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Prudence Versus Sophistication in Voting Strategy

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INTRODUCTION

Since voting schemes do induce strategical manipulation (by the Gibbard-Satterthwaite result) a natural but obscure question is -- what kind of manipulation? To answer this question amounts to describing the strategical behaviour of the agents, that is to select an equilibrium concept for games in strategical form. Most of the literature devoted to this problem (see [1], [5], [6], [8], [11]) has dealt with non-cooperative behaviour of the agents. This is because the major concern has been the implementation problem which is stated as follows: Given any social choice function (or correspondence) that is, a particular collectice decision making, is it possible to distribute privately the decision power among the agents in such a way that by exercising (non-cooperatively) this power, the agents eventually select the very outcome(s) recommended by the social choice function (correspondence). Any answer to this problem throws some light on the collective implications of non-cooperative behaviour, a question underlying most of the economic literature on collective decision mechanisms (see [4] and [3] for a survey of the literature).

In this paper we assume that the agents behave non-cooperatively. This assumption alone is not enough to determine unambiguously the outcome elected by the agents. Given a particular voting scheme and a particular preference profile of the agents, there are in general several Nash equilibriums in the corresponding game and accordingly several different outcomes are the possible result of an equilibrium vote of the agents. 1/ In order to make the behaviour of the agents well-defined we have to make specific assumptions about the information that the agents have on each other's preferences. In this paper we investigate the consequences of two extreme assumptions. At one extreme we assume that every agent has no information about any other agents' preference. This leads him to a prudent behaviour of the maximin type. At the other extreme we assume that every agent has full information about the preference profile; then, non-cooperation results in the sophisticated behaviour where the agents mutually anticipate their strategy by successively eliminating dominated strategies [see [2], [8], [9]).

To assume that everybody knows everybody elses utility does not avoid the implementation problem. The central legislator does not know the profile (at least when he must a priori choose a voting scheme that works for every profile); therefore in a world where no coercive device can prevent the individual agents from lying 2/ decentralization must cope with the traditional incentive compatibility requirement, even if information about utility profile is complete.

The paper is organized as follows. In Section 1 we define prudent and sophisticated voting behaviour and the corresponding implementation of social choice functions (correspondences). In Section 2 we illustrate these concepts on a particular class of voting schemes, namely voting by veto, in which we compare the prudent and sophisticated voting behaviours. The results stated

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1/ Actually Maskin did prove in [7] that only dictatorial voting schemes and voting schemes among two alternatives are such that for every profile the same outcome results from every Nash equilibrium of the associated game.

2/ If the freedom-of-speech principle prevails then lying becomes even an individual right.

in Section 2 are preliminary results for the next section. The main result is presented in Section 3. Namely, we exhibit a voting scheme, shortly described as voting by alternating veto, where both the prudent and the sophisticated voting of the agents yield the selection of the same alternative. The dictatorial voting schemes also share the property that the sophisticated voting and the prudent voting can not be distinguished. Thus, our result is worthwhile because voting by alternating veto in general is nearly anonymous (symmetrical among the players) and actually implements an exactly anonymous social choice function in some significant cases. These highly remarkable strategical properties of voting by veto makes from this familiar procedure a decision mechanism that can be defended on strong theoretical grounds, (see also reference [10]). Finally Section 4 is devoted to some open problems and concluding remarks.

1. PRUDENT AND SOPHISTICATED VOTING

Throughout the paper we will consider a collective decision problem of the following form: the set A of alternatives is finite with cardinality p. Each of the n agents has a preference ordering on A (we assume that no agent is indifferent between any two alternatives; this assumption is of crucial technical importance). Let U be the set of strict ordering on A. For the sake of convenience we denote the current element u of U as a utility function -- actually a one-to-one mapping from A into $\mathbb{R}$. All concepts and results will be purely ordinal.

A profile u is an n-tuple $u = (u_1,\ldots,u_n) \in U^n$ specifying the particular utility function of each agent.

Definition 1

An n-person voting scheme among A is a $(n+1)$-tuple $\mathcal{V} = (X_1,\ldots,X_n,\pi)$ where the strategy space, $X_1,\ldots,X_n$ are finite and π is a mapping from $X_1 \times\ldots\times X_n$ into A.

