}(t)) \in R(x(t),v(t),a(t)).$$

But then a comparison of the $\dot{x}_*, \dot{x}^*, \dot{v}_*$, and $\dot{v}^*$ equations to (61)-(63) and (66)-(68) (using (70) and (71)) shows us that

$$(72) \quad \dot{x}_{*ij}(t) \le \dot{x}_{ij}(t) \le \dot{x}^*_{ij}(t) \text{ and } \dot{v}_{*ij}(t) \le \dot{v}_{ij}(t) \le \dot{v}^*_{ij}(t)$$

for every i and j.

At time t=0 (71) holds with all <u>strict</u> inequalities (except that we may have $x_{ij}(0) = x_{*ij}(0) = 0$ for some i and j such that $i \notin L_j$). Thus, since (71) implies (72), (71) must continue to hold for all t between 0 and λ, because the distances between the components of $(x(t), v(t))$ and their respective upper-bound and lower-bound functions never diminish. (For any i and j such that $x_{ij}(0)=x_{*ij}(0) = 0$, we get $x_{*ij}(t) = 0$ for all t, and $x_{ij}(t)$ can never go negative, since $x_{ij}(t)<0$ and $i\notin L_j$ would imply $\dot{x}_{ij}(t)\geq 0$ in any solution to (69). So (71) holds in this case too.) Thus our solution satisfies

$$(\dot{x}(t),\dot{v}(t))\in R(x(t),v(t),a(t))$$

for almost all t between 0 and λ. So we have proven the existence of a solution on the interval $[0,\lambda]$.

In this solution, for every i and j, $v_{ij}(\lambda)>0$, and if $n_{ij}>0$ then $x_{ij}(\lambda)>0$. Furthermore, the vector of standard prices $p_j(t) = P_j(x(t),v(t))$ is a continuous function of $(x(t),v(t))$, so it is also continuous and bounded as a function of t in the interval from 0 to λ. Thus, if we relabelled the time axis by subtracting λ (so that $t=\lambda$ became $t=0$), then Assumption 4 would still be satisfied at the new "initial conditions". So the preceding argument also guarantees that our solution can be extended an additional λ units of time. Since $\lambda>0$, repeating this argument inductively guarantees that our solution can be extended over all $t\geq 0$. This proves Theorem 2.

<u>Proof of Theorem 3</u>. Let $(\hat{x}_i^G,\hat{y}_i)_{i=1}^I$ be a feasible stationary state; and let $(x_i^G,v_i,y_i)_{i=1}^I$ and p be the vectors of endowments, values, production rates, and prices in a stationary equilibrium. We shall prove that the stationary state cannot Pareto-dominate the stationary equilibrium by showing that

$$\sum_{i=1}^{I}(U_i(\hat{x}_i^G)/v_{i0})\leq\sum_{i=1}^{I}(U_i(x_i^G)/v_{i0}). \tag{73}$$

Let us consider now a fixed individual i. In the stationary equilibrium $a=0$ and $\dot{v}_{i0}=0$. So (33) implies that

$$v_{i0}p_g\geq v_{ig} \tag{74}$$

for every good g. (Recall $r_i=0$.)

Because the set $\{(\tilde{x}_i^G,\tilde{y}_i)\,|\,\tilde{y}_i \in Y_i(\tilde{x}_i^G)\}$ is convex, it follows that the function $W_i(\tilde{x}_i^G,v_i)$ is concave in $\tilde{x}_i^G$. (Recall (17).) Since U_i is also concave, we get

$$(75)\quad U_i(\hat{x}_i^G)+W_i(\hat{x}_i^G,v_i)\le U_i(x_i^G)+W_i(x_i^G,v_i)+ + \sum_{g=1}^{G}\left(\frac{\partial U_i}{\partial x_{ig}}(x_i^G)+\frac{\partial W_i}{\partial x_{ig}}(x_i^G,v_i)\right)(\hat{x}_{ig}-x_{ig}).$$

Substituting $\dot{v}_{ig}=0$ into (31), and using (74) and $r_i=0$, we get

$$\frac{\partial U_i}{\partial x_{ig}}(x_i^G) + \frac{\partial W_i}{\partial x_{ig}}(x_i^G,v_i) = c_g v_{ig} - n_{ig}(v_{i0}p_g - v_{ig}).$$

