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MAJORITY VOTING RULES AS BEHAVIOR TOWARDS RISK

by

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Section 1. Introduction

In the opening discussion of his original work, K. J. Arrow [1] mentioned the possibility of extending the model of rational choice he was to discuss to allow for randomized decision rules. In his words, "It is at least a possibility, to which attention should be drawn that the paradox to be discussed below might be resolved by such a broader concept of rationality." It is our hope that the model we are about to propose will be viewed as a step in this direction. Specifically, it can be argued that two quite distinct problems arise in the social choice area. Arrow and his followers pointed out a fundamental logical difficulty in the concept of social welfare when we choose to view it as an aggregate of individual preferences over social states. If we look for a social welfare function aggregating any set of citizens' preferences while meeting a certain set of a priori desirable ethical properties, e.g. Pareto optimality, citizens' sovereignty, non-dictatorship, etc - then this line of research has established the general non-existence of such aggregation mechanisms - unless individual preferences are very severely restricted. (See Black [4], Sen [14] for instance.) In the controversy that followed Arrow's discovery, however, several authors, especially A. Bergson, pointed out that the problem raised by Arrow can be interpreted in the following context: instead of "assuming that the concern of welfare economics is to counsel individual citizens generally," we would choose to restrict ourselves to counselling "public officials" only. "Furthermore, the values to be taken as data are not those which might guide the official if he were a private citizen. The official is envisaged as more or less neutral ethically. His one aim in life is to implement the values of other citizens as given by some rule of collective decision-making," [1]. In response to Bergson's statement, Arrow explains that his "interpretation of the social choice problem agrees fully with that given by Bergson," [1]. But another line of research can also be explored. If instead of posing the aggregation problem as a purely normative problem, we decide to consider it from the viewpoint of a concerned individual, we face a completely different question: what kind of collective decision rule (i.e. constitution) would a rational utility-maximizing individual choose? Here the optimization problem is shifted from the viewpoint of some ethically neutral outside "advisor" (as in Arrow's work) to that of a specific consumer who is personally involved in the consequences of his choice. Broadly speaking, the problem is one of decision-making under uncertainty. The main uncertainty arises from the partial or total ignorance faced by any single individual when it comes to describing (1) the issues at stake, (2) the preferences of the other citizens over the issues. Clearly, this brings out the "gaming" aspects of this problem.

Having thus described the distinction between Arrow's problem and what will be referred to thereafter as the "constitutional choice problem," we must note, first of all, a major dilemna: to choose a constitution, we, in turn, need a rule. The only way out consists in requiring a unanimity among the citizens at the stage of constitutional choice since this is the only case where social choice can be viewed as a special case of individual choice: if all agree on a rule, then the aggregation problem is solved ipso facto. In this paper we will explore the following working hypothesis: the frequent choice of some form of majority voting at the constitutional choice stage, far from being a mere historical accident, is the result of a rational economic calculation on the part of each individual citizen. In other words, given the basic uncertainty faced by each individual, majority voting possesses certain a priori desirable properties. It is these properties which we will now attempt to characterize.

Section 2. A General Framework of Analysis

Basic Assumptions

2.1. In order to view the constitutional choice problem in its full generality we must think of it at the outset as a decision-making problem under uncertainty. Imagine a group of individual citizens, a "society" S, where $S = \{h | h = 1,2,\ldots \ell\}$ trying to decide jointly --i.e. unanimously-- on a group decision-making rule, a "constitution." In this sense a "constitution" will be some kind of process or mechanism, (e.g. voting toss of a coin, spinning of a roulette wheel, etc.) agreed upon in advance and which will apply to all social decision problems--as opposed to individual decision problems. Roughly speaking these social decision problems will involve the production and/or distribution of public goods. As far as each individual in S is concerned, we assume that he behaves as an "homo-economicus" and all his decisions regarding his personal choices are guided by this principle of rationality. In particular when he is faced with the constitutional choice problem, his analysis of the problem and the solution he will choose will both be the result of this general rationality postulate. Needless to say one can address numerous criticisms to this "rationality" assumption but, at any rate, it seems highly plausible that such a would-be citizen should "think twice" and follow a selfish attitude when he is asked to give up some of his freedom of choice to the benefit of an unknown and, possibly, tyrannical "collective decisionmaker." After all as we have stated previously at this first stage of the social welfare problem, each individual is free to agree or not to agree on a "constitution" and, by the unanimity rule, even if only one person disagrees this constitution project cannot be adopted. In other words each individual can afford to be "ultra rational" in this particular decision as to the kind of social contract he should enter, since he alone forms a blocking coalition.

Given this economic rationality assumption, let us try to apply it to the problem at hand. At this point, a state of complete uncertainty prevails in regard to the nature of the collective issues that will be decided upon by our constitutional rule. Whether they concern one public good vs. another and/or one level of public spending vs another and/or one public policy vs. another or more, the most accurate formal representation we can give is the following: at any given time when a social decision is going to be made it will involve some set of "alternatives" $A = \{a_1, a_2, \ldots, a_m\}$ where m is a finite but arbitrary integer. Since the alternatives are completely unspecified at this point we can label them in any arbitrary order we wish.

Each individual $h \in S$ will be assumed to have a complete, asymmetric and transitive preference ordering on the alternative set A.

Definition 1: An individual preference ordering ($>_h$) is a complete, asymmetric and transitive binary relation defined on the product set $A \times A$.

Denoting this preference ordering by $>_h$, we have the following properties.