Definition 2

To every profile $u \in U^n$, the n-person voting scheme $\mathcal{V}$ among A associates the game in strategic form.

$$\mathcal{V}(u) = (X_1,\ldots X_n;\ u_1 \circ \pi,\ldots,u_n \circ \pi)$$

In game $\mathcal{V}(u)$ we define now the prodent and sophisticated behaviour of the agents.

Notation

If $z = (z_1, z_K)$ is a vector of $\mathbb{R}^K$ we denote by $\theta(z)$ the vector obtained by reordering the coordinates of z in increasing order:

$$\theta(z) = (z_{\sigma(1)},\ldots,z_{\sigma(K)}) \text{ where } \sigma \text{ is a permutation of}$$

$$\{1,\ldots,K\} \text{ and } z_{\sigma(1)} \leq z_{\sigma(2)} \leq \cdots \leq z_{\sigma(K)}.$$

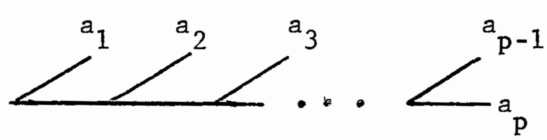
Definition 3

Let $\mathcal{V}$ be an n-person voting scheme among A and $u \in U^n$ be a particular profile. In the associated game $\mathcal{V}(u)$ we define the set $P_i(u_i)$ of prudent strategies of agent i as the set of those $x_i \in X_i$ which lexicographically maximize over X_i the vector $\theta\{u_i(x_i)\} = \theta\{u_i(x_i, x_{\hat{i}})_{x_{\hat{i}} \in X_{\hat{i}}}\}$

In particular a prudent strategy $x_i \in P_i(u_i)$ is a maximin strategy of player i.

$$\min_{x_{\hat{i}} \in X_{\hat{i}}} u_i(x_i, x_{\hat{i}}) = \max_{y_i \in X_i} \min_{x_{\hat{i}} \in X_{\hat{i}}} u_i(y_i, x_{\hat{i}})$$

The prudent behaviour of an agent is a decentralized one which is relevant when this agent has no information at all about the other agent's utility and consequently about their strategical choice. In many familiar voting schemes, like the plurality voting and the Borda count, the prudent behaviour is nothing else but the sincere voting (to announce his true peak alternative in plurality voting or his true preference ordering in the Borda procedure. The proof of these claims is left as an elementary exercise to the reader). In some other usual voting, like voting by binary choice, the notion of sincerity would not be enough to determine the prudent behaviour. Consider for instance the voting by successive amendments:



By majority voting the agents decide first to elect a_1 or to reject it. If a_1 is rejected then the election of a_2 is proposed. And so on... At $(p-1)^{th}$ round of voting (if any) the agents select, again by majority voting a_p or a_{p-1}. Clearly the prudent strategy of agent i requires him to vote for a_k at the k^{th} round unless he prefers a_k less than every following alternative $a_{k+1}, \ldots, a_p$. This behaviour very much favors the first ranked alternatives! On the other hand, we must be aware that there are some voting schemes where the agents have several prudent strategies that lead to the election of different alternatives. Consider for instance the following voting by veto procedure:

Player 1 vetoes first one alternative, next player 2 vetoes one of the remaining alternatives, next player 1 vetoes again one of the remaining alternatives, next player 3 is a dictator to select one of the remaining alternatives.

In any of his prudent strategies, player 1 must decide to eliminate for sure his two least preferred alternatives. But he has two prudent strategies: one is to eliminate first his least preferred alternatives and next -- after player 2's move -- his least preferred among the $(p-2)$ remaining alternatives. The other prudent strategy is to eliminate first the alternative he ranks $(p-1)$ and next, after player 2's move, his least preferred among the $(p-2)$ remaining alternatives. One checks that the corresponding vectors $\theta(u_1(x_1))$ coincide, so that player 1 can not distinguish these strategies in terms of prudence. However, these two strategies may clearly yield the election of distinct alternatives.

Definition 4

Let $\mathscr{V}$ be an n-person voting scheme among A. We will say that $\mathscr{V}$ p - implements (p stands for prudence) the following social choice correspondence SP

$$U^n \ni u \longrightarrow SP(u) = \pi(P_1(u_1) \times \ldots \times P_n(u_n)) \subset A$$

We turn now to the sophisticated behaviour of the agents.