Also, $W_i(x_i^G,v_i) = \sum_g v_{ig}y_{ig}$, because y_i was chosen to maximize (13); and $W_i(\hat{x}_i^G,v_i)\ge\sum_g v_{ig}\hat{y}_{ig}$, because $\hat{y}_i\in Y_i(\hat{x}_i^G)$. Thus (75) becomes

$$(76)\quad U_i(\hat{x}_i^G)+\sum_g v_{ig}\cdot(\hat{y}_{ig}-c_g\hat{x}_{ig}) + \sum_g(v_{i0}p_g-v_{ig})n_{ig}\hat{x}_{ig}$$

$$\le U_i(x_i^G) + \sum_g v_{ig}\cdot(y_{ig}-c_g x_{ig})+\sum_g(v_{i0}p_g-v_{ig})n_{ig}x_{ig}.$$

By (42), we know that for all g

$$(77)\quad \hat{y}_{ig}-c_g\hat{x}_{ig}\le n_{ig}\hat{x}_{ig}.$$

Also, if $v_{i0}p_g>v_{ig}$ then i must offer his entire endowment of g for sale ($s_{ij}=n_{ig}x_{ig}$) and demand no g ($d_{ig}=0$) in the stationary equilibrium. So $\dot{x}_{ig}=0$ implies that $y_{ig}-c_g x_{ig}=n_{ig}x_{ig}$ when the inequality in (74) is strict. Thus, for all g,

(78) $(v_{io}p_g - v_{ig})(y_{ig} - c_g x_{ig}) = (v_{io}p_g - v_{ig})n_{ig}x_{ig}.$

Then substituting (77) and (78) into (76) gives us

(79) $$U_i(\hat{x}_i^G) + \sum_g v_{io}p_g(\hat{y}_{ig} - c_g\hat{x}_{ig})$$
$$\leq U_{ig}(x_i^G) + \sum_g v_{io}p_g(y_{ig} - c_g x_{ig}).$$

Finally, if we divide (76) by v_{io} on both sides, sum over i, and use the fact that (by (41))

$$\sum_i (\hat{y}_{ig} - c_g\hat{x}_{ig}) = \sum_i (y_{ig} - c_g x_{ig}) = 0,$$

then we get (73), as desired. This proves the theorem.

Footnotes

[1]One might simply assume that $n_{ij} = \bar{n}_j$ for all j, where $\bar{n}_j$ is the common velocity of asset j, measuring its intrinsic liquidity. But allowing inventory velocities to depend on i enables us to describe situations where some individuals cannot sell certain goods. For example, if i is a farmer and j is penicillin then $n_{ij} = 0$. This flexibility will allow us to weaken Assumption 4 in Section 6.

[2]This idea is used in the model of Day and Cigno (1978, Chapter 12).

[3]When $r_i > 0$, some inefficiency can arise in our stationary equilibria, due to the effect of monetary intermediation. Grandmont and Younes (1973) have studied this effect in the context of a simpler model, without durable goods.

[4]To compute these results, all differential equations were approximated by discrete-time difference equations with a time interval of $\Delta t = \frac{1}{10}$. For example, equation (33) became

$$v_{io}(t+\tfrac{1}{10}) = v_{io}(t) + \frac{1}{10}\dot{v}_{io}(t)$$

$$= v_{io}(t) + \frac{1}{10}[\max_j (n_{io}(\frac{v_{ij}(t)}{p_j(t)} - v_{io}(t))^+) - (r_i + 2a(t))v_{io}(t)].$$

Similarly, the derivative required in equation (34) was approximated by

$$\frac{\dot{Q}(t)}{Q(t)} \approx \frac{(Q(t) - Q(t-.1))/(.1)}{(Q(t) + Q(t-.1))/2}, \qquad \text{where } Q(t) = p_1(t) + p_2(t).$$