(1) Completeness

$$\forall (a_i a_j) \in A \times A: \text{ either } a_i >_h a_j \text{ or } a_j >_h a_i \text{ but not both}$$

$$(i,j) = 1,2,\ldots,m;\ i \neq j$$

(2) Asymmetry

$$\forall (a_i a_j) \in A \times A : a_i >_h a_j \Rightarrow a_j \not>_h a_i$$

$$(i,j = 1,2,\ldots,m;\ i \neq j)$$

(3) Transitivity

$$\forall (a_i a_j) \text{ and } (a_j a_k) \in A \times A$$

$$(a_i >_h a_j \text{ and } a_j >_h a_k) \Rightarrow (a_i >_h a_k)$$

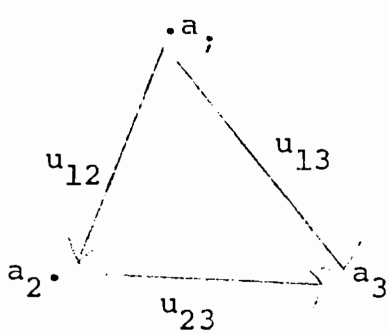
$$(i, j,k = 1,2,\ldots,m;\ i \neq j \neq k)$$

These preference orderings can also be viewed as complete, asymmetric and transitive graphs $G_h = (A,U)$ where A denotes the sets of nodes of the graph G_h and U denotes the set of arcs between all the pairs (a_i, a_j) $i,j=1,2,\ldots,m$; $i \neq j$. An alternative and equivalent way to deal with these graphs consists in using the "<u>associated matrix</u>" P_h for each such graph.

<u>Definition 2</u>: The matrix P_h associated with a graph G_h is a square $(m \times m)$ nonnegative matrix whose (ij)th entry is

$$p_h^{ij} = \begin{cases} 1 \text{ if and only if } u_{ij} \in U \Longleftrightarrow a_i >_h a_j \\ 0 \text{ if and only if } u_{ij} \notin U \Longleftrightarrow a_i \not>_h a_j \end{cases}$$

For instance, for a set of three alternatives (m=3), the hth individual preference ordering could be $a_1 >_h a_2 >_h a_3$ i.e.



$$P_h = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad (3.3)$$

The existence of a bijective mapping between the set of all such graphs G_h and the set of all square boolean matrices P_h enables us to pursue our discussion in terms of the P_h matrices, representing the various preference orderings $>_h$. Such an approach will prove to be both mathematically convenient and intuitively appealing; in effect it amounts to considering that each individual when confronted with the problem of strictly ordering many alternatives, proceeds sequentially by paired comparisons and thus <u>reveals his preferences indirectly</u> through his choice of a single alternative in each pair(*). This individual pairwise choice process is completely described by the associated matrices P_h(**).

Let us now examine these matrices P_h more closely. In the absence of any properties on the G_h graphs, it is clear that the class $\mathcal{C}$ of all such matrices consists of all the ($m \times m$) boolean matrices, i.e. all the vertices of the unit hypercube in an m^2 dimensional space; and there are exactly 2^{m^2} such vertices. However in the problem at hand the properties that we have assumed for the preference orderings $>_h$ and hence, also for the graphs G_h, allow us to restrict ourselves to a subclass $\mathcal{S} \subset \mathcal{C}$ and to cut down on the dimensionality requirements of our problem.

(i) In regard to the characterization of this subclass $\mathcal{S} \subset \mathcal{C}$ it follows directly from the completeness and asymmetry assumptions on G_h that:

$$\mathcal{S} = \{P_h \in \mathcal{C} \mid p^h_{ij} + p^h_{ji} = 1,\ \forall i,\ j=1,2,\ldots,m,\ i \neq j\}^{(***)}$$

If we now add the transitivity requirement, we further restrict

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(*) In the literature this indirect approach to preference revelation is known as the "<u>choice function</u>" concept. A priori it would seem that we would have to consider all conceivable subsets of A, i.e., the power set $\mathcal{P}(A)$ instead of just the two-elements subsets as we have done; however, under the assumptions of rational choice it is easy to show that the choice in any environment is uniquely determined by a knowledge of the choices in two-element environments. It is also clear that only a <u>single</u> alternative will be chosen by each individual in any two-element environment since the $>_h$ relation is a strict ordering. See [1], page 16 for a discussion of these two points.

(**) This description of the individual determination of a strict ordering through elementary pairwise comparisons seems to conform with the actual procedure followed by most individuals, as many psychological experiments have suggested.

(***) The elements of this subclass $\mathcal{S}$ are known in the mathematical literature as "tournament matrices."

ourselves to a proper subclass $\hat{\mathcal{P}} \subset \mathcal{P}$ whose elements $\hat{P}_h$ are characterized by the property that they can all be written in the simple form

$$\hat{P}_h = \begin{bmatrix} 0 & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & 0 & 1 & & & \vdots \\ \cdot & & \ddots & \ddots & & \vdots \\ \cdot & & & \ddots & \ddots & 1 \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 \end{bmatrix}$$

(ii) In regard to the dimensionality requirements of our problem is it apparent that some of the information in the matrices P_h can be discarded without any loss of generality. Specifically, because of the completeness and asymmetry assumptions, only $\binom{m}{2}$ entries p^h_{ij} are needed to characterize any ordering $>_h$. For instance if we have $a_2 >_h a_3 >_h a_1$ i.e.

$$P_h = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{tikzcd} & a_2 \arrow[dl, "u_{23}"'] \arrow[dr, "u_{21}"] & \\ a_3 \arrow[rr, "u_{31}"'] & & a_1 \end{tikzcd}$$

we can discard all the entries on or below the main diagonal and keep the upper triangular reduced matrix $P'_h = \begin{bmatrix} 0 & 0 \\ & 1 \end{bmatrix}$

Geometrically the set of all such upper triangular reduced matrices P'_h consists of the $2^{\binom{m}{2}}$ vertices of the $\binom{m}{2}$-dimensional unit hypercube in $\binom{m}{2}$-space. Adding the transitivity requirements amounts to ruling out some of the vertices of the unit hypercube. In the sequel these points P_h will be referred to as individual binary preference patterns.