Notation

Let $\mathscr{V} = (X_1,\ldots,X_n;\pi)$ be an n-person voting scheme among A and let $u = (u_1,\ldots,u_n) \in U^n$ be a fixed preference profile. The associated successive elimination of dominated strategies if the decreasing sequence

$(X_1^t, \ldots X_n^t)$ defined inductively as follows:

$$X_1^o = X_1, \ldots, X_n^o = X_n$$

for every $t = 0,1,2,\ldots$ and every $i \in \{1,\ldots,n\}, X_i^{t+1}$ is the set of undominated strategies of agent i in the normal form game $(X_1^t, \ldots, X_n^t; u_1 \circ \pi, \ldots, u_n \circ \pi)$. That is to say

$$X_i^{t+1} = \{x_i \in X_i^t / \nexists y_i \in X_i^t \left\{ \begin{array}{l} \forall x_{\hat{i}} \in X_{\hat{i}}^t \; u_i \circ \pi(x_i, x_{\hat{i}}) \leq u_i \circ \pi(y_i, x_{\hat{i}}) \\ \\ \exists \; x_{\hat{i}} \in X_{\hat{i}}^t u_i \circ \pi(x_i, x_{\hat{i}}) < u_i \circ \pi(y_i, x_{\hat{i}}) \end{array} \right\}$$

By definition, the sequence $(X_i^t)_{t=1,2,\ldots}$ is decreasing for every i, hence by the finitness of X_i the sequence $(X_1^t, \ldots, X_n^t)$ is stationary for t large enough.

Definition 5

We say that $\mathcal{V}$ is dominant solvable if for every profile $u = (u_1, \ldots, u_n) \in U^n$ the associated successive elimination of dominated strategies satisfies:

(1) for some t, $\pi(X_i^t \times \ldots \times X_n^t)$ contains a single alternative.

In this case we will say that X_i^t is the set of sophisticated strategies of player i and denote it by $SO_i(u)$. Moreover we say that $\mathcal{V}$ d-implements (where d stands for dominance) the following social choice function SD

$$U^n \ni u \longrightarrow SD(u) = \pi(SO_1(u) \times \ldots \times SO_n(u)) \in A$$

Let us briefly comment on the concept of dominance-solvable voting schemes (introduced first by Farqharson [2] and next studied in [8], [9]). This concept is intended to describe the voting behaviour of the agent being both completely informed of the utility profile and unable to cooperate in any way. Therefore, they mutually anticipate their behaviour by successively eliminating dominated strategies. The set $SO_1(u) \times \ldots \times SO_n(u)$ of sophisticated n-tuples of strategies (if it exists, that is if (1) holds true) is a subset of the Nash equilibrium set and contains every equilibrium made up of dominating strategies (if any). Roughly speaking, the alternative selected by sophisticated voting of the players is a single-valued selection among the alternative selected by the various Nash equilibriums.

The class of dominance-solvable voting schemes is large. Essentially every voting scheme defined as a game in extensive form with perfect information is d-solvable, and the sophisticated voting is simply the perfect equilibrium of the game. Thus voting by binary choice, voting by veto, and the king-maker procedure are d-solvable, whereas plurality voting or the Borda procedure are not. Given a particular d-solvable voting scheme, it p-implements a social choice correspondence SP and it d-implements a social choice function SD. To compare SD and SP amounts to comparing the sophisticated and prudent behaviour of the agents. This will throw some light on the kind of manipulation involved in a particular voting scheme. In Section 2 we describe SD and SP. in the particular class of voting by veto procedure where these two social choice functions are different and in some sense symmetrical. Next in Section 3 we concentrate on voting schemes where the social choice function SD is a selection of the social choice correspondence SP; that is to say where the sophisticated voting can not be distinguished from a prudent behaviour.

2 SOPHISTICATED AND PRUDENT VOTING BY VETO

Definition 6.

Let $p_1,\ldots,p_n$ be non negative integers such that $p_1+\ldots+p_n = p - 1$ (remember that p is the cardinality of A). We denote by $\mathcal{V}[p_1,\ldots,p_n]$ the n-person voting scheme among A defined as follows:

- First player 1 vetoes any p_1 alternatives he wishes among A. Denote $A_1 \subset A$ the set of alternatives that he vetoes.

- Next player 2 vetoes any p_2 alternatives among $A \setminus A_1$. Denote A_2 the set of alternatives he vetoes.

- Then player 3 vetoes any p_3 alternatives among $A \setminus \{A_1 \cup A_2\}$.