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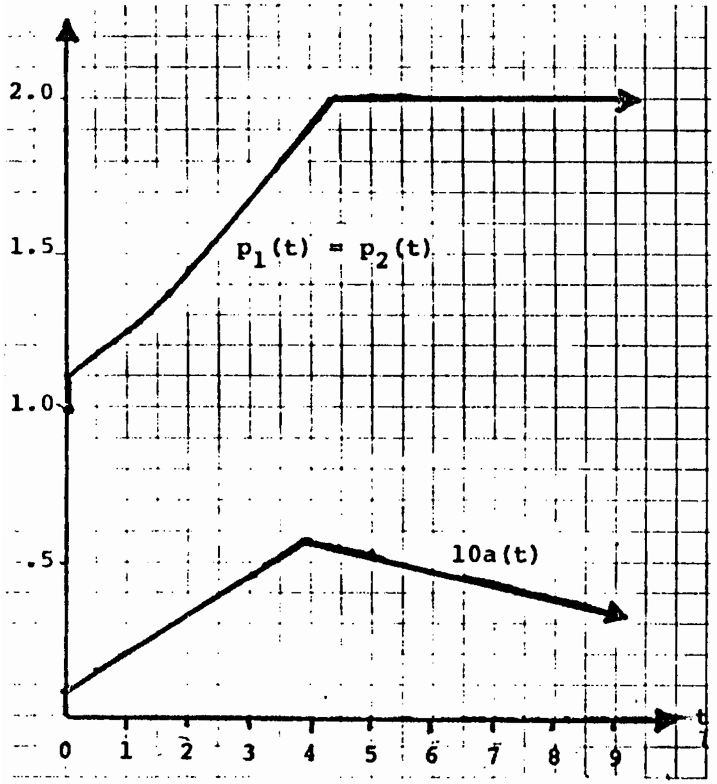
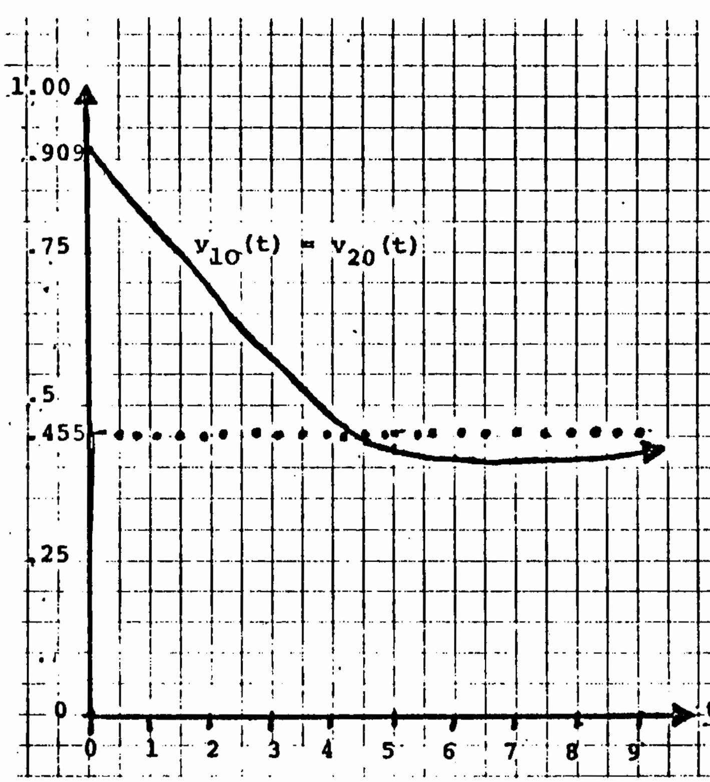