<u>Definition 3</u>: An individual <u>binary preference pattern</u> is an $\binom{m}{2}$-dimensional boolean vector representing the actual choices made by a citizen $h \in S$ in a sequence of $\binom{m}{2}$ paired comparisons.

(*) This may entail a simultaneous relabeling of the rows and columns of P_h. For a proof of this result see [4].

In order to further our understanding of the geometry of the problem, these notions will now be illustrated in a three alternative case.

2.2 An illustration

Let $A = \{a_1\ a_2\ a_3\}$.

There are $\binom{3}{2} = 3$ paired comparisons. For instance let the order of these comparisons be

$$\begin{cases} a_1 \text{ vs } a_2 \\ a_1 \text{ vs } a_3 \\ a_2 \text{ vs } a_3 \end{cases}$$

Hence there are 2^3 a priori conceivable binary preference patterns, i.e. disregarding the transivity requirement for the time being. Writing these binary patterns P'_h directly in vector form instead of using the upper triangular matrices as previously, we have:

$$P'_1 = \begin{pmatrix}0\\0\\0\end{pmatrix};\quad P'_2 = \begin{pmatrix}1\\0\\0\end{pmatrix};\quad P'_3 = \begin{pmatrix}1\\1\\0\end{pmatrix};\quad P'_4 = \begin{pmatrix}0\\1\\0\end{pmatrix};\quad P'_5 = \begin{pmatrix}0\\1\\1\end{pmatrix};$$

$$P'_6 = \begin{pmatrix}0\\0\\1\end{pmatrix};\quad P'_7 = \begin{pmatrix}1\\0\\1\end{pmatrix};\quad P'_8 = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

Geometrically in order to generate the set of all such conceivable preference patterns, we have simply followed a Hamiltonian path through the 8 vertices of the unit cube in three dimensional space as the figure below indicates. (The choice of the point of departure (000) and arrival (111) is, of course, purely arbitrary.)

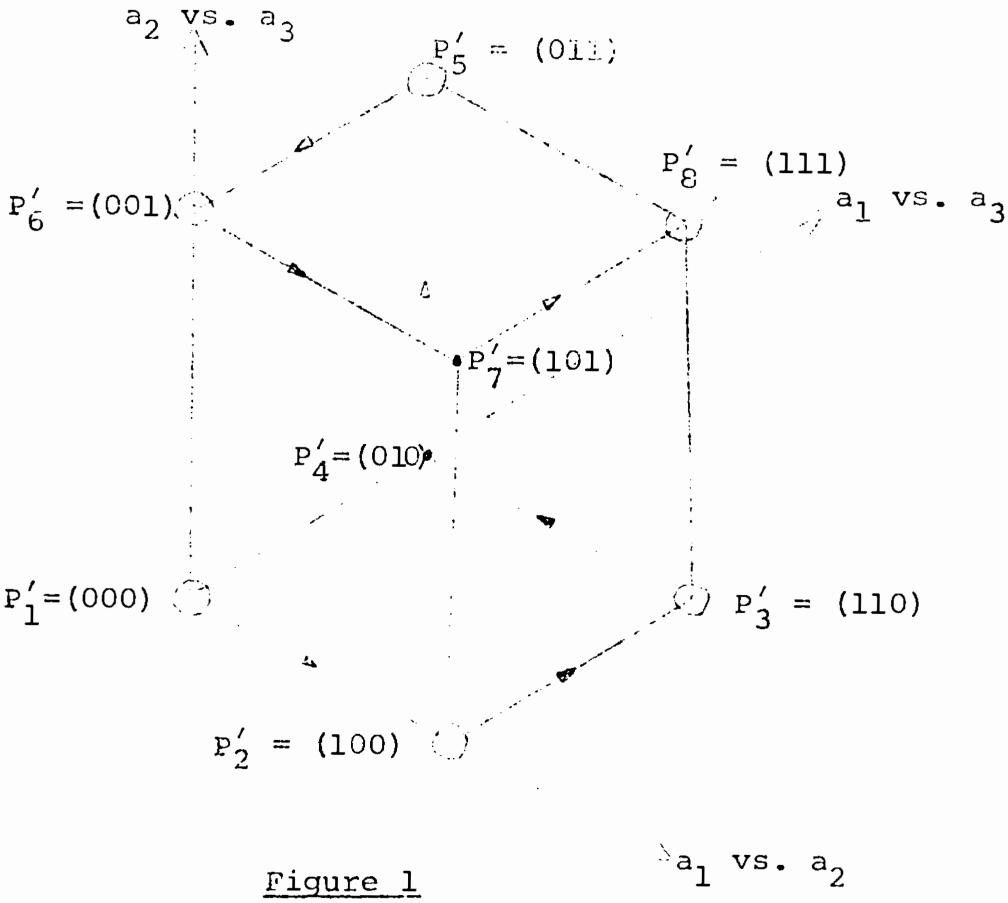

Figure 1

The existence of such a Hamiltonian path through the nodes $P_1' \to P_8'$ only illustrates the fact that one can move from any binary preference pattern to any other and "cover" the whole class $\mathscr{P}$ of binary patterns. These moves, of course, are effected in a discrete fashion since we do not allow individuals to express stochastic orderings. (If we did, then the whole hypercube, i.e. the interior and its boundary would form the space of conceivable preference patterns in such a society.(*)) However, a word of caution is in order at this point: since we restricted the individual preference orderings $>_h$ to be transitive, some of the binary preference patterns in the set $\mathscr{P} = \{P_1', P_2', \ldots, P_8'\}$