And so on. If we denote by A_k the p_k alternatives vetoed by player k then the selected alternative is the single element of $A \setminus \{A_1 \cup \ldots \cup A_n\}$.

Lemma 1.

In the voting scheme $\mathcal{V}[p_1,\ldots,p_n]$ the prudent behavior of the players selects an unambiguous alternative: in other words, the social choice correspondence $SP[p_1,\ldots,p_n]$ that this voting scheme p-implements actually is single valued. To prove Lemma 1 it suffices to describe the prudent strategies of the game $\mathcal{V}[p_1,\ldots,p_n](u)$ for every profile $u \in U^n$.

Clearly player 1's unique prudent strategy is to veto his p_1 least preferred alternatives: it is actually his unique maximin strategy. Similarly, player 2's unique prudent strategy is to veto his p_2 least preferred alternatives among the $(p-p_1)$ remaining ones.

Thus the algorithm describing $SP[p_1,\ldots,p_n]$ is as follows:

Let u be given and define inductively the subsets $A_1,\ldots,A_n$ of A by:

$$(2) \quad \begin{cases} A_1 = \text{the } p_1 \text{ least preferred alternatives of } u_1 \text{ among } A, \\ A_k = \text{the } p_k \text{ least preferred alternatives of } u_k \text{ among} \\ \qquad A \setminus \{A_1 \cup \ldots \cup A_{k-1}\}. \end{cases}$$

Then we have:

$$SP[p_1,\ldots,p_n](u) = a \Leftrightarrow \{a\} = A \setminus \{A_1 \cup \ldots \cup A_n\}.$$

Lemma 2.

In voting scheme $\mathscr{V}[p_1,\ldots,p_n]$ the sophisticated behavior of the players selects the same alternative as the prudent behavior of the players does in the voting scheme $\mathscr{V}[p_n,p_{n-1},\ldots,p_1]$. That is to say, we have the formula:

$$SD[p_1,p_2,\ldots,p_n] = SP[p_n,p_{n-1},\ldots,p_1]. \tag{3}$$

The above Lemma claims that given the profile $u = (u_1,\ldots,u_n)$ the sophisticated voting of the players is defined by the following algorithm:

$$\begin{cases} \text{Player } n \text{ vetoes the set } B_n \text{ of the } p_n \text{ least preferred} \\ \quad \text{alternatives of } u_n \text{ among } A. \\ \text{For } k = n-1,n-2,\ldots,1, \text{ player } k \text{ vetoes the set } B_k \text{ of} \\ \quad \text{the } p_k \text{ least preferred alternatives of } u_k \text{ among} \\ \quad A \setminus \{B_n \cup B_{n-1} \cup \ldots \cup B_{k+1}\} \end{cases} \tag{4}$$

Then we have

$$SD[p_1,\ldots,p_n](u) = a \quad \Leftrightarrow \quad \{a\} = A \setminus \{B_n \cup B_{n-1} \cup \ldots \cup B_1\}.$$

The sophisticated voting of player 1 when the profile is u is to veto the alternatives of B_1: in order to compute B_1 he must then compute the whole sequence $B_n,B_{n-1},\ldots,B_2$, thus using his complete knowledge of the other players' preferences. This illustrates the high complexity of sophisticated voting.

Proof of Lemma 2.

We prove by induction on n, the number of players, that the sophisticated voting in game $\mathscr{V}[p_1,\ldots,p_n](u)$ is described by algorithm (4). This claim is trivial for n = 1. We assume it holds true up to (n-1). We compute now the sophisticated strategy of player 1. If he vetoes some set

C_1 of p_1 alternatives, then he can (by the inductive assumption) assume that the (n-1) other players will successively veto the subset $C_2, \ldots, C_n$ of A defined inductively as:

(5)

C_n is the set of the p_n least preferred alternatives of u_n among $A \setminus C_1$.

For $k = (n-1), (n-2), \ldots, 2$, C_k is the set of the p_k least preferred alternatives of u_k among $A_1 \setminus \{C_1 \cup C_n \cup C_{n-1} \cup \cdots \cup C_{k+1}\}$.