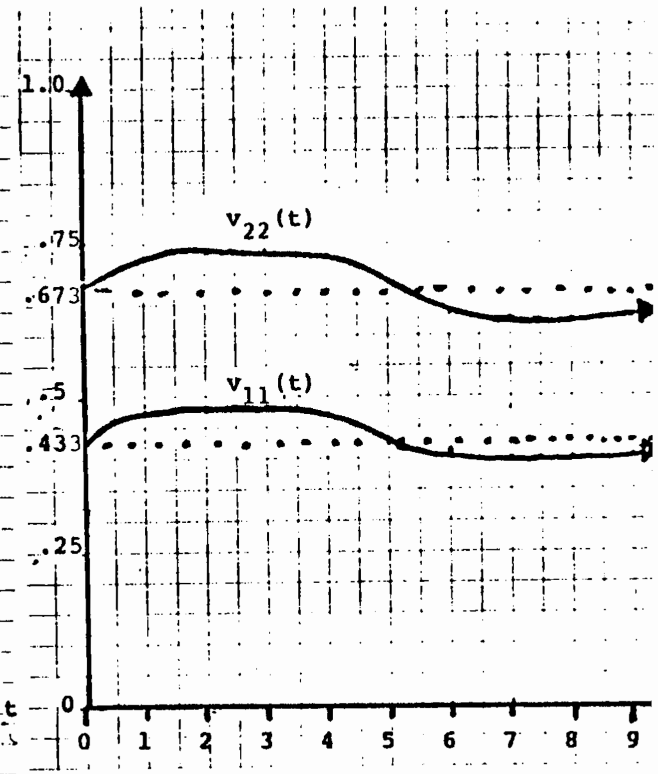
<u>Figure 1</u>

Prices and expected inflation, after doubling the money supply .



<u>Figure 2</u>

Marginal value of money, after doubling the money supply.