(*)Actually this line of thought has many ramifications, some of which have been investigated elsewhere. See [4].

must violate the transitivity property: more specifically two such patterns ($2^3 - 3! = 2$) must be intransitive viz. P'_4 and P'_7 in this case, since $P'_4 = a_2 > a_1 > a_3 > a_2$ and $P'_7 = a_1 > a_2 > a_3 > a_1$ (*). One very interesting point should also be noted: these two intransitive patterns are not adjacent vertices; as a matter of fact they are symmetric to each other with respect to the diagonal $P'_4 \; P'_7$ (**). This fact does not depend upon the order of the paired comparisons: this symmetry property is invariant under any possible permutation of the axes. For instance if we had compared a_1 vs. a_2, a_3 vs. a_2 and a_1 vs. a_3 in this order, the two intransitive patterns would have been (000) and (111) (***). It is thus seen that the set $\hat{\mathcal{P}}$ of "admissible" preference patterns--under our assumptions--is: $\hat{\mathcal{P}} = \mathcal{P} - \{P'_4 \; P'_7\}$. Let us now summarize these developments. The set $\hat{\mathcal{P}}$ of admissible individual preference pattern can be viewed as an $\binom{m}{2} \times m!$ boolean matrix (B), the rows of which represent the elementary paired evaluations (a_i vs. a_j) (for each admissible preference pattern), and the columns of which represent the admissible individual preference patterns. In our example this B matrix reads as follows:

(*) (**) The fact that only two such intransitive patterns occur simply corresponds to the fact that one can go through a 3-cycle in two directions only: clockwise and counterclockwise.

(***) An important implication of non adjacency of the intransitive patterns should also be mentioned: in order to separate the transitive patterns from the intransitive patterns we need more than two classes; we need two hyperplanes to rule out these points. This seemingly innocuous fact has far-reaching consequences in regard to the theory of constrained (here, transitive) aggregation. See [4], [5].

$$B = [P'_1\ P'_2\ P'_3\ P'_5\ P'_6\ P'_8]$$

or

$$\underset{(3.6)}{B} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{array}{l} \leftarrow (a_1 \text{ vs. } a_2) \\ \leftarrow (a_1 \text{ vs. } a_3) \\ \leftarrow (a_2 \text{ vs. } a_3) \end{array}$$

We should note at this point that: (1) the number of columns of any such B matrix is always even ($m! = m(m-1)\ldots\underline{2}.1$) and (2) in this examp.e each row of B has an equal number of 1's and 0's (3 in this case)--which means that among all admissible patterns, as far as any pair of alternatives is concerned, there are as many patterns that rank a_i over a_j as there are patterns that rank a_j over a_i for any $i \neq j$ and $i,j = 1,2,3$. This is also true, of course, for the case $m = 2$.

The following theorem will show that this property holds for any number ($m!$) of admissible preference patterns.

Theorem 1: Let $\mathcal{B}$ be the set of **all** $\binom{m}{2} \times m!$ boolean matrices obtained as **described above**. If $B \in \mathcal{B}$ then the **following** relation always holds:

$$B.1 = \frac{1}{2}\begin{pmatrix} m! \\ \cdot \\ \cdot \\ m! \end{pmatrix}$$

(where 1 denotes the $m!$-dimensional unit vector).

(A proof of a similar theorem has been provided in [6].

2.3. A statement of the constitutional choice problem

In its most general form the choice of a constitution for any society S composed of ℓ citizens can be viewed as the problem of reaching a unanimous agreement on a mapping $f \in \mathcal{F}$, where $\mathcal{F}$ denotes the space of all mappings from the set $\mathcal{M}$ of all $\binom{m}{2} \times \ell$ boolean matrices (each column of which represent an admissible individual binary preference pattern) to the set $\mathcal{V}$ of all $\binom{m}{2}$-dimensional boolean vectors (the social binary preference patterns) Formally, we can now state:

Definition 4: A constitution is a triple $[S;A;f] \in \{S \times A \times \mathcal{F}\}$ where $\mathcal{F} = \{f | f: \mathcal{M} \rightarrow \mathcal{V}\}$. The elements of $\mathcal{M}$ can be interpreted as a proper subset of the vertices of the unit $\binom{m}{2} \times \ell$-dimensional hypercube and the elements of $\mathcal{V}$ consist of the vertices of the unit $\binom{m}{2}$-dimensional hypercube. In the definition of a constitution as given above, it should be noted that we require a specification of more than just the aggregation mapping f: we also require a statement of who qualifies as a citizen--i.e. who belongs to S, and which matters are within the realm of public decisions-making--i.e. what does the set A of collective issues consist of. The reasons for including these two specifications as an integral part of a "constitution" are two-fold: (i) on the one hand it is clear that, even before reaching a unanimous agreement on some aggregation mapping $f \in \mathcal{F}$, we must, first, agree on whose preferences are to enter in the domain $\mathcal{M}$ (this is, in fact, a prior problem to the one of deciding whether or not f will be such as to make these preferences included in $\mathcal{M}$ effectively count.)