The selected alternative is finally c defined by

$$\{c\} = A \setminus \{C_1 \cup C_n \cup C_{n-1} \cup \cdots \cup C_2\}.$$

We must now prove that c is not preferred by u_1 to a defined by (4). For that purpose we compare the sequences B_n, B_{n-1}, ... and C_n, C_{n-1}, ... by which a and c have been successively defined. We remark that if B and C are any two subsets of A such that $B \subset C$ and if we set

$\tilde{B}$ = the p_k least preferred alternatives of u_k among $A \setminus B$

$\tilde{C}$ = the p_k least preferred alternatives of u_k among $A \setminus C$

then we have $\tilde{B} \subset C \cup \tilde{C}$.

Applying this remark to $B = \phi \subset C_1 = C$ and $k = n$ we obtain:

$$B_n \subset C_1 \cup C_n.$$

Applying the remark again to the above inclusion and $k = (n-1)$, we obtain:

$$B_{n-1} \subset C_1 \cup C_n \cup C_{n-1} \Rightarrow B_n \cup B_{n-1} \subset C_1 \cup C_n \cup C_{n-1}.$$

Clearly we can apply the remark inductively for decreasing k, so that for every k we obtain:

$$B_n \cup \dots \cup B_k \subset C_1 \cup C_n \cup \dots \cup C_k.$$

And finally:

$$B_n \cup \dots \cup B_2 \subset C_1 \cup C_n \cup \dots \cup C_2,$$

that is to say:

$$c \in A \setminus \{B_n \cup \dots \cup B_2\} = B_1 \cup \{a\}.$$

By definition of B_1 we have:

$$\forall\, a' \in B_1 \cup \{a\}, \qquad u_1(a) \geqslant u_1(a').$$

Therefore $u_1(a) \geqslant u_1(c)$, which was to be proved. By Lemmas 1 and 2 the sophisticated and prudent behaviors of the players in $[p_1,\dots,p_n]$ are both deterministic and are deduced from one another by simply reversing the ordering of the players.

3. Voting by Alternating Veto

Voting by veto procedures introduce a strong dissymmetry among agents: even if the veto powers $p_1, \ldots, p_n$ are all equal, or nearly equal, the ordering of the agents has a strong influence on the outcome of both the sophisticated and the prudent voting (this obvious intuition is made clear by the two algorithms above, (2) and (4)).

To reduce and sometimes avoid this dissymmetry we will replicate the alternatives and at the same time alternate the strategical vetoes of the players.

Definition 7

Let r and, for every alternative a, k_a be non-negative integers such that:

$$(6) \qquad n \cdot r = \left(\sum_{a \in A} k_a \right) - 1$$

To every r, k_a, verifying (6), we associate a family of procedures defined as follows:

Each agent is endowed with r token. There are r rounds of vetoing. At each round the agents are successively asked (the ordering of the agents can change at each round, but the agents cannot influence it) to throw one token over one of the alternatives, a, on which at most $(k_a - 1)$ tokens have already been thrown. The elected alternative is the alternative a which after r rounds received only $(k_a - 1)$

tokens (every other alternative b having in view of (6) received exactly k_b tokens). We will call these procedures voting by alternating veto. This terminology is made clear by the following interpretation of the above procedure: each alternative is replicated, a certain number of times, alternative a being replicated k_a times. At each round the players must successively veto one replica of one of the remaining alternatives.

The basic property of voting by alternating veto is that if we order carefully the successive vetoes by the agents, then the sophisticated behaviour of the agents cannot be distinguished from a prudent one.

Definition 8

Let $\mathcal{V}=(X_1, \ldots, X_n, \pi)$ be a d - solvable n person voting scheme among A. We say that $\mathcal{V}$ is exactly solvable if for every profile $u \in U^n$, the sophisticated voting of the players also is a prudent voting. That is to say:

$$(7) \qquad \forall \quad u \in U^n \quad SD(u) \in SP(u) \ .$$

If (7) holds true we say that V exactly implements the social choice function SD.

This terminology is derived from Peleg's concept of an exactly consistent voting scheme (see [11]). In an exactly consisteng voting scheme, there exists for every profile an equilibrium n- uple which cannot be distinguished from the 'tell-the-truth' strategy. However, there is no obvious comparison of the two concepts.