<u>Figure 3</u>

Marginal values of goods to their pro ducers, after doubling the money supply

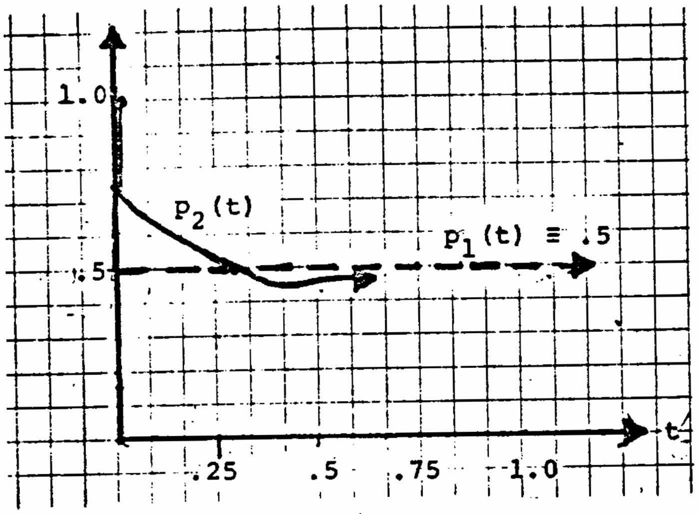


Figure 4

Price adjustments after halving the money supply

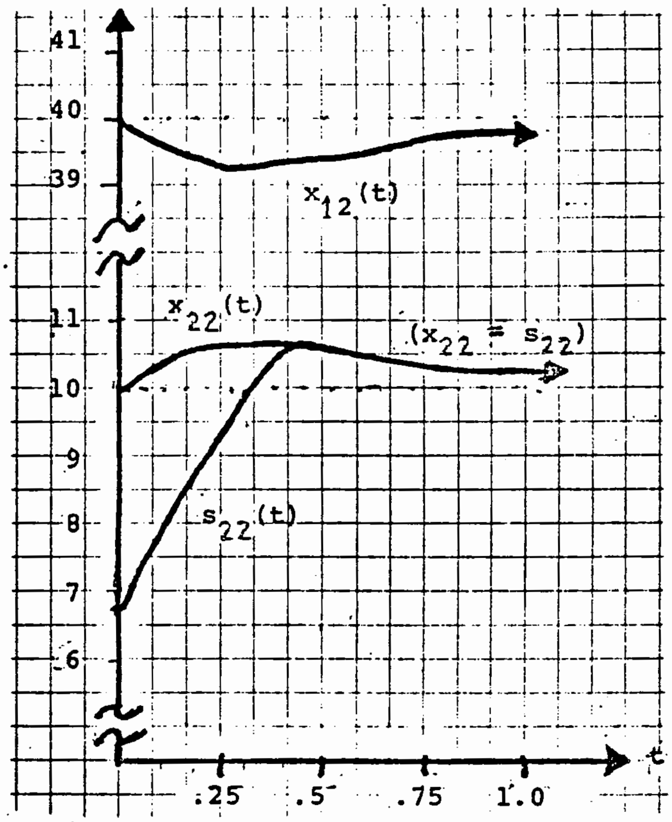


Figure 5

Sales and inventories of good 2 after halving the money supply

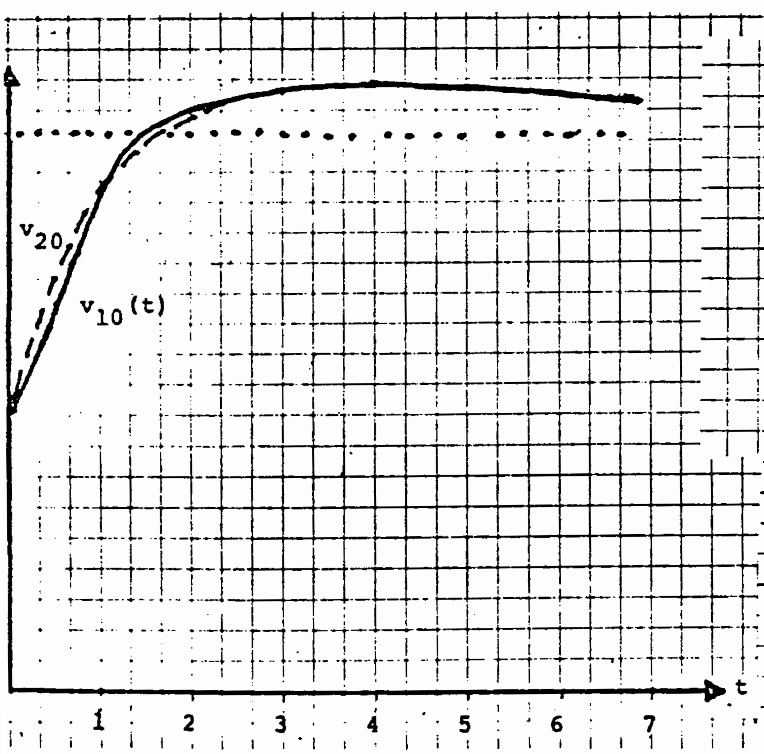


Figure 6

Marginal value of money, after halving the money supply

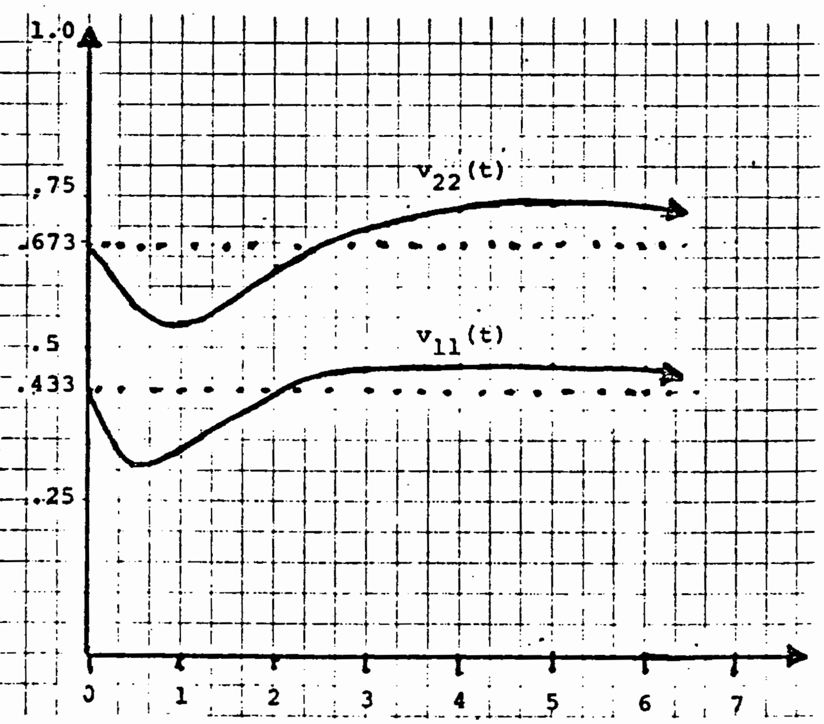


Figure 7

Marginal values of goods to their producers, after halving the money supply