(ii) On the other hand, it seems reasonable to require that the constitution provide a clear-cut way of deciding which "issues" are to be called public and thus to fall under the jurisdiction of the decision-maker f. Furthermore, as we said earlier, it is clear that, at the constitutional choice stage, the aggregation rule to be followed is that of unanimity; anything short of unanimity appears arbitrary at best, whereas if a unanimity of citizens agree on a constitution, the aggregation problem is solved ipso facto. As most constitutional theoreticians since Rousseau have argued, anybody who is considering entering such a "social contract" should be allowed to block any decision he objects to; after all if he is to give up freely some of his future freedom of choice over public issues then he must be sure that (1) no decision will be reached without his consent and (2) no strategy of coalition formation within the constitutional assembly will be beneficial to him or to anybody else--unless of course the coalition consists of the whole society S. At this point one might be tempted to conclude that the only constitution that would be unanimously approved would be the unanimity rule itself. But the problem is, that such a rule would block any social decision to be reached at all except the most trivial ones--when no disagreement exists whatsoever. Now, presumably, the whole point of agreeing upon a constitution a priori without any knowledge of the issues or, at least, of the various preference patterns on the issues--and their respective strengths--is to avoid a stalemate and the resulting inaction that would ensue. Furthermore, if the status quo is properly included among the alternatives--as it should be--then under the unanimity rule, whenever there is the slightest disagreement, the whole society would break down and the various protagonists of the conflict would have to resort to "meta-constitutional" means to enforce their views. This, of course, may occur: revolutions are one of the few "invariants" in history. But their cost is usually high: as one of the most conspicuous forms of public good, they are eminently indivisible and the price to be paid is usually prohibitive even for their would-be beneficiaries--as Condorcet himself would have attested! The essence of the constitutional choice dilemna is to find "cheaper" methods of conflict resolution, provided that such methods look "unbiased" enough i.e. they do not favor, a priori, any particular group in the society--to warrant their unanimous adoption. This last statement seems to point out, very precisely in which direction our search for a constitutional rule should go: if we visualize how a representative citizen faced with this kind of constitutional choice ought to behave--if he is to be rational--it is clear that he is taking a gamble in the sense that once a rule has been adopted whenever a social choice will be made he may turn out either to "win" in the sense that his preferences will turn out to be society's preference or to lose if he is the underdog on the issue at stake. Whether he happens to win or to lose in a particular contest, in both cases the consequences of the outcome can be represented by a payoff of some kind, positive if he wins and negative if he loses (for instance "subsidies" or "benefits" he would receive in the former case and taxes he would have to pay in the latter). Furthermore we could convert these payoffs into utilities but, for the sake of simplicity, if such an interpretation is chosen, we shall assume that the individual is neutral towards risk so that the monetary payoffs can be used directly as proxies for their corresponding utilities. Assuming a rational expected utility maximizer, the formal problem faced by this citizen is that of choosing a constitution which is "fair" in the sense of a "fair" gamble. Because of the basic uncertainty involved in this choice it is the "best" policy for any one individual having to make this choice; and, furthermore, this is the only "unbiased" policy--which thus warrants its unanimous adoption. To illustrate further, these rather intuitive arguments seem to indicate that a completely randomized decision rule is called for: for instance, on each paired comparisons the constitution could state that the "social" choice will be made by flipping a fair coin. This rather startling rule may appear quite impersonal and totally irresponsive to the many spectra of individual opinions that could arise at any time in the future. And even though it would possess the *a priori* unbiasedness necessary for its unanimous adoption, it is not hard to imagine that the members of the constitutional assembly would want some rule more directly linked to the individual preference patterns--whatever these might be. The result we are about to discuss shows that under certain very reasonable conditions, the simple majority rule constitutes such a completely randomized decision rule!

Section 3: Randomization, Fairness And Majority Voting Rules

The discussion in this section will proceed as follows: we will first introduce random elements into the problem through the concept of a "culture"; this will then be used to show that simple majority voting can be viewed as a randomized decision rule; as a natural corollary, it will then be seen that under certain symmetry assumptions the choice of the simple majority voting rule ensures that the constitutional gamble is a fair gamble. And finally various extensions and generalizations of this result will be provided.

3.1. Win and loss probabilities and the concept of a "culture"

We have previously made use of the expression "probability of winning" (or losing) in the constitutional choice gamble. Clearly for such an expression to make sense we must first assign a probability measure to the set $\hat{\mathscr{P}} = \{P_k\}$ $(k = 1,2,...m!)$ of $\binom{m}{2}$-dimensional admissible individual preference patterns--i.e. a subset of the vertices of the $\binom{m}{2}$-dimensional hypercube as shown above (see Figure 1 and 2.2)

Definition 5: A culture is a probability measure Γ on the set $\hat{\mathscr{P}}$ of admissible preference patterns.

This is where the fundamental uncertainty of the constitutional choice problem comes into play. At this stage the individual citizen can be reasonably sure of only two things: (1) the fact that public decisions will some day have to be faced and (2) that, except in trivial cases when everyone will agree, it is to be expected that divergent opinions will co-exist over the whole planning horizon where the rule will apply. Beyond this rather meager information, the individual citizen is totally unable to predict how individual opinions will be distributed over a given set of issues--even if he can predict the general nature of those issues. We could first a priori argue that, from the principle of insufficient reason, (Laplace's Law of Equal Ignorance) each admissible pattern P_k' is equally likely. Another possibility is to start from the information-theoretic notion of uncertainty and its measure. This concept of an uncertainty measure, as suggested by Shannon and Wiener, is known as the <u>entropy function</u>. For a discrete probability space the entropy function $H(P_k')$ is defined by:

$$H(P_k') = -\sum_{k=1}^{m!} \gamma_k(P_k') \cdot \mathrm{Ln}\, \gamma_k(P_k')$$

It is a simple matter to show that for this entropy function H to be maximized under the constraint that $\sum_{k=1}^{m!} \gamma_k = 1$, we must choose a set of <u>equal</u> individual probabilities: $\gamma_k = \frac{1}{m!}$ for k=1,2,...m!