A strategy proof voting scheme is in particular exactly solvable since both the sophisticated and the prudent behaviour of a player amount to selecting a dominant strategy. In general exact-solvability is a weakening of strategy-proofness which amounts to saying that the completely informed, non-cooperative agents eventually select the same alternative as completely uninformed risk-averse agents. The next theorem displays a family of exactly solvable voting schemes:

Theorem 1

Suppose r is an even integer and k_a , $a \in A$ are non-negative integers verifying (6) . Then the following associated voting by veto procedure is exactly solvable: During the first $\frac{r}{2}$ rounds, the ordering of the players is fixed, say $1, \ldots, n$. During the last $\frac{r}{2}$ rounds this ordering is simply reversed, say $n, (n-1), \ldots, 1$.

Proof

We first introduce a useful notation: for every finite sequence $i_1, \ldots, i_{q-1}$ with values in the set $\{1, \ldots, n\}$ of agents, we denote by $W[i_1, \ldots, i_{q-1}]$ the n-person voting scheme among q alternatives (where q will be in fact much greater than p) in which the players successively veto one of the remaining alternatives. More precisely the procedure works as follows: player i_1 vetoes first one alternative, say a_1; the next player i_2 vetoes one alternative, say a_2 among $A - \{a_1\}$; and so on . . . ; at stage t , player i_t vetoes one alternative, say a_t, among $A - \{a_1, \ldots, a_{t-1}\}$; because a contains q elements, there is exactly one alternative a_q in $A - \{a_1, \ldots, a_{q-1}\}$: this is the elected alternative.

For instance, the voting scheme $\mathcal{V}[p_1, \ldots, p_n]$ is now equivalently written as:

$$W[\underbrace{1, \ldots, 1}_{p_1 \text{ times}}, \underbrace{2, \ldots, 2}_{p_2 \text{ times}}, \ldots, \underbrace{n, \ldots, n}_{p_n \text{ times}}]$$

In the voting scheme described in the statement of Theorem 1 there are $q = \sum_{a \in A} k_a$ alternatives: alternative a is replicated k_a - times. The procedure is then:

$$(8) \quad W[\underbrace{12, \ldots, n}_{1}, \underbrace{1, \ldots, n}_{2}, \ldots, \underbrace{1, \ldots, n}_{r/2}, \underbrace{n(n-1), \ldots, 1}_{1}, \underbrace{n, \ldots, 1}_{2}, \ldots, \underbrace{n, \ldots, 1}_{r/2}]$$

In order to prove the theorem we must describe the sophisticated voting of the players in (8) and check that it coincides with a prudent behaviour. In view of algorithm (4) describing the sophisticated voting in the voting scheme $\mathcal{V}[p_1, \ldots, p_n]$ we obtain easily the sophisticated voting in $W[i_1, \ldots i_{q-1}]$. Denote by B the set of q (different) alternatives, and $u = (u_1, \ldots, u_n)$ a particular profile. Then SD(u) is described by the following algorithm:

Let b_{q-1} be the least preferred alternative of $u_{i_{q-1}}$ among B

Let b_{q-2} be the least preferred alternative of $u_{i_{q-2}}$ among $B \setminus \{b_{q-1}\}$

. . .

(9)

Let b_k be the least preferred alternative of u_{i_k} among $B \setminus \{b_{q-1}, \ldots, b_{k+1}\}$

. . .

Then $b = SD(u) \Leftrightarrow \{b\} = B \setminus \{b_{q-1}, \ldots, b_1\}$

Remark that in order that each alternative b_k is well defined we have assumed that each utility function is one-to-one over B. Actually we can weaken this assumption and still give sense to algorithm (6). For instance, suppose that if one agent is indifferent between two alternatives of B, then every other agent also is indifferent:

(10) $\forall i, j \in \{1, \ldots, n\} \ \forall a, b \in B (u_i(a) = u_i(b)) \Leftrightarrow (u_j(a) = u_j(b))$.

Then two alternatives a,b such that every agent is indifferent between them can be identified so that algorithm (9) yields undiscernable alternatives. In the case of voting scheme (8), the set B has the form:

$$B = \{ \underbrace{a, \ldots, a}_{k_a \text{ times}}, \underbrace{b, \ldots, b}_{k_b \text{ times}}, \ldots \}$$

By assumption the utility functions u_i are one-to-one on A so that on B property (10) holds. Therefore a straightforward application of algorithm (9) yields the sophisticated voting in scheme (8) .

On the other hand, we remark that in voting scheme $W(i_1, \ldots, i_{q-1})$ it is a prudent behaviour for any player to veto his least preferred alternative among the remaining ones whenever he has to veto some alternative. Thus the following algorithm defines a selection of the social choice correspondence p - implemented by $W(i_1, \ldots, i_{q-1})$:

Let b_1 be the least-preferred alternative of u_{i_1} among B .