<u>Definition</u> 6:[(*)] An "impartial culture" Γ_{imp} is a culture Γ where all preference patterns are equally likely:

$$\Gamma_{imp} = \{\Gamma | \gamma_k = \tfrac{1}{m!}\} \; k = 1,2,\ldots m!\}$$

For a given impartial culture Γ_{imp}, consider now the expected value of the random variable P_k'. It follows directly from Theorem 1 above that $E(P_k') = \begin{pmatrix} \frac{1}{2} \\ \vdots \\ \frac{1}{2} \end{pmatrix}$.

<u>Corollary 1</u>: Under the impartial culture assumption, the random variable P_k' is such that:

(*) The concepts of culture and impartial culture were first introduced by Garman and Kamien. See [9]

$$E(P_k') = \begin{bmatrix} \frac{1}{2} \\ \vdots \\ \frac{1}{2} \end{bmatrix}$$

<u>Proof</u>: From Theorem 1 we know that

(1) $$\frac{(B.1)}{\binom{m}{2} \times m!} = \begin{bmatrix} \frac{m!}{2} \\ \vdots \\ \frac{m!}{2} \end{bmatrix}$$

The impartial culture assumption states

(2) $$\gamma_k = \gamma_g = \frac{1}{m!} \qquad \forall k,g = 1,2,\ldots m! \quad k \neq g$$

Whence

(3) $$E(P_k') = \begin{bmatrix} \frac{1}{2} \\ \vdots \\ \frac{1}{2} \end{bmatrix}$$

The geometrical interpretation of this result is quite simple. Using the same illustration as in 2.2 above (m=3) we see that the expected social preference pattern $E(P_k')$ is the center of gravity of the cube $\mathscr{S}$ representing all conceivable individual preference pattern. (See Figure 2, below)

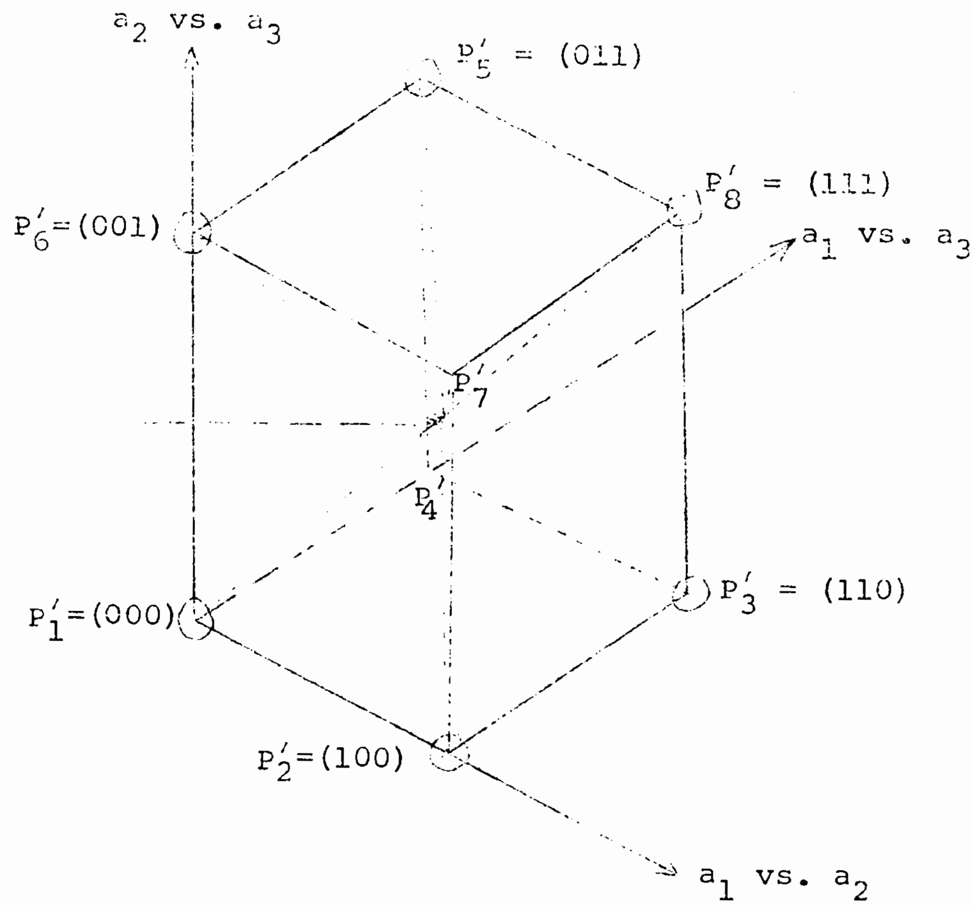

Figure 2

3.2. Majority voting as a randomized decision rule

Actually although this result provides a very useful insight into the problem, there exists another way to look at it which will bring out even more clearly the randomized nature of majority voting as a collective decision rule. This can be stated as a simple corollary to Theorem 1 above.

Corollary 2: In an impartial culture Γ_{imp} and under the simple majority voting rule, alternative i is as likely to win over alternative j as it is to lose, for any $i \neq j$ and $i,j = 1,2...\binom{m}{2}$. This is such a simple direct consequence of the above theorem that we only need sketch the proof.

<u>Proof</u>: Consider any row (a_i vs a_j) ($i \neq j$) of the B matrix as defined previously. Such a row consists of two kinds of possible individual opinions over this paired comparison: 1 if $a_i >_h a_j$ or 0 if $a_j >_h a_i$ and from theorem 1 we know that there is an equal number of 1's and 0's in any row of B.