Let b_2 be " " " u_{i_2} among $B \setminus \{b_1\}$.

. . . .

(11)

Let b_k " " " u_{i_k} among $B \setminus \{b_1, \ldots b_{k-1}\}$.

Then if $\{b\} = B \setminus \{b_1, \ldots, b_{q-1}\}$, we have $b \in SP(u)$

Comparing algorithm (9) and (11) it is clear that $W(i_1, \ldots, i_{q-1})$ is exactly solvable if the two sequences $\{i_1, i_2, \ldots, i_{q-1}\}$ and $\{i_{q-1}, i_{q-2}, \ldots, i_2, i_1\}$ coincide. It is clearly the case for the voting scheme (8); this completes the proof of Theorem 1 .

The property of exact-solvability is not by itself sufficient to make a voting scheme ethically attractive. An exactly solvable voting scheme is nothing but a procedure wherein the collective implications of non-cooperation are easily predictable, just as they are in a strategy-proof dictatorial voting scheme. To make the voting scheme more attractive, we would like that the social choice function that it exactly implements shares some additional properties. Among these, the three most usually desired are efficiency, anonymity and neutrality. We say that the social choice function S is

-- efficient if $S(u)$ is a Pareto optimum alternative of the profile u ;

-- anonymous if $S(u_1, \dots, u_n)$ is a symmetrical function of $(u_1, \dots, u_n)$ (S does not discriminate among players) ;

-- neutral if S does not discriminate among alternatives, which is formalized as follows: for every one-to-one mapping σ from A into itself, and every profile $(u_1, \dots, u_n)$:

$$\sigma(S(u_1 \circ \sigma , \dots, u_n \circ \sigma)) = S(u_1, \dots, u_n) .$$

Lemma 3

For every r , k_a $(a \in A)$ verifying (6) every associated voting by alternating veto d-implements an efficient social choice function.

Proof

It suffices to show that in any voting scheme of the form $W[i_1, \dots, i_{q-1}]$ the alternative b selected by algorithm (9) is efficient. By (9) every alternative other than b is one of the b_k 's and therefore is less preferred than b by at least one utility function u_{i_k} . Therefore it cannot dominate b .

Lemma 4

The social choice function exactly implemented by the voting by alternating veto (8) is neutral if and only if $k_a = k$ does not depend on a . This is possible only if p is odd and relatively prime with respect to n .

Proof

Suppose r , k_a $(a \in A)$ are given such that k_a does depend on a . Then we fix $a,b \in A$ such that $k_a < k_b$. In this case we prove that any corresponding voting by alternating veto implements a non-neutral social choice function. Suppose that the profile $(u_1, \ldots, u_n)$ is such that:

$$\forall\, i = 1, \ldots, n \quad \forall\, c \in A \qquad c \neq a,b \Rightarrow u_i(c) < \inf \{u_i(a), u_i(b)\} \ .$$

Then we set $k = k_a + k_b - 1$ and we develop our particular voting by alternating veto which can be written as:

$$W[i_1, \ldots, i_k, i_{k+1}, \ldots, i_{n \cdot r}] \ .$$

In view of the profile that we consider, the sophisticated voting of the players, described by algorithm (9) , guarantees that every player after stage $k+1$ will veto one replica of one alternative of $A \setminus \{a,b\}$. Since

$n \cdot r - k = \sum_{c \in A \setminus \{a,b\}} k_c$ we deduce that at every stage before stage k (including stage k) the players will veto one replica either of a or of b. We can then select a profile such that from i_1 to i_k, the players who prefer a more than b are as many as the players who preferr b more than a , or these two numbers differ at most by one:

$$(12) \quad \left| \#\{j=1,..,k/U_{i_j}(a) > U_{i_j}(b)\} - \#\{j=1,\dots,k/U_{i_j}(b) > U_{i_j}(a)\} \right| \leq 1 .$$

Then one checks easily that alternative a is selected by the sophisticated voting of the players. If we consider the permutation σ : $\sigma(a) = b$; $\sigma(b) = a$; $\sigma(c) = c$ otherwise, then property (12) still holds true for the profile $(u_1 o \sigma, \dots, u_n o \sigma)$ so that alternative a is again selected by sophisticated players. This contradicts the neutrality of the social choice function implemented by this voting by alternating veto procedure.