(i) If ℓ, the number of voters is even:

(1) $\ell = 2q$

and, from theorem 1, simple majority voting leads to a tie since q voters are distributed in each of the two groups under the impartial culture assumption

(ii) If ℓ is odd

(2) $\ell = 2q + 1$

and the last $(2q + 1)$th individual is also as likely to belong to any one of the two classes under our assumptions.

Q.E.D.

In the light of this result it is now clear that simple majority voting on paired issues in an impartial culture amounts to a completely randomized decision rule: "winning" or "losing" are equally likely ($\frac{1}{2}$) as far as each individual is concerned. Flipping a coin would have been as effective as voting, except, of course, that it is doubtful that many members of the consitutional assembly would be so lucid in their decision as to realize that this fact would allow them to cut down even more drastically on the cost of collective decision-making. For one thing, of course, the impartial culture assumption may not always be met in all the actual applications of the constitution, over the whole planning horizon.

And at any rate, even though the flipping of a (fair) coin is obviously a much cheaper and equally "unbiased" alternative to majority voting in such societies, its apparent total disregard for individual opinions per se would no doubt be condemned by the firm believers in Jeffersonian democracy as a clear mark of disrespect towards the democratic creed and a fatal blow to the cause of constitutional decency!

3.3. Simple majority voting as a "fair" decision rule

We stated earlier that any one individual in the constitutional assembly trying to evaluate the consequences of the choice of any rule could not help recognizing the fact that, short of an infinite dictatorship in his favor, on any pair of issues two outcomes were conceivable: a "win" or a "loss" and either would entail a payoff of some kind for him.

An exact specification of these payoffs is obviously rendered difficult by the basic indeterminacy of the issues and of the spectrum of individual opinions on them. Again, in this situation, the only sure thing is that "winning" will be more rewarded in terms of individual payoffs than losing. To simplify our discussion let us consider that the payoffs to the representative citizen are symmetric. More specifically for any arbitrary pairs of alternatives (a_i vs a_j) let $\pi_{ij}^{w_h}$ be the positive payoff associated with a "win" on (ij) for any individual $h \in S$; then the negative payoff associated with a loss on (ij) for the same individual h will be

$$\pi_{ij}^{l_h} = -\pi_{ij}^{w_h}$$

In general, then, for any set of $\binom{m}{2}$ paired comparisons of the m alternatives, to any individual h, there is associated an $\binom{m}{2} \times 2$ payoff matrix $[\pi^h] = [\pi^{w_h} \; \pi^{l_h}]$ where the two vectors π^{w_h} and π^{l_h} are symmetric to each other with respect to the origin in $\mathbb{R}^{\binom{m}{2}}$ space. Needless to say it may well be the case that the expected payoff matrix $[\pi^h]$ imagined by an individual h differs from that of another individual $[\pi^g]$, but as far as any one individual $h \in S$ is concerned, when making his decision as to whether to accept or reject a proposed constitution, the single important assumption is that of symmetry which itself is justified on the basis of total uncertainty--the only thing which is certain, being the knowledge that an external cost will be borne by the loser and symmetrically the winner will enjoy an external benefit.

With this added assumption we can now state and prove an additional property of the majority voting rule.

Corollary 3: In an impartial culture and under the assumption of symmetric individual payoffs the choice of the simple majority voting rule makes the constitutional gamble a fair gamble for any one individual $h \in S$.

Proof: This follows directly from Theorem 1 and Corollary 2 above since in an impartial culture, for any number of citizens even or odd a win or a loss are equally likely on any pair of issues. Hence:

$$E([\pi^h]) = [\pi^{w_h}, \pi^{l_h}] \cdot (\tfrac{1}{2}, \tfrac{1}{2})^{tr} = [0]$$

3.4. Generalized majority voting, "important issues" and "biased" cultures

3.4.1. If we reflect upon the previous results we can readily see why the uncertainty aspect in constitutional choice is so crucial. Although this point has long been recognized in the literature, the above developments aim at providing a formal framework for such an analysis. It is also clear that several generalizations can now be outlined. As we said earlier the general validity of the impartial culture assumption is an empirical question outside the scope of this paper. At any rate, and without rejecting the basic uncertainty that characterizes the constitutional choice problem, it appears interesting to reflect for a while upon the sensitivity of these results to specific changes in the environment faced by the constitutional decision-maker. More specifically, two factors can separately or jointly alter the environment: for one thing, it may well be the case that one (or several) issue(s) on the set of alternatives to be judged, will turn out to be much more crucial than the others; such a forecast may be viewed, in a sense, as an extra safeguard against any unexpected occurence. On the other hand, some "biased cultures" may come into existence as a result of social life which acts in such a way as to create behavioral similarities among individuals; in fact, this integrative role is what characterizes a society in the first place and would be reflected in the distribution of individual preference patterns.[(*)] This possibility is no doubt worth investigating but in this paper we shall restrict ourselves to a study of the first-type of alteration (existence of an "important issue") in the environment.

[(*)] All experimental evidence seems to point in this direction. (See Coombs [7])

Furthermore, it will become clear that the same analysis would also apply to the second type of alteration ("biased cultures").(*)

3.4.2. We shall now examine how the existence of important issues can be accounted for in our model. Basically if some alternative in the set A is expected to be an "important question," this will be revealed by the individual payoff matrix $[\pi^h]$. One simple way of expressing this fact would consist in admitting asymmetrical payoff entries in the payoff matrix $[\pi^h]$.