Conversely, if k_a does not depend on a , it is obvious that the corresponding voting by alternating veto are neutral procedures. Coming back to equation $r \cdot n = \sum_{a \in A} k_a - 1$ this yields:

$$(13) \qquad r \cdot n = k \cdot p - 1 .$$

An elementary result of arithmetic (known as the Bezout identity) yields the following statement: if n and p are given, then there exist non-negative integers r, k with r even, verifying (13) if and only if

p is odd and relatively prime with respect to n .

We come now to the anonymity property: in our context it is the most desirable since the dictatorial social choice functions are bad only with respect to anonymity arguments. At this point the use of a rather complex procedure like the one we considered becomes fully justified. Notice first that voting by alternating veto procedures are essentially anonymous because every agent is endowed with the same number of tokens, that is to say the same 'veto power.'[1] Moreover, when r , the number of rounds , becomes large, intuition suggests that the ordering of the players becomes less relevant so that the implemented social choice function becomes nearly anonymous. This intuition actually is a rigorous statement for some value of n :

Lemma 5

Let n be a prime integer greater than p . Let r and k be non-negative integers such that

$$(14) \quad r \cdot n = k \cdot p - 1 \ .$$

Then for r , k large enough verifying (14) , the voting by alternating veto procedures where $k_a = k$ for every a d-implement an anonymous social choice function. The proof of this Lemma, which is by no means trivial, can be found in [9] .

Gathering the results of the three above Lemmas, we obtain the following

[1] Notice that when $r < k$, i.e. when $p < n$, no agent can alone veto even one alternative

theorem which summarizes the desirable properties of voting by alternating veto.

Theorem 2

Let n be a prime integer, let p be odd and strictly smaller than n. Then there exists a pair r, k of non-negative integers verifying

$$r \cdot n = k \cdot p - 1$$

and an associated voting by alternating veto procedure, which exactly implements an efficient, anonymous and neutral social choice function. The arithmetic condition (n should be prime and greater than p) is very natural since there does not exist an efficient anonymous and neutral social choice function unless every prime factor of n is greater than p. Let us mention finally that the social choice function exactly implemented by alternating veto is also a monotonic social choice function (this is obvious in view of algorithm (9)), a property shared by most of the social choice functions implemented by dominance solvable voting schemes.

4. Conclusion and Open Problems

The very concept of exact implementation should be hopefully strengthened to make its strategical interpretation more convincing. Instead of requiring that $SD(u)$, the alternative selected by the sophisticated voting of the agents, should be a member of $SP(u)$, the possibly numerous alternatives that can be selected by their prudent behaviour, it would be undoubtedly more convincing to demand $SD(u) = SP(u)$, which entails the additional requirement that the prudent behaviour actually is non-ambiguous. This is not satisfied by voting by alternating veto since the prudent behaviour might be ambiguous, even for large values of r .

Suppose for instance $p = 3$ and $n = 5$, and choose a pair of (large) integers r,k such that

$$5r = 3k - 1 \quad .$$

Suppose agent 1 's preferences are:

$$u_1(a) > u_1(b) > u_1(c)$$

Because $r < k$ player 1 cannot *a priori* prevent c from being elected. Therefore his prudent behaviour during the first rounds of vetoing is to systematically veto one replica of alternative c . Suppose now that after t rounds of vetoing, $(k - r + t + 1)$ tokens have been thrown on c by player 1 himself as well as by the other players. Because player 1 is left with $r - t$ tokens he has <u>now</u> the power to prevent the election of alternative c and still being left with one token; namely:

$$(r - t) + (k - r + t + 1) = k + 1 \ .$$

Accordingly he has several options within his very prudent behaviour: he can decide to throw first one token over b and then his $(r - t - 1)$ remaining tokens over c ifnobody else contributes to the elimination of c ; alternatively he can decide to prevent as soon as possible the election of c . These two options are not equivalent and this explains why $SP(u)$ is not in general single-valued. The author failed to determine whether or not Theorem 1 (as well as Theorem 2) still hold true for some reasonable family of voting procedures, if we use the stronger version of exact implementation mentioned above. We feel however that Theorem 2 allows us to support voting by alternating veto as a highly valuable voting scheme both with respect to its strategical and equity properties.

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