Let $i \in A$ be such an "important issue" and let the $(m-1)$ rows of $[\pi]$ where i is involved be such that: $\pi_{ik}^{l_h} = - \lambda \, \pi_{ik}^{w_h}$ where λ is a positive scalar in the real field ($\lambda \in \,]0, +\infty[$). The problem of finding a "fair" decision rule can now be stated:

Find $p_1, p_2 \in \,]0, 1[$

such that $p_1 + p_2 = 1$

$$E([\pi^h]) = p_1 \, \pi_{ik}^{w_h} + p_2 \, \pi_{ik}^{l_h} = 0$$

i.e. $$p_1 \, \pi_{ik}^{w_h} - \lambda (1-p_1) \, \pi_{ik}^{w_h} = 0$$

The solution is $\dfrac{p_1}{(1-p_1)} = \lambda$

which expresses p_1 as an implicit function of λ. As an illustration, if $\lambda = 2$ we find $p_1 = \frac{2}{3}$ --which is nothing else but the well-known $\frac{2}{3}$ majority rule as adopted, for instance, by the U.N.

(*) One should also keep in mind that, logically speaking, no society is supposed to have existed prior to the choice of a constitution: stricto sensu, no matter how much integration is achieved by social life, this only occurs <u>after</u> a society has been formed and grown for some time--i.e. <u>after</u> a constitution of some sort has been adopted.

Charter for the so-called "important questions."(*) The case $\lambda = 1$ (symmetric payoffs) yields, of course, the rule $p_1 = \frac{1}{2}$ i.e. simple majority as we discussed previously. More generally we have:

$$\forall \lambda \in]0, 1[\Rightarrow p_1 \in]0; \tfrac{1}{2}[$$

$$\forall \lambda \in]1, +\infty[\Rightarrow p_1 \in]\tfrac{1}{2}; 1[$$

As $\lambda \to \infty$ we approach the unanimity rule asymptotically--which only reflects the fact that the more disproportionate the payoffs (as between a "win" and a "loss") the larger majority one should require. In this way it appears that a simple scaling operation for the payoffs leads to any conceivable decision rule in the interval $]0, 1[$.

(*) In the U.N. Charter, however, there is an interesting loophole which can deprive the two-third majority safeguard of some--possibly all--of its substance: According to Chapter IV, article 18 of the Charter "decisions of the General Assembly on important questions shall be made by a two-third majority." The Charter then proceeds to enumerate such important questions but this list is not meant to be exhaustive since "decisions on other questions, including <u>the determination of additional categories of questions to be decided</u> by a two-third majority, shall be made by a [simple] majority of the members present and voting." Because of this rule, however, questions that, put in proper perspective, could rightly be viewed as "important" may be decided upon by a simple majority just because a simple majority is required <u>to determine whether or not</u> they should be treated as "important."

As a matter of fact, it should be clear from our model that, for this two-third majority rule to be a truly effective and meaningful protection of "minority rights," would <u>symmetrically</u> require that the decision, as to which majority applies, be reached by as little as a 1/3 majority, say.

4.2.3. As we said earlier another way to extend our results would involve the use of exogenous information on the issues and the citizens' opinion: if some opinion patterns are more likely than others i.e. if we face a biased culture, then the win and loss probabilities are no longer equal, even if the payoffs are assumed exactly symmetric. To make the constitutional gamble "fair" we would be led to use a system of head taxes and subsidies for the individual citizens in order to remedy the possible positive or negative bias that would result from simple majority voting.(*) This would lead to "shadow market" pricing decisions for the public issues at stake. Such a line of research would take us too far and will be pursued elsewhere. At any rate the crucial conclusion of the above discussion remains: under the assumptions of this model, any majority voting rule can be generated.

4. Conclusion

At the close of this discussion, one might be tempted to view these results as a rehabilitation of majority rule. In fact they only provide an a priori/possible theoretical justification for a historical phenomenon; hopefully they go some way toward explaining why one could "rationally" choose some majority rule at the constitutional choice stage, in the face of total or partial uncertainty about the future issues and the shades of

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(*) Such a system is, in fact, an alternative to the qualified majority rules proposed before. Whether we face asymmetrical payoffs or a "biased culture" we have a choice of "instruments" to be used to generate a fair gamble: either by manipulating the win (and loss) probabilities p_1 (and p_2) or by manipulating the payoffs $_{\pi}w_{h}{}_{ik}$ (and $_{\pi}l_{h}{}_{ik}$).

opinions over those issues. But of course intransitive social preference patterns may arise under such majority rules, as we have illustrated in section 3 above. We have pointed out how and why such intransitivities may (and will) actually occur in practice; and, after all, this should surprise nobody: the rule was specifically chosen in the context of binary choices and its repetition is not necessarily the best way to make ternary (or n-ary) choices. In a binary choice no voting paradox occurs for the simple reason that there are but two groups of citizens forming a partition of S, viz:

$$S^+_{a_{ik}} = \{h \mid a_i >_h a_k \text{ for } h = 1,2,\ldots \ell\}$$

$$S^-_{a_{ik}} = \{h' \mid a_k >_{h'} a_i \text{ for } h' = 1,2,\ldots \ell\}$$

and $S^+ \cup S^- = S$; $S^+ \cap S = \emptyset$.

In words, all those individuals who show a preference for a_i over a_k show a dual preference against a_k over a_i. But as soon as we consider a triple of alternatives e.g. (a_i, a_k, a_j) then social intransitivities may arise because of the mere fact that the group of citizens who, on the one hand, prefer a_i over a_k is actually heterogeneous since it includes citizens who, on the other hand, prefer a_k over a_j, but also others who prefer a_j over a_k. In losing the homogeneity of the (S^+, S^-) partition of S we have lost the only universal safeguard we had against intransitivities! In his words: "It is at least a possibility, to which attention should be drawn, that the paradox to be discussed below might be resolved by such a broader concept of rationality"? It is to be hoped that the model we have just proposed, will be viewed as a step in this direction.

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